THE PROBLEMS OF MODERN COSMOLOGY

A volume in honour of Professor S.D. Odintsov
in the occasion of his 50th birthday

Tomsk 2009

Professor S.D. Odintsov among the participants of DSU International Conference in Egypt (2008) (from left to right: Profs. J. Valle, S. Zerbini, S.D. Odintsov, E. Elizalde)

The Problems of Modern Cosmology.
A volume in honour of Professor S.D. Odintsov in the occasion of his 50th birthday.
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The volume contains the papers on the problems of modern cosmology, gravity and theoretical physics written by colleagues, coauthors and friends of Professor S.D. Odintsov. Most of the papers are the reviews devoted to the current state of modern cosmology and gravity. The volume is intended for researchers and PhD students in area of modern cosmology and theoretical physics.

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Preface

On March 2, 2009 Professor Sergey Dmitrievich Odintsov, the well-known and internationally recognized researcher in area of cosmology and theoretical physics, is celebrating his fiftieth birthday. The current special volume represents the collection of papers of his friends and colleagues from all over the world. The volume is published in the relation with this significant event and is devoted to the discussion of modern cosmology problems.

Scientific interests by S.D. Odintsov are quite wide: quantum gravity, quantum field theory in curved space-time, higher dimensional (Kaluza-Klein and Randall-Sundrum-type) theories, renormalization group and phase transitions, inflationary and late-time Universe evolution, modified gravity, black holes, astrophysics, etc. The fundamental results some of which became internationally famous have been obtained in all these areas of high energy physics and cosmology.

The scientific investigation by Sergey Odintsov is closely related with Tomsk State Pedagogical University (TSPU) where he started as an assistant professor and then became full professor, head of the Laboratory of Fundamental Study and leader of the research area on cosmology and astrophysics at TSPU. TSPU Professor and ICREA Research Professor S.D. Odintsov has supervised number of PhD students at Tomsk State Pedagogical University. In his scientific work S.D. Odintsov collaborates with researchers from different countries. It is worth to mention his specially close scientific relations with Japan (Nagoya University, Hiroshima University and Yukawa Institute of Theoretical Physics, Kyoto University), Spain (Space Research Institute (ICE) and Barcelona University), Norway (University of Trondheim) and Italy (Trento University). The academical/research exchange agreements have been established between TSPU and most of above international centers thanks to efforts by Prof. S.D. Odintsov.

On his 50th birthday Sergey devotes all his time to science as it was when he has been PhD student at Tomsk State University. His research plans are ambitious and he is extremely scientifically productive.

His friends, colleagues, TSPU professors and contributors present this volume from their hearts as a gift on his 50th anniversary. All of them wish him even better scientific achievements and good luck and good health for many years ahead!

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Brane-Wor(l)ds within Brans-Dicke Gravity

Dedicated to the 50 year Jubilee of Professor S. D. Odintsov

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Abstract

We review some recent results obtained from the application of the Gauss-Codazzi formalism to brane-worlds models in the Brans-Dicke gravity. The cases of 4-branes embedded in a six-dimensional with and without $\mathbb{Z}_2$ symmetry are both analyzed.

1 Introduction

Soon after the establishment of the modern concepts of brane-worlds models [9] it was realized their importance in the study of gravitational systems [2], as well as their application in the analysis of cosmological problems [3]. For codimension one brane-world models there is a very powerful tool — the so-called Gauss-Codazzi formalism — developed in order to project the gravitational field equations from the bulk to the brane [4]. This procedure allows the exploration of brane cosmological signatures in a deep way. Roughly speaking, the existence of extra dimensions, within gravitational context, added new source terms to the brane projected gravitational field equation. Obviously, this also happens in the Brans-Dicke [10] gravity. Besides, new "source terms" appear due to the dynamics of the additional gravitational scalar field. These terms appearing in the projected gravitational equations in the Brans-Dicke theory certainly lead to new cosmological signatures. The reason to analyze brane-world models in such scalar-tensorial theory rests on the fact that, at least sufficiently high energies, the General Relativity is not able to fully describe mostly of the puzzled gravitational phenomena [19]. Apart of that, there is a strong interplay between Brans-Dicke theory and the gravity theory recovered from string theory at low energies [7]. In this vein, it is possible to obtain information about some systems in string theory by the analysis in

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Brans-Dicke framework. The relationship between the two theories is given by a — model depended — relabel of the Brans-Dicke parameter.

From the topological point of view, the brane(s) in the standard Randall-Sundrum model [9] is (are) performed by domain wall(s) and the extra dimension is a $S^1/\mathbb{Z}_2$ orbifold. It is familiar for cosmologists, however, that domain walls (with symmetry breaking scales greater than 1 MeV) are problematic [8] and should not appear in a complete scenario. The program of an exotic compactification of extra dimensions using topological defects continued with global cosmic strings in General Relativity [9] and with local and global cosmic strings in the Brans-Dicke gravity [10]. Because of the defect used to generate the bulk-brane structure (the cosmic string), these models hold in six-dimensions. Models studied in the Brans-Dicke framework [10] present only one transverse extra dimension (codimension one models), while the brane has five-dimensions with topology given by $\mathbb{R}^4 \times S^1$. The presence of a transverse and a curled dimension in the same model is called hybrid compactification [11].

The aim of this work is to review the application of the Gauss-Codazzi formalism for hybrid compactification models in the Brans-Dicke framework. In Section II we develop the main lines of this approach in the case where the spacetime is endowed with $\mathbb{Z}_2$ orbifold symmetry. In the Section III, we work in the case without such symmetry and in Section IV, we conclude with some final remarks and possible applications. We stress that the main results delivered here, as well as the details, are somehow described in [12, 13].

2 Projected gravitational field equations in $\mathbb{Z}_2$ symmetric brane-worlds

From now on we consider the brane as a five-dimensional submanifold embedded in a manifold of six-dimensions — the bulk. As remarked before, the motivations to work in such dimensionality can be found in [9, 10] (and references therein). Denoting the covariant derivative in the bulk by ∇_μ and the one associated to the brane by D_μ the Gauss equation reads [14]

$$^{(5)}R^{\alpha}_{\beta\gamma\delta} = {}^{(6)}R^{\mu}_{\nu\rho\sigma} q_{\mu}^{\ \alpha} q_{\beta}^{\ \nu} q_{\gamma}^{\ \rho} q_{\delta}^{\ \sigma} + K^{\alpha}_{\ \gamma} K_{\beta\delta} - K^{\alpha}_{\ \delta} K_{\beta\gamma}, \tag{1}$$

where $q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric on the brane (n_μ being the orthogonal unitary vector along to the extra transverse dimension) and $K_{\mu\nu} = q_{\mu}^{\ \alpha} q_{\nu}^{\ \beta} \nabla_\alpha n_\beta$ is the extrinsic curvature, which gives information about the embedding of the brane. Starting from equation (1.1), it is easy to see that the Einstein tensor on the brane in given by

$$\begin{aligned} ^{(5)}G_{\beta\delta} &= {}^{(6)}G_{\nu\sigma} q_{\beta}^{\ \sigma} q_{\delta}^{\ \sigma} + {}^{(6)}R_{\nu\sigma} n^\nu n^\sigma q_{\beta\delta} + K K_{\beta\delta} - K_{\delta}^{\ \gamma} K_{\beta\gamma} \\ &- \frac{1}{2} q_{\beta\delta}(K^2 - K^{\alpha\gamma} K_{\alpha\gamma}) - \tilde{E}_{\beta\delta}, \end{aligned} \tag{2}$$

where $\tilde{E}_{\beta\delta} = {}^{(6)}R^{\mu}_{\nu\rho\sigma} n_\mu n^\rho q_{\beta}^{\ \nu} q_{\delta}^{\ \sigma}$. Now, taking into account that the relation between the Riemann, Ricci and Weyl tensors in an arbitrary dimension (n) is

$$\begin{aligned} ^{(n)}R_{\alpha\beta\mu\nu} &= {}^{(n)}C_{\alpha\beta\mu\nu} + \frac{2}{n-2}\left({}^{(n)}R_{\alpha[\mu} g_{\nu]\beta} - {}^{(n)}R_{\beta[\mu} g_{\nu]\alpha} \right) \\ &- \frac{2}{(n-1)(n-2)} {}^{(n)}R g_{\alpha[\mu} g_{\nu]\beta}, \end{aligned} \tag{3}$$

we can rewrite the equation (1.2) in the form

$$\begin{aligned}
{}^{(5)}G_{\beta\delta} &= \frac{1}{2}\,{}^{(6)}G_{\nu\sigma}q_\beta^{\ \nu}q_\delta^{\ \sigma} - \frac{1}{10}\,{}^{(6)}Rq_{\beta\delta} - \frac{1}{2}\,{}^{(6)}R_{\nu\sigma}q^{\nu\sigma}q_{\beta\delta} + KK_{\beta\delta} - K^\gamma_{\ \delta}K_{\beta\gamma} \\
&- \frac{1}{2}q_{\beta\delta}(K^2 - K^{\alpha\gamma}K_{\alpha\gamma}) - E_{\beta\delta},
\end{aligned} \tag{4}$$

where $E_{\beta\delta} ={}^{(6)} C^\mu_{\ \nu\rho\sigma}n_\mu n^\rho q_\beta^{\ \nu}q_\rho^{\ \sigma}$.

The idea, hereafter, is to express the geometric quantities of the brane in terms of the stress-tensor and the scalar dynamics of the bulk, in order to apply the Gauss-Codazzi formalism to the case in question. With this purpose, we remember that the Einstein-Brans-Dicke equation is given by

$$\begin{aligned}
{}^{(6)}G_{\mu\nu} &= \frac{8\pi}{\phi}T_{M\mu\nu} + \frac{w}{\phi^2}\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi\Big) \\
&+ \frac{1}{\phi}\Big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box^2\phi\Big),
\end{aligned} \tag{5}$$

where ϕ is the Brans-Dicke scalar field — the dilaton —, w a dimensionless parameter and $T_{M\mu\nu}$ the matter energy-momentum tensor, everything except ϕ and gravity, in the bulk. The scalar equation of Brans-Dicke theory is given by

$$\Box^2\phi = \frac{8\pi}{3+2w}T_M. \tag{6}$$

Inserting the equation (6) into (0.3) and founding the Ricci tensor and the scalar of curvature, it is possible to express the equation (4) as

$$\begin{aligned}
{}^{(5)}G_{\beta\delta} &= \frac{1}{2}\left[\frac{8\pi}{\phi}T_{M\nu\sigma} + \frac{1}{\phi}\nabla_\nu\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\nu\phi\nabla_\sigma\phi\right](q_\beta^{\ \nu}q_\delta^{\ \sigma} - q^{\nu\sigma}q_{\beta\delta}) \\
&+ \frac{2\pi}{5\phi}q_{\beta\delta}T_M\left(\frac{13+27w}{3+2w}\right) - \frac{7w}{20\phi^2}q_{\beta\delta}\nabla_\alpha\phi\nabla^\alpha\phi + KK_{\beta\delta} - K^\gamma_{\ \delta}K_{\beta\gamma} \\
&- \frac{1}{2}q_{\beta\delta}(K^2 - K^{\alpha\gamma}K_{\alpha\gamma}) - E_{\beta\delta}.
\end{aligned} \tag{7}$$

In order to extract information about this system we have to compute the quantities on the brane. It can be implemented by taking the limit of the extra transverse dimension tending to the brane, but we have to specify the behavior of the extrinsic curvature under such limit. This is a central piece in the application of the Gauss-Codazzi formalism and strongly depends whether or not the spacetime is endowed with a $\mathbb{Z}_2$ symmetry. In the case where there is such symmetry, it is possible to show, by application of distributional calculus tools, that the extrinsic curvature in one side of the brane, $K^+_{\mu\nu}$, is related with it's other side partner, $K^-_{\mu\nu}$ by (see reference [12] for all the details)

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = \frac{4\pi}{\phi}\left(- T_{\mu\nu} + \frac{q_{\mu\nu}(1+w)T}{2(3+2w)}\right), \tag{8}$$

and

$$K^+ = K^- = \frac{2\pi}{\phi}\left(\frac{w-1}{3+2w}\right)T. \tag{9}$$

The relation (0.4) is the generalization of the so-called Israel-Darmois matching conditions [15] to the Brans-Dicke gravity. It is possible to split the matter stress-tensor in

$$T_{M\mu\nu} = -\Lambda g_{\mu\nu} + \delta(y) T_{\mu\nu}, \tag{10}$$

and

$$T_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu}, \tag{11}$$

where Λ is the cosmological constant of the bulk and λ the tension of the brane[4]. Now, substituting the equations (0.4)-(0.7) into (7) we obtain

$$\begin{aligned} {}^{(5)}G_{\beta\delta} &= \frac{1}{2}\left[\frac{1}{\phi}\nabla_\nu\nabla_\sigma\phi + \frac{w}{\phi^2}\nabla_\nu\phi\nabla_\sigma\phi\right](q_\beta^{\ \nu} q_\delta^{\ \sigma} - q^{\nu\sigma} q_{\beta\delta}) + 8\pi\Omega\tau_{\beta\delta} - \Lambda_5 q_{\beta\delta} \\ &+ 8\left(\frac{\pi}{\phi}\right)^2 \Sigma_{\beta\delta} - E_{\beta\delta} \ , \end{aligned} \tag{12}$$

where

$$\Omega = \frac{3\pi(w-1)\lambda}{\phi^2(3+2w)}, \tag{13}$$

$$\Lambda_5 = \frac{-4\pi\Lambda(21-41w)}{5\phi(3+2w)} + \left(\frac{\pi}{\phi}\right)^2\left[\frac{7w}{20\pi^2}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{24(w-1)\lambda}{(3+2w)^2}[(w-1)\lambda + \tau]\right] \tag{14}$$

and

$$\Sigma_{\beta\delta} = q_{\beta\delta}\tau^{\alpha\gamma}\tau_{\alpha\gamma} - 2\tau^{\gamma}_{\delta}\tau_{\gamma\beta} + \left(\frac{3+w}{3+2w}\right)\tau\tau_{\beta\delta} - \frac{(w^2+3w+3)}{(3+2w)^2} q_{\beta\delta}\tau^2. \tag{15}$$

The main property to be noted from equation (0.8) is that we do not recover Brans-Dicke gravity on the brane if the dilaton depends only on the extra transverse dimension. Instead, Einstein equation is recovered with subtle but important modifications coming from both extra dimensions and dilaton dynamics. Hence, equation (0.8) can be used to extract deviations from usual General Relativity. We refer again the reader to reference [12] for more analysis and comments on the implications of the result encoded in (0.8)-(0.11).

3 Lifting the $\mathbb{Z}_2$ symmetry

The $\mathbb{Z}_2$ symmetry has a multiple role in brane-worlds scenarios [13]. In what concerns to the gravitational aspects it determines univocally the jump of the extrinsic curvature across the brane (0.4). In this vein, it is not surprising that the absence of such symmetry makes the calculations a little more involved. In this Section we shall present the guidelines of how to project the Einstein-Brans-Dicke equation on the brane without the $\mathbb{Z}_2$ symmetry. More details can be found in reference [13] for the Brans-Dicke case and in [16] for the context of General Relativity[5]. Let us start defining two quite important tools which determine the mean value of any tensorial quantity, say X,

$$\langle X \rangle = \frac{1}{2}(X^+ + X^-), \tag{16}$$

[4]We remark that the delta term appearing in such decomposition can lead to complications in a complete cosmological scenario.

[5]Actually, the reference [13] is a first generalization of the work presented in [16] to the Brans-Dicke framework.

and the jump across the brane

$$[X] = X^+ - X^-, \tag{17}$$

where $X^\pm$ are both limits of X approaching the brane from both $\pm$ sides. It is not difficult to see that the quantities defined by equations (0.12) and (0.13) lead to the algebra

$$[AB] = \langle A\rangle[B] + [A]\langle B\rangle, \tag{18}$$

$$\langle AB\rangle = \langle A\rangle\langle B\rangle + \frac{1}{4}[A][B]. \tag{19}$$

Note that from the Gauss equation (1.1), we can write down the Ricci tensor on the brane in a more convenient way

$$^{(5)}R_{\mu\nu} = Y_{\mu\nu} + KK_{\mu\nu} - K_\mu^\lambda K_{\lambda\nu}, \tag{20}$$

where

$$Y_{\mu\nu} \equiv \frac{3}{4}\,{}^{(6)}R_{\alpha\beta}q_\mu^\alpha q_\nu^\beta + \frac{1}{4}\,{}^{(6)}R_{\alpha\beta}q^{\alpha\beta}q_{\mu\nu} - \frac{1}{5}\,{}^{(6)}Rq_{\mu\nu} + E_{\mu\nu}. \tag{21}$$

In order to obtain the projected equation on the brane, one needs to apply the limits defined in (0.12) and (0.13) into (0.17). Starting with $[{}^{(5)}R_{\mu\nu}]$, one has

$$[{}^{(5)}R_{\mu\nu}] = 0 = [Y_{\mu\nu}] + \langle K\rangle[K_{\mu\nu}] + [K]\langle K_{\mu\nu}\rangle - \langle K_{[\mu}{}^\alpha\rangle[K_{\nu]\alpha}]. \tag{22}$$

Now, by using the same decomposition showed in equations (0.6) and (0.7), the equations (0.4) and (0.5) give, respectively

$$[K_{\mu\nu}] = -\frac{8\pi}{\phi}\left(\tau_{\mu\nu} + \frac{q_{\mu\nu}}{2(3+2w)}((w-1)\lambda - (w+1)\tau)\right), \tag{23}$$

and

$$[K] = \frac{8\pi(w-1)}{2\phi(3+2w)}(\tau - 5\lambda). \tag{24}$$

Hence, in the light of (0.19) and (0.20), the equation (0.18) results in

$$\begin{aligned} -\left(\frac{8\pi}{\phi}\right)^{-1}[Y_{\mu\nu}] &= \langle K_{\alpha[\mu}\rangle\tau_{\nu]}^\alpha - \left(\tau_{\mu\nu} + \frac{q_{\mu\nu}}{2(3+2w)}((w-1)\lambda - (w+1)\tau)\right)\langle K\rangle \\ &+ \frac{(3(1-w)\lambda - (w+3)\tau)}{2(3+2w)}\langle K_{\mu\nu}\rangle\ . \end{aligned} \tag{25}$$

This last equation will be useful helping to find the mean value of the extrinsic curvature. Firstly, however, let us derive the full projected equation. The mean operator acting on (0.17) gives

$$\langle {}^{(5)}R_{\mu\nu}\rangle = {}^{(5)}R_{\mu\nu} = \langle Y_{\mu\nu}\rangle + \frac{1}{4}\Big([K][K_{\mu\nu}] - [K_\mu{}^\alpha][K_{\nu\alpha}]\Big) + \langle K\rangle\langle K_{\mu\nu}\rangle - \langle K_\mu{}^\alpha\rangle\langle K_{\nu\alpha}\rangle. \tag{26}$$

Using the following decomposition of $Y_{\mu\nu}$ and $K_{\mu\nu}$ in the trace and traceless parts

$$Y_{\mu\nu} = \frac{Y}{5}q_{\mu\nu} + \varpi_{\mu\nu}, \tag{27}$$

$$K_{\mu\nu} = \frac{K}{5}q_{\mu\nu} + \zeta_{\mu\nu}, \tag{28}$$

the projected equation on the brane reads

$$^{(5)}G_{\mu\nu} = -\bar{\Lambda}_5 q_{\mu\nu} + 8\pi\Omega\tau_{\mu\nu} + 8\Big(\frac{\pi}{\phi}\Big)^2\Sigma_{\mu\nu} + \langle\varpi_{\mu\nu}\rangle + \frac{3}{5}\langle K\rangle\langle\zeta_{\mu\nu}\rangle - \langle{\zeta_\mu}^\alpha\rangle\langle\zeta_{\nu\alpha}\rangle, \tag{29}$$

where $\bar{\Lambda}_5$ is given by

$$\Lambda_5 = \frac{3}{10}\langle Y\rangle + \frac{6}{25}\langle K\rangle^2 - \frac{1}{2}\langle\zeta^{\alpha\beta}\rangle\langle\zeta_{\alpha\beta}\rangle + \frac{3}{8}\Big(\frac{8\pi}{\phi}\Big)^2\frac{\lambda(w-1)}{(3+2w)^2}(\tau + \lambda(w-1)), \tag{30}$$

and

$$\begin{aligned}\langle Y\rangle &= -2\Bigg(\Bigg\langle\frac{8\pi}{\phi}T_{M\mu\nu}n^\mu n^\nu\Bigg\rangle + \Bigg\langle\frac{w}{\phi^2}\Big(\phi,_\mu\phi,_\nu n^\mu n^\nu - \frac{1}{2}\phi,_\alpha\phi,^\alpha\Big)\Bigg\rangle \\ &+ \Bigg\langle\frac{1}{\phi}\Big(\phi,_{\mu;\nu}n^\mu n^\nu - \frac{8\pi}{3+2w}T_M\Big)\Bigg\rangle\Bigg),\end{aligned} \tag{31}$$

with $\phi,_\mu \equiv \nabla_\mu\phi$. Note the appearance of $\langle\zeta_{\mu\nu}\rangle$-like terms in (0.25). According to the decomposition (0.24) those terms arise only due to the absence of the $\mathbb{Z}_2$ symmetry and encodes information about the shear of the curled on-brane dimension. Therefore, it seems that the application of the Gauss-Codazzi formalism into non-$\mathbb{Z}_2$ symmetric brane-worlds describes hybrid compactification scenarios in a more natural way. Of course, by orbifolding the extra transverse dimension one reobtains the previous Section results.

It is necessary to go one step further in order to determine the mean value of the extrinsic curvature appearing in (0.25). To do so, let us define a convenient new brane matter stress-tensor by $\hat{\tau}_{\mu\nu} = \tau_{\mu\nu} + \frac{(3(1-w)\lambda-(w+3)\tau)}{4(3+2w)}q_{\mu\nu}$, in terms of which the equation (0.18) reads

$$0 = [Y_{\mu\nu}] + \langle K\rangle[K_{\mu\nu}] + \frac{8\pi}{\phi}\langle K_{[\mu}{}^\alpha\rangle\hat{\tau}_{\nu]\alpha}. \tag{32}$$

Now, after expressing $[K_{\mu\nu}]$ and $[K]$ in terms of that new stress-tensor $\hat{\tau}_{\mu\nu}$ and inserting it in equation (0.28) we find

$$\frac{8\pi}{\phi}\langle K\rangle = \frac{3(\hat{\tau}^{-1})^{\mu\nu}[Y_{\mu\nu}]}{9-(\hat{\tau}^{-1})^\mu_\mu\hat{\tau}^\nu_\nu}, \tag{33}$$

and again from (0.28) we arrive at

$$-\frac{8\pi}{\phi}\langle K_{[\mu}{}^\alpha\rangle\hat{\tau}_{\nu]\alpha} = [Y_{\mu\nu}] + \frac{3(\hat{\tau}^{-1})^{\alpha\beta}[Y_{\alpha\beta}]}{9-(\hat{\tau}^{-1})^\sigma_\sigma\hat{\tau}^\gamma_\gamma}(-\hat{\tau}_{\mu\nu} + \frac{\hat{\tau}}{3}q_{\mu\nu}), \tag{34}$$

or, in a more compact way,

$$\frac{8\pi}{\phi}\langle K_{[\mu}{}^\alpha\rangle\hat{\tau}_{\nu]\alpha} \equiv -[\hat{Y}_{\mu\nu}]. \tag{35}$$

The complete decoupling of $\langle K_{\mu\nu}\rangle$ can be obtained from the vielbein decomposition. Therefore, let us introduce a complete basis $h^{(i)}_\mu$ $(i = 0, 1, ..., 4)$ of orthonormal vectors constructed by the contraction of an orthonormal matrix set which represents a local Lorentz transformation and turns $\hat{\tau}_{\mu\nu}$ (and consequently $\tau_{\mu\nu}$) diagonal. The orthonormality conditions are given by

$$\begin{aligned} h^\mu{}_{(i)}h_{\mu(j)} &= \eta_{(i)(j)}, \\ \sum_{i,j=0}^{4}\eta_{(i)(j)}h_\mu{}^{(i)}h_\nu{}^{(j)} &= \sum_{j=0}^{4}h_\mu{}^{(j)}h_{\nu(j)} = q_{\mu\nu},\end{aligned} \tag{36}$$

where $\eta_{(i)(j)}$ is the Minkowski metric[6]. Expressing $\hat{\tau}_{\mu\nu}$ in terms of the vielbein, $\hat{\tau}_{\mu\nu} = \sum_i \hat{\tau}_{(i)} h_\mu{}^{(i)} h_{\nu(i)}$, we have from (35)

$$\frac{8\pi}{\phi}\langle K_{\mu\nu}\rangle = -\sum_{i,j} \frac{h_\mu{}^{(i)} h_\nu{}^{(j)}}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}}[\hat{Y}_{(i)(j)}], \tag{37}$$

after a contraction with $h^\mu{}_{(i)} h^\nu{}_{(j)}$. In the equation (0.31), $[\hat{Y}_{(i)(j)}] \equiv h^\mu{}_{(i)} h^\nu{}_{(j)}[\hat{Y}_{\mu\nu}]$. So, since the diagonal term of (0.31) is given by $\sum_{i=j} \frac{h_\mu{}^{(i)} h_\nu{}^{(j)}}{\hat{\tau}_{(i)}+\hat{\tau}_{(j)}}[\hat{Y}_{(i)(j)}] = \frac{1}{2}(\hat{\tau}^{-1})_\mu{}^\alpha [\hat{Y}_{\alpha\nu}]$, the generalized matching condition to the mean value of the extrinsic curvature reads

$$\begin{aligned}\frac{8\pi}{\phi}\langle K_{\mu\nu}\rangle &= \frac{1}{2}(\hat{\tau}^{-1})_\mu{}^\alpha [Y_{\alpha\nu}] + \frac{3(\hat{\tau}^{-1})^{\beta\gamma}[Y_{\beta\gamma}]}{2(9-(\hat{\tau}^{-1})^\sigma_\sigma \hat{\tau}^\rho_\rho)}\left(q_{\mu\nu} - \frac{\hat{\tau}^\rho_\rho(\hat{\tau}^{-1})_{\mu\nu}}{3}\right) \\ &- \sum_{i\neq j} \frac{h_\mu{}^{(i)} h_\nu{}^{(j)}}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}}[\varpi_{(i)(j)}].\end{aligned} \tag{38}$$

From equation (0.32) one can find the expression for $\langle K\rangle$, while from the decomposition (0.24) one finds $\langle\zeta_{\mu\nu}\rangle$. After all, the projected Einstein-Brans-Dicke equation in the orthonormal frame has the following diagonal terms

$${}^{(5)}G_{(i)(i)} = -\bar{\Lambda}_5 + 8\pi\Omega\tau_{(i)} + \Sigma_{(i)} + \langle\varpi_{(i)(i)}\rangle + \frac{3}{5}\langle K\rangle\langle\zeta_{(i)(i)}\rangle + \sum_k \langle\zeta_{(i)}{}^{(k)}\rangle\langle\zeta_{(i)(k)}\rangle, \tag{39}$$

where $\Sigma_{(i)} = \frac{1}{4}\left(\frac{8\pi}{\phi}\right)^2 \left(\frac{(w+3)}{2(3+2w)}\tau\tau_{(i)} - \tau^2_{(i)} + \frac{1}{2}\left(\sum_j \tau^2_{(j)}\right) - \frac{(w^2+3w+3)}{2(3+2w)^2}\tau^2\right)$. Moreover, the absence of the $\mathbb{Z}_2$ symmetry allows the existence of off-diagonal terms of the Einstein brane tensor, given by

$${}^{(5)}G_{(i)(j)} = \langle\varpi_{(i)(j)}\rangle + \frac{3}{5}\langle K\rangle\langle\zeta_{(i)(j)}\rangle + \sum_k \langle\zeta_{(i)}{}^{(k)}\rangle\langle\zeta_{(j)(k)}\rangle. \tag{40}$$

the equations (0.34) and (0.3) are the result of the generalization of the Gauss-Codazzi formalism to brane-worlds without the orbifold symmetry. A general characteristic of this procedure is the appearance of terms proportional to the shear of the extrinsic curvature, as well as the existence of off-diagonal elements in the Einstein projected tensor. These last two properties enables one to say that this formalism can extract more physical information when applied to hybrid compactification models.

4 Outlooks

This work can be positioned in the middle road between the pure formalism and the application. A more formal approach must take into account the advices arising in the study of consistence conditions for Brans-Dicke brane-worlds (BDBW) [17]. Nevertheless, the use of Gauss-Codazzi formalism is necessary in order to construct a bridge between formalism and phenomenology.

Apart of the rather technical approach focused in this work, the result encoded in the equations (0.8), (0.34) and (0.3) is exhaustive: the presence of the dilatonic field, by all means,

[6] Note that we are not assuming Einstein's summation convention over the tangent indices.

brings new signatures if applied to cosmological systems. In this vein, several possibilities are open to further investigation in the scope of BDBW. A systematic study of the problems analyzed, for instance, in [3] in the context of BDBW models can certainly provide new insights about high energy physics, as well as about the physics of extra dimensions itself. Moreover, we hope that in the study of specific cosmological problems, as the galactic rotation curves for example, the ubiquitous presence of the Brans-Dicke parameter restricts the wide range of possible adjustments, coming from the projected gravitational field equations, and points out to a more phenomenologically viable scenario. Of course, such restriction is possible only if one is willing to accept that the current lower bound of the Brans-Dicke parameter, coming from Solar System experiments [18], is also valid in the brane-world framework.

To finalize we should remark that, since the Brans-Dicke theory can mimic gravity recovered from string theory at low energy, at least in some regimes, the study of the cosmological aspects of BDBW can also bring some information about problems in string cosmology.

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Cosmological Dark Energy and the Quantum Conformal Factor

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Abstract

We review an approach to the problem of vacuum energy in cosmology, based on dynamical screening of Λ on the horizon scale. After motivating the idea of its screening by particle creation and vacuum polarization effects, we discuss the relevance of the quantum trace anomaly. The latter implies additional terms in the low energy effective theory of gravity, which amounts to a non-trivial modification of the classical Einstein theory, fully consistent with the Equivalence Principle. We consider possible signatures of the restoration of conformal invariance predicted by the fluctuations of these new scalar degrees of freedom on the spectrum and statistics of the CMB.

1 Vacuum Fluctuations and the Cosmological Term

In classical general relativity, the requirement that the field equations involve no more than two derivatives of the metric tensor allows for the possible addition of a constant term, the cosmological term Λ, to Einstein's equations,

$$R_a^{\ b} - \frac{R}{2}\,\delta_a^{\ b} + \Lambda\,\delta_a^{\ b} = \frac{8\pi G}{c^4}\,T_a^{\ b}. \tag{1}$$

If transposed to the right side of this relation, the Λ term corresponds to a constant energy density $\rho_\Lambda = c^4\Lambda/8\pi G$ and isotropic pressure $p_\Lambda = -c^4\Lambda/8\pi G$. Hence, even if the matter $T_a^{\ b} = 0$, a cosmological term causes spacetime to become curved with a radius of curvature of order $|\Lambda|^{-\frac{1}{2}}$.

In purely classical physics there is no natural scale for Λ. Indeed if $\hbar = 0$ and $\Lambda = 0$, there is no fixed length scale at all in the vacuum Einstein equations, G/c^4 being simply a conversion factor between the units of energy and those of length. Hence Λ may take on any value whatsoever with no difficulty (and with no explanation) in classical general relativity.

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As soon as we allow $\hbar \neq 0$, there is a quantity with the dimensions of length that can be formed from $\hbar, G$, and c, namely the Planck length

$$L_{pl} \equiv \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} = 1.616 \times 10^{-33}\,\text{cm}. \tag{2}$$

Hence when quantum theory is considered in a general relativistic setting, the quantity

$$\lambda \equiv \Lambda L_{pl}^2 = \frac{\hbar G \Lambda}{c^3} \tag{3}$$

becomes a dimensionless pure number, whose value one might expect a theory of gravity incorporating quantum effects to address.

W. Pauli was apparently the first to consider the question of the effects of quantum vacuum fluctuations on the the curvature of space [1]. Pauli recognized that the sum of zero point energies of the two transverse electromagnetic field modes *in vacuo*,

$$\rho_\Lambda = 2\int^{L_{min}^{-1}} \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\hbar\omega_{\mathbf{k}}}{2} = \frac{1}{8\pi^2}\frac{\hbar c}{L_{min}^4} = -p_\Lambda \tag{4}$$

contribute to the stress-energy tensor of Einstein's theory as would an effective cosmological term $\Lambda > 0$. Since the integral (4) is quartically divergent, an ultraviolet cutoff L_{min}^{-1} of (4) at large $\mathbf{k}$ is needed. Taking this short distance cutoff L_{min} to be of the order of the classical electron radius e^2/mc^2, Einstein's theory with this large a Λ would lead to a universe so curved that its total size "could not even reach to the moon." If instead of the classical electron radius, the apparently natural but much shorter length scale of $L_{min} \sim L_{pl}$ is used to cut off the frequency sum in (4), then the estimate for the cosmological term in Einstein's equations becomes vastly larger, and the entire universe would be limited in size to the microscopic scale of L_{pl} (2) itself, in even more striking disagreement with observation.

The vacuum zero point fluctuations being considered in (4) are the same ones that contribute to the Casimir effect, but this estimate quartically dependent on a short distance cutoff L_{min}, is certainly *not* relevant for the effect observed in the laboratory. In calculations of the Casimir force between conductors, one subtracts the zero point energy of the electromagnetic field in an infinitely extended vacuum from the modified zero point energies in the presence of the conductors. This subtracted energy density $\rho_v = -\frac{\pi^2}{720}\frac{\hbar c}{d^4}$ is of the opposite sign as (4), leading to an attractive force per unit area between the plates of 0.013 dyne/cm^2 $(\mu m/d)^4$, a value which is both independent of the ultraviolet cutoff L_{min}^{-1}, and the microscopic details of the atomic constituents of the conductors. This is a clear indication, confirmed by experiment, that the *measurable* effects associated with vacuum fluctuations are *infrared* phenomena, dependent upon macroscopic boundary conditions, which have little or nothing to do with the extreme ultraviolet modes or cutoff of the integral in (4).

In recent years the vacuum energy problem has evolved from a fundamental question in theoretical physics to a central one of observational cosmology as well. Observations of type Ia supernovae at moderately large redshifts ($z \sim 0.5$ to 1) have led to the conclusion that the Hubble expansion of the universe is *accelerating* [2]. According to Einstein's equations this is possible if and only if the energy density and pressure of the dominant component of the universe satisfies the inequality

$$\rho + 3p \equiv \rho\,(1 + 3w) < 0\,. \tag{5}$$

A vacuum energy with $\rho > 0$ and $w \equiv p/\rho = -1$ leads to an accelerated expansion, a kind of "repulsive" gravity in which the relativistic effects of a negative pressure can overcome a positive energy density in (5). Taken at face value, the observations imply that some 74% of the energy in the universe is of this hitherto undetected $w = -1$ dark variety. This leads to a non-zero inferred cosmological term in Einstein's equations of

$$\Lambda_{\rm meas} \simeq (0.74)\,\frac{3H_0^2}{c^2} \simeq 1.4\times 10^{-56}\ {\rm cm}^{-2} \simeq 3.6\times 10^{-122}\ \frac{c^3}{\hbar G}\,. \tag{6}$$

Here H_0 is the present value of the Hubble parameter, approximately 73 km/sec/Mpc $\simeq 2.4\times 10^{-18}\,{\rm sec}^{-1}$. The last number in (6) expresses the value of the cosmological dark energy inferred from the SN Ia data in terms of Planck units, $L_{\rm pl}^{-2} = \frac{c^3}{\hbar G}$, *i.e.* the dimensionless number in (3) has the value $\lambda \simeq 3.6\times 10^{-122}$. Explaining this smallest number in all of physics is the basic form of the "cosmological constant problem."

The treatment of quantum effects at distances much larger than any ultraviolet cutoff is precisely the context in which effective field theory (EFT) techniques should be applicable [3]. Note that the value (6) inferred from observations would seem to imply that we are living in a very special epoch in the history of the universe, when the vacuum energy has grown to be an appreciable fraction (74%) of the total, but just before the matter and radiation have been redshifted away completely, as conventional adiabatic theory indicates they soon will be, exponentially rapidly. This "cosmic coincidence problem," independently of the naturalness problem of the very small value of λ also suggests that something basic may be missing from current cosmological models. The first indication as to what that element may be emerges from consideration of quantum effects in curved spacetimes such as de Sitter spacetime.

2 Quantum Effects in de Sitter Spacetime

The simplest example of accelerated expansion is a universe composed purely of vacuum energy. This is de Sitter spacetime with $H^2 = \Lambda c^2/3$ a constant, which is the maximally symmetric solution to Einstein's equations (14) with $\Lambda > 0$ and $T_a^{\ b} = 0$. In spatially flat, homogeneous and isotropic Robertson-Walker (RW) form, it has the line element,

$$ds^2 = -c^2 d\tau^2 + a^2(\tau)\, d\mathbf{x}^2\,, \tag{7}$$

where τ is the proper time of a freely falling observer and $a(\tau)$ is the RW scale factor

$$a_{deS}(\tau) = e^{H\tau} \quad ; \quad H^2 = \left(\frac{\dot a}{a}\right)^2. \tag{8}$$

Quantum fluctuations in de Sitter and their backreaction were studied extensively in the past [4, 5, 6, 7]. Several of these studies indicated that fluctuations at the horizon scale c/H could be very important, and a mechanism for relaxing the effective value of the vacuum energy to zero over time dynamically was proposed [4, 8]. Although a satisfactory cosmological model based on these ideas does not yet exist, a dynamical theory of vacuum energy appears to be the most viable alternative to fine tuning or purely anthropic considerations for the very small but non-zero value of λ.

2.1 Particle Creation in de Sitter Space

A space or time dependent electric field creates particles. J. Schwinger studied this effect in QED in a series of classic papers [9]. It was later realized that a time dependent gravitational metric should also produce particles [10]. Let us consider the electromagnetic case in more detail, noticing that there is an analogy to the vacuum energy problem in electromagnetism as well, the "cosmological electric field problem" [8]. It consists of the elementary observation that Maxwell's equations in vacuum admit a solution with constant, uniform electric field $\mathbf{E}_{cosm}$ of arbitrary magnitude and direction. Why then do we not observe such a macroscopic uniform electric field in the universe?

A non-zero $\mathbf{E}_{cosm}$ selects a preferred direction in space. Nevertheless, relativistic particle motion in a uniform constant electric field has precisely the *same* number of symmetry generators (ten) as those of the usual zero field vacuum. These 10 generators define a set of modified spacetime symmetry transformations, leaving $\mathbf{E}_{cosm}$ fixed, whose algebra is isomorphic to that of the Poincaré group [11]. In this respect field theory in a spacetime with a constant, non-dynamical $\mathbf{E}_{cosm} \neq 0$ is similar to field theory in de Sitter spacetime with a constant, non-dynamical $\Lambda \neq 0$. The isometry group of de Sitter spacetime is $O(4,1)$, which also has 10 generators, exactly the same number of flat Minkowski spacetime. In the absence of a unique choice of vacuum in a spacetime dependent background, the point of view often adopted is to choose the state with the largest possible symmetry group permitted by the background. In de Sitter spacetime this maximally symmetric state is called the Bunch-Davies (BD) state [10].

Is there any evidence that quantum effects are relevant to the question of the cosmological electric field or of Λ? The answer is yes, due to particle creation effects. In fact, the electric field is diminished by the Maxwell equation,

$$\frac{\partial \mathbf{E}}{\partial t} = -\langle \mathbf{j} \rangle \tag{9}$$

in the case of exact spatial homogeneity of the average current. The important question is that of the time scale of the effective dissipation to effectively explain why there is no observed $\mathbf{E}_{cosm}$ today. The Schwinger calculation of the decay rate of the vacuum into charged pairs involves a tunneling factor, $\exp(-\pi m^2 c^3/eE\hbar)$ for the creation of the first pair from the vacuum, which would greatly suppress the effect. However, if any charged matter is present, the charges are accelerated, radiate, and pair produce without any tunneling suppression factor. Hence on physical grounds there is an *instability* of the vacuum in a background electric field and extreme sensitivity to boundary conditions.

These observations are of a very general nature, and apply equally well to vacuum fluctuations in a gravitational background field. In cosmology the Friedman equation together with the equation of covariant energy conservation, $\dot{\rho} + 3\,H\,(\rho + p) = 0$, imply

$$\dot{H} = -\frac{4\pi G}{c^2}\,(\rho + p). \tag{10}$$

This relation from Einstein's equations is to be compared to the Maxwell eq. (9). In both cases there is a classical static background that solves the equation trivially, namely H or $\mathbf{E}$ a constant with zero source terms. In the case of (10) this is de Sitter spacetime with $\rho_\Lambda + p_\Lambda = 0$. In order to exhibit explicitly its static nature, we make a coordinate transformation to bring the de Sitter line element (7), (8) into the form,

$$ds^2\Big|_{deS} = -(c^2 - H^2 r^2)\,dt^2 + \frac{dr^2}{1 - H^2 r^2/c^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{11}$$

In this representation the static nature of de Sitter space and the existence of an observer horizon at $r_{\scriptscriptstyle H} = c/H$ are manifest, but a particular point in space $r = 0$ is chosen as the origin, so that the spatial homogeneity of the RW coordinates (8) is no longer manifest.

Although a full calculation including self-interactions and backreaction has not been done, even in electromagnetism, the formal similarity between the two cases suggests that a dissipative relaxation of the vacuum energy into ordinary matter and radiation is possible via this mechanism.

2.2 Thermodynamic Instability of de Sitter Spacetime

A second set of considerations points to the role of dynamical quantum effects on the horizon scale in de Sitter space. The existence of the observer horizon at $r = r_H$ leads in conventional treatments to a Hawking temperature for freely falling observers [13]. The Hawking temperature is closely related to the particle creation effect and reads

$$T_{_H} = \frac{\hbar H}{2\pi k_{_B}} = \frac{\hbar c}{2\pi k_{_B}}\sqrt{\frac{\Lambda}{3}}\,. \tag{12}$$

Although this temperature is very small for $\lambda \ll 1$, a thermodynamic argument similar to Hawking's original one for black holes implies that the BD equilibrium state in de Sitter space is thermodynamically *unstable* [5].

In fact, both the energy within one horizon volume, and the entropy of the de Sitter horizon S_H are *decreasing* functions of the temperature, *i.e.*,

$$E_{_H} = \rho_\Lambda V_H = \frac{\hbar c^5}{4\pi G k_{_B} T_H}\,; \quad S_{_H} = k_{_B}\frac{A_H}{4L_{pl}^2} = \frac{\hbar c^5}{4\pi G k_{_B} T_H^2} = \frac{3\pi}{\lambda}\,k_{_B}\,. \tag{13}$$

Hence, by considering a small fluctuation in the Hawking temperature, we find a behavior of a system with negative heat capacity [5],

$$\frac{dE_{_H}}{dT_{_H}} = -\frac{E_{_H}}{T_{_H}} = -\frac{3\pi\, k_{_B}}{\lambda} < 0\,. \tag{14}$$

However, negative heat capacity is impossible for a stable system in thermodynamic equilibrium. It corresponds to a runaway process in which any infinitesimal heat exchange between the regions interior and exterior to the horizon will drive the system further away from its equilibrium configuration.

2.3 Graviton Fluctuations in de Sitter Spacetime

A third route for investigating quantum effects in de Sitter spacetime leading to the same qualitative conclusions is through studies of the fluctuations of the metric degrees of freedom. If one requires de Sitter invariance by computing the Euclidean propagator on S^4 and then analytically continuing to de Sitter, one obtains a graviton propagator with rather pathological properties. Both the transverse-tracefree (spin-2) and trace (spin-0) projections of the Feynman propagator grow without bound at large spacelike and timelike separations [14, 15, 16]. Since this leads to infrared divergences in physical scattering processes [15, 17], the large distance behavior of the propagator function cannot be removed by a gauge transformation. This infrared behavior is a striking violation of cluster decomposition properties of the de Sitter invariant vacuum state.

A similar situation had been encountered in the quantization of a massless, minimally coupled free scalar field. A covariant construction of the propagator meets the obstacle that the wave operator $\Box$ has a normalizable (constant) zero mode on S^4. Hence a de Sitter invariant propagator does not exist [18]. Formally projecting out the problematic mode leads to a propagator function which grows logarithmically for large spacelike or timelike separations of the points x and x',

$$G_{\rm dS}(x,x') = -\frac{1}{8\pi^2}\left(\frac{2}{\zeta^2} - H^2 \ln\frac{H^2\zeta^2}{4}\right) \quad ; \quad \zeta(x,x') \equiv \frac{2}{H}\sinh\frac{Hs(x,x')}{2}\,, \tag{15}$$

where $s(x,x')$ is the geodesic distance between x and x'. Since the wave equation for spin-2 gravitons is identical to that of two massless, minimally coupled scalars (one for each polarization) in a certain gauge [14, 19], it shares many of the same features.

The closest analog of this behavior in flat spacetime is that of massless scalar field in two dimensions. The Lorentz invariant Feynman propagator $G(x,x')$ is

$$G(x,x') = -\frac{\hbar}{4\pi}\ln[\mu^2(x-x')^2] \tag{16}$$

with μ an arbitrary constant acting as an infrared cutoff. The logarithmic growth at large distances implies that free asymptotic particle states do not exist for a massless scalar field in two dimensions. In the generic case, a mass and other interaction terms are generically allowed, and are expected to control the behavior of the theory at large distances and late times. Conversely, if masslessness is protected by a global symmetry, then that symmetry is restored by quantum fluctuations and there is no Lorentz invariant massless Goldstone scalar in the physical spectrum [20, 21]. The similar logarithmic behavior of the graviton propagator indicates that infrared quantum fluctuations of the gravitational field are important in de Sitter space, and self-interactions or additional relevant terms in the effective action will control the late time behavior.

3 Quantum Theory of the Conformal Factor

The sensitivity to horizon scale quantum fluctuations in de Sitter spacetime strongly suggests that there is an infrared relevant operator in the low energy effective theory of gravity, not contained in the classical Einstein-Hilbert action. There are several hints for the source of such terms. First it is the fluctuations in the scalar or conformal sector of the metric field which are the most infrared divergent in de Sitter spacetime [22]. These are associated with the trace of the polarization tensor, and are parameterized by the conformal part of the metric tensor,

$$g_{ab}(x) = e^{2\sigma(x)}\bar{g}_{ab}(x), \tag{17}$$

where e^{σ} is called the conformal factor and $\bar{g}_{ab}(x)$ is a fixed fiducial metric. The RW scale factor $a(\tau)$ in (7) is an example of a conformal factor, fixed classically by the Friedman equation. The second clue that this is the important sector to look for non-perturbative infrared effects is that σ couples to the trace of the energy-momentum tensor $T_a^{\ a}$, an operator known to have an anomaly for massless quantum fields in curved spacetime [23]. An anomaly implies that the effects of quantum fluctuations can remain relevant at the longest length and time scales, and therefore modify the purely classical theory. This is certainly the lesson of two dimensional quantum gravity, which we review next. The classically constrained scalar field σ acquires *dynamics* through the trace anomaly. It is the effective action and dynamics of this field which we have proposed as the essential new ingredient to gravity at cosmological distance scales which can provide a natural mechanism for screening the cosmological term [22].

3.1 Quantum Gravity in Two Dimensions

In two dimensions the trace anomaly takes the simple form,

$$\langle T_a{}^a \rangle = \frac{N}{24\pi} R\,, \qquad (d=2) \tag{18}$$

where $N = N_S + N_F$ is the total number of massless fields, either scalar (N_S) or (complex) fermionic (N_F). A non-local action corresponding to (18) can be found by introducing the conformal parameterization of the metric (17) and noticing that the scalar curvature densities of the two metrics g_{ab} and $\bar{g}_{ab}$ are related by

$$R\sqrt{-g} = \bar{R}\sqrt{-\bar{g}} - 2\sqrt{-\bar{g}}\,\bar{\Box}\sigma\,, \qquad (d=2)\,. \tag{19}$$

Multiplying (18) by $\sqrt{-g}$, using (19) and noting that $\sqrt{-g}\langle T_a{}^a\rangle$ defines the conformal variation, $\delta\Gamma^{(2)}/\delta\sigma$ of an effective action $\Gamma^{(2)}$, we conclude that the σ dependence of $\Gamma^{(2)}$ can be at most quadratic in σ. Hence the Wess-Zumino effective action [24] in two dimensions, $\Gamma^{(2)}_{WZ}$ is

$$\Gamma^{(2)}_{WZ}[\bar{g};\sigma] = \frac{N}{24\pi}\int d^2x\,\sqrt{-\bar{g}}\left(-\sigma\bar{\Box}\sigma + \bar{R}\,\sigma\right)\,. \tag{20}$$

It is straightforward in fact to find a *non-local* scalar functional $S_{anom}[g]$ such that

$$\Gamma^{(2)}_{WZ}[\bar{g};\sigma] = S^{(2)}_{anom}[g = e^{2\sigma}\bar{g}] - S^{(2)}_{anom}[\bar{g}]\,. \tag{21}$$

By solving (19) formally for σ, we find

$$S^{(2)}_{anom}[g] = \frac{Q^2}{16\pi}\int d^2x\,\sqrt{-g}\int d^2x'\,\sqrt{-g'}\,R(x)\,\Box^{-1}(x,x')\,R(x')\,. \tag{22}$$

The coefficient $Q^2 = -N/6$ becomes $Q^2 = (25-N)/6$ if account is taken of the contributions of the metric fluctuations themselves in addition to those of the N matter fields [25]. In the general case Q^2 is arbitrary, related to the matter central charge, and can be treated as an additional free parameter of the low energy effective action.

The anomalous effective action (22) is quite different from the classical terms in the action, and describes different physics. Since the integral of R is a topological invariant in two dimensions, the classical Einstein-Hilbert action contains no propagating degrees of freedom, and it is S_{anom} which contains the *only* kinetic terms of the low energy EFT. It is clear that S_{anom} describes an additional scalar degree of freedom σ, not contained in the classical action $S^{(2)}_{cl}$. Quantum gravity in two dimensions acquires new dynamics in its conformal sector, not present in the classical theory. Extensive study of the stress tensor and its correlators, arising from this effective action established that the two dimensional trace anomaly gives rise to a modification or gravitational “dressing" of critical exponents in conformal field theories at second order critical points [25]. These dressed exponents are evidence of the infrared fluctuations of the additional scalar degree of freedom σ which are absent in the classical action. The appearance of the gravitational dressing exponents and the anomalous effective action (22) itself have been confirmed in the large volume scaling limit of two dimensional simplicial lattice simulations in the dynamical triangulation approach [26, 27]. The action (22) due to the anomaly is exactly the missing relevant term in the low energy EFT of two dimensional gravity responsible for non-perturbative fluctuations at the largest distance scales. This modification of the classical theory is *required* by general covariance and quantum theory, and essentially *unique* within the EFT framework.

4 Quantum Conformal Factor in Four Dimensions

The line of reasoning in $d = 2$ dimensions just sketched to find the conformal anomaly and construct the effective action may be followed also in four dimensions. In $d = 4$ the trace anomaly takes the somewhat more complicated form,

$$\langle T_a{}^a \rangle = bF + b' \left(E - \frac{2}{3} \Box R \right) + b'' \Box R \,, \tag{23}$$

in a general curved spacetime. In eq. (23) we employ the notation,

$$E \equiv {}^*\!R_{abcd}\, {}^*\!R^{abcd} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \,, \qquad \text{and} \tag{24}$$

$$F \equiv C_{abcd}C^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{R^2}{3} \,. \tag{25}$$

with R_{abcd} the Riemann curvature tensor, ${}^*\!R_{abcd} = \varepsilon_{abef} R^{ef}{}_{cd}/2$ its dual, and C_{abcd} the Weyl tensor. Note that E is the Gauss-Bonnet combination whose integral gives the Euler number of the manifold, analogous to the Ricci scalar R in $d = 2$. For free massless particles, the coefficients b and b' are given by [10, 23]

$$b = \frac{1}{120(4\pi)^2} \left(N_S + 6N_F + 12N_V \right) , \tag{26}$$

$$b' = -\frac{1}{360(4\pi)^2} \left(N_S + \frac{11}{2} N_F + 62N_V \right) , \tag{27}$$

with (N_S, N_F, N_V) the number of fields of spin $(0, \frac{1}{2}, 1)$ respectively. Notice also that $b > 0$ while $b' < 0$ for all fields of lower spin for which they have been computed.

Three local fourth order curvature invariants E, F and $\Box R$ appear in the trace of the stress tensor (23), but only the first two (b and b') terms cannot be derived from a local effective action of the metric alone. The third b'' coefficient is not part of the true anomaly, since it can be shifted by adding in the effective action a local R^2 counterterm. As in the case of the chiral anomaly in QCD, or two dimensional gravity, the trace anomaly can have significant new infrared effects, not captured by a purely local metric description.

To find the WZ effective action corresponding to the b and b' terms in (23), we introduce as in two dimensions the conformal parameterization (17), and compute

$$\sqrt{-g}\, F = \sqrt{-\bar{g}} \bar{F} \ ; \quad \sqrt{-g} \left(E - \frac{2}{3} \Box R \right) = \sqrt{-\bar{g}} \left(\overline{E} - \frac{2}{3} \overline{\Box} \overline{R} \right) + 4\sqrt{-\bar{g}} \bar{\Delta}_4 \sigma \,, \tag{28}$$

whose σ dependence is no more than linear. The differential operator [22, 28, 29]

$$\Delta_4 \equiv \Box^2 + 2R^{ab}\nabla_a\nabla_b - \frac{2}{3} R\Box + \frac{1}{3} (\nabla^a R)\nabla_a \,, \tag{29}$$

appearing in (28) is the unique conformally covariant fourth order scalar operator, $\sqrt{-g}\, \Delta_4 = \sqrt{-\bar{g}}\, \bar{\Delta}_4$. One can then easily deduce from (23) the effective action

$$\Gamma_{WZ}[\bar{g}; \sigma] = b \int d^4x \sqrt{-\bar{g}}\, \bar{F}\, \sigma + b' \int d^4x \sqrt{-\bar{g}} \left\{ \left(\bar{E} - \frac{2}{3} \bar{\Box} \bar{R} \right) \sigma + 2\, \sigma \bar{\Delta}_4 \sigma \right\} , \tag{30}$$

up to terms independent of σ. By solving (28) formally for σ and substituting the result in (30) we obtain

$$\Gamma_{WZ}[\bar{g}; \sigma] = S_{anom}[g = e^{2\sigma}\bar{g}] - S_{anom}[\bar{g}], \tag{31}$$

with the *non-local* anomalous action,

$$S_{anom}[g] = \frac{1}{2}\int d^4x\sqrt{g}\int d^4x'\sqrt{g'}\left(\frac{E}{2}-\frac{\Box R}{3}\right)_x \Delta_4^{-1}(x,x')\left[bF + b'\left(\frac{E}{2}-\frac{\Box R}{3}\right)\right]_{x'} \qquad (32)$$

and $\Delta_4^{-1}(x,x')$ denoting the inverse of the fourth order differential operator (29).

Notice from the derivation of S_{anom} that although the σ independent piece of the gravitational action cannot be determined from the trace anomaly alone, the σ dependence is *uniquely* determined. Thus, whatever else may be involved in the full quantum theory of gravity at short distance scales, the anomalous effective action (32) should be included at large distances. Graviton (spin-two) fluctuations of the metric should give rise to an effective action of precisely the same form as S_{anom} with new coefficients b and b', which can be checked at one-loop order [30]. The fluctuations generated by S_{anom} define a non-trivial infrared fixed point, with anomalous dimensions analogous to the two dimensional case [22, 31].

5 Finite Volume Scaling and Infrared Screening of λ

Let us consider the dynamical effects of the anomalous terms in the simplest case that the fiducial metric is conformally flat, *i.e.* $g_{ab} = e^{2\sigma}\eta_{ab}$. Then the Wess Zumino effective action simplifies to

$$\Gamma_{WZ}[\eta;\sigma] = -\frac{Q^2}{16\pi^2}\int d^4x\ (\Box\sigma)^2 \quad ; \quad Q^2 \equiv -32\pi^2 b'\,. \qquad (33)$$

This action quadratic in σ is the action of a free scalar field, albeit with a kinetic term that is fourth order in derivatives. The propagator is a logarithm,

$$G_\sigma(x,x') = -\frac{1}{2Q^2}\ \ln\left[\mu^2(x-x')^2\right] \qquad (34)$$

the same as (16) in two dimensions. Because of the logarithmic behavior we must expect the similar sort of infrared fluctuations, conformal fixed point and dressing exponents as those obtained in two dimensional gravity.

Indeed, any effective local operator of non-negative mass dimension p acquires an anomalous dimension. Its associated coupling of classical dimension $4-p$ follows an evolution according to an anomalous scaling dimension β_p given by

$$\beta_p = 4 - p + \frac{\beta_p^2}{2Q^2} \quad \Rightarrow \quad \beta_p = Q^2\left(1-\sqrt{1-\frac{(8-2p)}{Q^2}}\right)\,. \qquad (35)$$

In the limit $Q^2 \to \infty$ the fluctuations of σ are suppressed and we recover the classical scale dimension $4-p$ for fixed sign of the square root, valid for $Q^2 \geq 8-2p$ for all $p \geq 0$, and thus $Q^2 \geq 8$. The Newtonian coupling corresponds to $p = 2$ while the cosmological term to $p = 0$. These scaling dimensions were computed both by covariant and canonical operator methods. In the canonical method we also showed that the anomalous action for the conformal factor does not have unphysical ghost or tachyon modes in its spectrum of physical states [32].

By normalizing to a fixed four volume $V = \int d^4x$ one can show that the finite volume renormalization of $\lambda = \hbar G\Lambda/c^3$ is controlled by the anomalous dimension,

$$2\delta - 1 \equiv 2\,\frac{\beta_2}{\beta_0} - 1 = \frac{\sqrt{1-\frac{8}{Q^2}}-\sqrt{1-\frac{4}{Q^2}}}{1+\sqrt{1-\frac{4}{Q^2}}} \leq 0\,, \qquad (36)$$

which enters the infrared renormalization group volume scaling relation [30],

$$V\frac{d}{dV}\lambda = 4\,(2\delta - 1)\,\lambda \quad \Rightarrow \quad \lambda \propto V^{2\delta-1} \to 0\,. \tag{37}$$

The anomalous scaling dimension (36) is negative for all $Q^2 \geq 8$, starting at $1-\sqrt{2} = -0.414$ at $Q^2 = 8$ and approaching zero as $-1/Q^2$ as $Q^2 \to \infty$. This implies that the dimensionless cosmological term λ has an infrared fixed point at zero as $V \to \infty$. Thus the cosmological term is dynamically driven to zero as $V \to \infty$ by infrared fluctuations of the conformal part of the metric. We emphasize that no free parameters enter except Q^2, which is determined by the trace anomaly coefficient b'. Once Q^2 is assumed to be positive, then $2\delta - 1$ is negative, and λ is driven to zero at large distances by the conformal fluctuations of the metric, with no additional assumptions.

However, the application of this screening mechanism to cosmology, in which we presume a classical or semiclassical line element of the form (7), is unclear. Near the conformal fixed point the inverse Newtonian constant G^{-1} is also driven to zero when compared to some fixed mass scale m, as $Gm^2 \propto V^{(\beta_2-2\beta_3)/\beta_0} \to 0$ [33]. This is clearly different from the situation we observe in our local neighborhood. Under what conditions and where exactly (32) can dominate the classical Einstein terms, and moreover how the screening mechanism could be used to relax the vacuum energy to zero in a realistic cosmological model are questions not answered by our considerations to this point.

6 Conformal Invariance and the CMB

We already argued that the fluctuations responsible for the screening of λ take place at the horizon scale. Then, the microwave photons in the CMB reaching us from their surface of last scattering should retain some imprint of the effects of these fluctuations. It then becomes natural to extend the classical notion of scale invariant cosmological perturbations [34] to full conformal invariance. In that case the classical spectral index of the perturbations should receive corrections due to the anomalous scaling dimensions at the conformal phase [35, 3]. In addition to the spectrum, the statistics of the CMB should reflect the non-Gaussian correlations characteristic of conformal invariance [35].

Indeed, the Harrison-Zel'dovich observation that the primordial density fluctuations should be characterized by a spectral index $n = 1$ is equivalent to the statement that the observable giving rise to these fluctuations has engineering or naive scaling dimension $p = 2$. This is because the density fluctuations $\delta\rho$ are related to the metric fluctuations by Einstein's equations, $\delta R \sim G\delta\rho$, which is second order in derivatives of the metric. Hence, the two-point spatial correlations $\langle\delta\rho(x)\delta\rho(y)\rangle \sim \langle\delta R(x)\delta R(y)\rangle$ should behave like $|x-y|^{-4}$, or $|k|^1$ in Fourier space, according to simple dimensional analysis. In general, one should expect to find well-defined logarithmic deviations from naive scaling, corresponding (upon resummation) to a dimension $\Delta \neq p$. The deviation from naive scaling $\Delta - p$ is the "anomalous" dimension of the observable due to quantum fluctuations. Once Δ is fixed for a given observable, the requirement of conformal invariance determines the form of its two- and three-point correlation functions up to an arbitrary amplitude, without reliance on any particular dynamical model.

Consider first the two-point function of any observable $\mathcal{O}_\Delta$ with dimension Δ. Conformal invariance requires [36, 37]

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle \sim |x_1 - x_2|^{-2\Delta} \tag{38}$$

at equal times in three dimensional flat spatial coordinates. In Fourier space this gives

$$G_2(k) \equiv \langle\tilde{\mathcal{O}}_\Delta(k)\tilde{\mathcal{O}}_\Delta(-k)\rangle \sim |k|^{2\Delta-3}\,. \tag{39}$$

Thus, we define the spectral index of this observable by

$$n \equiv 2\Delta - 3\ . \tag{40}$$

In the case that the observable is the primordial density fluctuation $\delta\rho$, and in the classical limit where its anomalous dimension vanishes, $\Delta \to 2$, we recover $n = 1$.

In order to convert the power spectrum of primordial density fluctuations to the spectrum of fluctuations in the CMB at large angular separations we follow the standard treatment [38] relating the temperature deviation to the Newtonian gravitational potential φ at the last scattering surface, $\frac{\delta T}{T} \sim \delta\varphi$, which is related to the density perturbation in turn by $\nabla^2\delta\varphi = 4\pi G\,\delta\rho$. Hence, in Fourier space,

$$\frac{\delta T}{T} \sim \delta\varphi \sim \frac{1}{k^2}\frac{\delta\rho}{\rho}\ , \tag{41}$$

and the two-point function of CMB temperature fluctuations is given by

$$C_2(\theta) \equiv \left\langle \frac{\delta T}{T}(\hat{r}_1)\frac{\delta T}{T}(\hat{r}_2)\right\rangle \propto \Gamma(2-\Delta)(r_{12}^2)^{2-\Delta} \sim (1-\cos\theta)^{2-\Delta}\ , \tag{42}$$

where $r_{12} \equiv (\hat{r}_1 - \hat{r}_2)r$ assuming that the CMB photons were emitted at the last scattering surface at equal cosmic time; thus, $r_{12}^2 = 2(1-\cos\theta)r^2$.

If the conformal fixed point behavior described previously dominates at these scales then the scaling dimension of an observable with classical dimension p is given by [39]

$$\Delta_p = 4\,\frac{\sqrt{1-\frac{(8-2p)}{Q^2}} - \sqrt{1-\frac{8}{Q^2}}}{1-\sqrt{1-\frac{8}{Q^2}}} = p + \frac{1}{2Q^2}\,p\,(4-p) + \cdots \tag{43}$$

In the limit $Q^2 \to \infty$, the effects of fluctuations in the metric are suppressed and one recovers the classical scaling dimension p. The quantity Q^{-2} is determined in principle by the trace anomaly coefficient b' through (23) and (33), but we may regard it as simply a free parameter characterizing the universality class of the conformal metric fluctuations, which should be determined from the observations. With $p = 2$, we find a definite prediction for deviations from a strict Harrison-Zel'dovich spectrum according to Eqns. (40) and (43) in terms of the parameter Q^2 [39]. The resulting spectral index is always greater than unity for all finite $Q^2 \geq 8$, approaching one as

$$n = 1 + \frac{4}{Q^2} + \cdots \tag{44}$$

The latest WMAP CMB results favor a spectral index for scalar perturbations of about 0.95, some three standard deviations lower than unity. From (23) and (33), the value of Q^2 for free conformally invariant fields is

$$Q^2 = \frac{1}{180}(N_S + \tfrac{11}{2}N_F + 62N_V - 28) + Q^2_{grav}\ , \tag{45}$$

where Q^2_{grav} is the contribution of spin-2 gravitons and -28 is that of the conformal or spin-0 part of the metric itself. The main theoretical uncertainty in determining Q^2_{grav} is that the Einstein theory is neither conformally invariant nor free, so that a method for evaluating the infrared effects of spin-2 gravitons is required which is insensitive to ultraviolet physics. A purely one-loop computation gives $Q^2_{grav} \simeq 7.9$ for the graviton contribution [30]. Taking this estimate at face value and including all known fields of the Standard Model of particle physics (for which $N_F = 45$ and $N_V = 12$) we find

$$Q^2_{SM} \simeq 13.2 \qquad \text{and} \qquad n \simeq 1.45 \;, \tag{46}$$

which is now firmly excluded by the WMAP data. If we require that n be within 0.05 of unity, then $Q^2 > 80$ is needed.

Turning now from the two-point function of CMB fluctuations to higher point correlators, we find a second characteristic prediction of conformal invariance, namely non-Gaussian statistics for the CMB. The first correlator sensitive to this departure from gaussian statistics is the three-point function of the observable $\mathcal{O}_\Delta$, which takes the form [37]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle \sim |x_1 - x_2|^{-\Delta}|x_2 - x_3|^{-\Delta}|x_3 - x_1|^{-\Delta} \,. \tag{47}$$

In the general case of three different angles, the expression for the three-point correlation function (47) is quite complicated. In the special case of equal angles $\theta_{ij} = \theta$, the three-point correlator becomes simply

$$C_3(\theta) \sim (1 - \cos\theta)^{\frac{3}{2}(2-\Delta)} \,. \tag{48}$$

Expanding the function $C_3(\theta)$ in multiple moments ℓ and taking the limit $\Delta \to 2$, we obtain $\ell(\ell+1)c^{(3)}_\ell = 6c^{(3)}_2$, which is the same result as for the moments $c^{(2)}_\ell$ of the two-point correlator but with a different quadrupole amplitude.

The value of this quadrupole normalization cannot be determined by conformal symmetry considerations alone. A naive comparison with the two-point function which has a small amplitude of the order of 10^{-5} leads to a rough estimate of $c^{(3)}_2 \sim \mathcal{O}(10^{-7.5})$, which would make it very difficult to detect. However, if the conformal invariance hypothesis is correct, then these non-Gaussian correlations should exist at some level, in distinction to the simplest inflationary scenarios. Their amplitude is model dependent and possibly much larger than the above naive estimate. Any detection of non-Gaussian statistics of the CMB would be an important clue to their origin and possibly an important test for the hypothesis of conformal invariance.

Thus the conformal invariance hypothesis applied to the primordial density fluctuations predicts deviations both from the classical spectrum and Gaussian statistics, which should be imprinted on the CMB anisotropy. A particular realization of this hypothesis is provided by the metric fluctuations induced by the known trace anomaly of massless matter fields which gives rise to fixed point with a spectral index $n > 1$. Although this is disfavored by the WMAP data, lacking a complete cosmological model which takes dark energy, dark matter, the CMB and possibly other effects into account in a consistent way, it is premature to draw a final conclusion on the conformal invariance hypothesis. The possibility of explaining the small value of λ in a natural way is a strong reason to pursue a more complete cosmological model within this framework.

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Non-minimal Einstein-Yang-Mills-dilaton theory

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Abstract

We establish a new non-minimal Einstein-Yang-Mills-dilaton model, for which the Lagrangian is linear in the curvature and contains eight arbitrary functions of the scalar (dilaton) field. The self-consistent system of equations for the non-minimally coupled gauge, scalar and gravitational fields is derived. As an example of an application we discuss the model with pp-wave symmetry. Two exact explicit regular solutions of the whole system of master equations, belonging to the class of pp-wave solutions, are presented.

1 Introduction

The Einstein-Maxwell-dilaton theory and its non-Abelian generalization, Einstein-Yang-Mills-dilaton (EYMd) theory, attract serious attention, since they have a supplementary, dilatonic, degree of freedom for the structure modeling of the gravitationally coupled systems. A lot of impressive results are obtained in the framework of the *minimal* EYMd theory in the cosmological and string contexts, as well as in the application to the colored static spherically and axially symmetric objects (see, e.g., [10] - [5] and references therein). A new impetus to the development of the EYMd theory has been given by the discovery of the accelerated expansion of the Universe. Numerous attempts have been made to consider modified theories of gravitational interaction, such as $F(R)$-gravity, Gauss-Bonnet-gravity, etc., as alternatives to the dark energy (see, e.g., [6] and references therein). One of the directions in such a generalization of the Einstein theory of gravity is connected with a *non-minimal* extension of the field theory. Some historical details, review and references, related to the non-minimal interaction of gravity with scalar and electromagnetic fields, can be found, e.g., in [7]. As for the non-minimal generalization of the Einstein-Yang-Mills theory, there are two different approaches. The first one is based on the dimensional reduction of the Gauss-Bonnet action [8], the alternative way is connected with the non-Abelian generalization of the non-minimal Einstein-Maxwell theory [9, 10] along the lines proposed by Drummond and Hathrell for the

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linear electrodynamics [11]. Recently in [20] one of the variants of the non-linear non-minimal generalizations of the Einstein-Yang-Mills theory has been developed, which combines, in fact, two ideas: the idea of $F(R)$-gravity, on the one hand, and the idea of Ricci-dilaton, on the other hand. The paper [20] has stimulated in some respects the preparation of this note.

2 Fundamentals of the non-minimal EYMd model

2.1 General concepts and definitions

We consider a non-minimal Einstein-Yang-Mills-dilaton model quadratic in the Yang-Mills field strength $F^{(a)}_{ik}$, quadratic in the derivative of the scalar (dilaton) field $\nabla_k\Phi$, and linear in the curvature. Such a model can be described self-consistently using the action functional of the type

$$S_{(\mathrm{EYMd})} = \int d^4x\sqrt{-g}\ \left\{\frac{R{+}2\Lambda}{\kappa}{+}\frac{1}{2}C^{ikmn}_{(a)(b)}(\Phi)\ F^{(a)}_{ik}F^{(b)}_{mn}{-}\mathcal{C}^{ik}(\Phi)\nabla_i\Phi\nabla_k\Phi{+}\mathcal{F}(\Phi)R{+}V(\Phi)\right\}\,, \quad (1)$$

which is, on the one hand, a dilatonic extension of the non-minimal Einstein-Yang-Mills model [9], and, on the other hand, a reduction of the Einstein-Yang-Mills-Higgs functional [10] to the case, when the $SU(N)$ Higgs miltiplet $\Phi^{(a)}$ degenerates into the scalar singlet Φ.

The used definitions are standard: $g = \det(g_{ik})$ is the determinant of a metric tensor g_{ik}, R is the Ricci scalar, Λ is the cosmological constant, the multiplet of real tensor fields $F^{(a)}_{ik}$ describes the strength of gauge field, the symbol Φ denotes the real scalar field, associated with a dilaton, ∇_k denotes the covariant derivative, $V(\Phi)$ is a potential of the scalar field. Latin indices without parentheses run from 0 to 3, (a) and (b) are the group indices, running from (1) to (N^2-1) for the model with $SU(N)$ symmetry. The quantities $C^{ikmn}_{(a)(b)}$ and $\mathcal{C}^{ik}$ denote the so-called constitutive tensors for the gauge and scalar fields, respectively. They contain neither Yang-Mills strength tensor $F^{(a)}_{ik}$, nor the derivative of the scalar field. Depending on the model under consideration, these tensors can be constructed using space-time metric, its first derivatives (through the covariant derivative ∇_k), second derivatives (through the Riemann tensor $R^i{}_{klm}$, Ricci tensor R_{km} and Ricci scalar R), etc. In addition, the scalar field Φ, time-like velocity four-vector U^k and space-like director(s) N^k, and their covariant derivatives, $\nabla_k U_l$ and $\nabla_k N_l$, can be constructive elements of the constitutive tensors.

We follow the definitions of the book [13] and consider the Yang-Mills field $\mathbf{F}_{mn}$ taking values in the Lie algebra of the gauge group $SU(N)$ (adjoint representation):

$$\mathbf{F}_{mn} = -i\mathcal{G}\mathbf{t}_{(a)}F^{(a)}_{mn}\,, \quad \mathbf{A}_m = -i\mathcal{G}\mathbf{t}_{(a)}A^{(a)}_m\,. \quad (2)$$

The generators $\mathbf{t}_{(a)}$ are Hermitian and traceless. The symmetric tensor $G_{(a)(b)}$ defined as

$$G_{(a)(b)} \equiv 2\mathrm{Tr}\ \mathbf{t}_{(a)}\mathbf{t}_{(b)}\,, \quad (3)$$

plays a role of a metric in the group space. The tensor $F^{(a)}_{mn}$ is connected with the potentials of the gauge field $A^{(a)}_i$ by the formulas [13, 14]

$$F^{(a)}_{mn} = \nabla_m A^{(a)}_n - \nabla_n A^{(a)}_m + \mathcal{G}f^{(a)}_{\cdot(b)(c)}A^{(b)}_m A^{(c)}_n\,. \quad (4)$$

The symbols $f^{(a)}_{\cdot(b)(c)}$ denote the real structure constants of the gauge group $SU(N)$. The tensor $F^{ik}_{(a)}$ satisfies the relation

$$\hat{D}_k{}^*F^{ik}_{(a)} \equiv \nabla_k{}^*F^{ik}_{(a)} - \mathcal{G}f^{(c)}_{\cdot(b)(a)}A^{(b)}_m{}^*F^{ik}_{(c)} = 0\,. \quad (5)$$

The symbol $\hat{D}_k$ denotes the gauge-covariant derivative. For the derivative of arbitrary tensor defined in the group space we use the following rule [15]:

$$\hat{D}_m Q^{(a)\cdots}_{\cdots(d)} \equiv \nabla_m Q^{(a)\cdots}_{\cdots(d)} + \mathcal{G} f^{(a)}_{\cdot(b)(c)} A^{(b)}_m Q^{(c)\cdots}_{\cdots(d)} - \mathcal{G} f^{(c)}_{\cdot(b)(d)} A^{(b)}_m Q^{(a)\cdots}_{\cdots(c)} + \dots . \tag{6}$$

The asterisk relates to the dual tensor

$$^*F^{ik}_{(a)} = \frac{1}{2}\epsilon^{ikls} F_{ls(a)} , \tag{7}$$

where $\epsilon^{ikls} = \frac{1}{\sqrt{-g}} E^{ikls}$ is the Levi-Civita tensor, E^{ikls} is the completely antisymmetric symbol with $E^{0123} = -E_{0123} = 1$.

2.2 Non-minimal EYMd model based on eight arbitrary functions of the dilaton field

In the papers [9, 10] we focused on the three-, five-, six- and seven-*parameter* models, considering the parameters q_1, q_2, etc., as phenomenological non-minimal coupling constants. Now, following the main idea of dilatonic extension of the Einstein-Maxwell and Einstein-Yang-Mills theories (see, e.g., [16]), we assume that the (non-minimal) constitutive tensors $C^{ikmn}_{(a)(b)}$ and $\mathcal{C}^{ik}$ contain arbitrary functions of the dilaton field Φ instead of coupling constants. Our ansatz for the action functional of the non-minimal EYMd model is the following

$$S_{(\mathrm{NMEYMd})} = \int d^4x\sqrt{-g}\ \left\{ \frac{R+2\Lambda}{\kappa} + \frac{1}{2} f_0(\Phi) F^{(a)}_{ik} F^{ik}_{(a)} - \mathcal{F}_0(\Phi)\nabla_m\Phi\nabla^m\Phi + V(\Phi) + \right.$$

$$\left. + \mathcal{F}(\Phi)R + \frac{1}{2}\mathcal{R}^{ikmn}(\Phi) F^{(a)}_{ik} F_{mn(a)} - \Re^{mn}(\Phi)\nabla_m\Phi\nabla_n\Phi \right\} . \tag{8}$$

The function $f_0(\Phi)$ describes a multiplier of the dilaton-type, which is traditional for the *minimal* theory; it is placed in front of the first invariant $F^{(a)}_{ik} F^{ik}_{(a)}$ of the gauge field and is considered to be positive. In analogy with $f_0(\Phi)$ we introduce the multiplier $\mathcal{F}_0(\Phi)$ in front of the invariant $\nabla_m\Phi\nabla^m\Phi$. The function $\mathcal{F}(\Phi)$ is typical for the non-minimal extension of the scalar field theory; traditionally, one uses the term $\mathcal{F}(\Phi)R$ in the well-known form $\xi R\Phi^2$. The so-called susceptibility tensor $\mathcal{R}^{ikmn}$

$$\mathcal{R}^{ikmn}(\Phi) \equiv \frac{1}{2} f_1(\Phi) R\left(g^{im}g^{kn} - g^{in}g^{km}\right) + f_3(\Phi) R^{ikmn} +$$

$$+ \frac{1}{2} f_2(\Phi)(R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) , \tag{9}$$

contains three arbitrary functions $f_1(\Phi)$, $f_2(\Phi)$, $f_3(\Phi)$ instead of curvature coupling parameters q_1, q_2, q_3 (compare with [9, 10, 10]). The tensor $\Re^{mn}$

$$\Re^{mn}(\Phi) \equiv f_4(\Phi) R g^{mn} + f_5(\Phi) R^{mn} , \tag{10}$$

describes the so-called derivative coupling of the scalar field to the curvature (see, e.g., [18]), but now we consider arbitrary functions $f_4(\Phi)$ and $f_5(\Phi)$ instead of coupling parameters q_4 and q_5 (see [10]). Finally, it is worth noting, that the constitutive tensors, introduced in (1), have now the following form

$$C^{ikmn}_{(a)(b)}(\Phi) = \left[\frac{1}{2} f_0(\Phi)(g^{im}g^{kn} - g^{in}g^{km}) + \mathcal{R}^{ikmn}(\Phi)\right] G_{(a)(b)} , \quad \mathcal{C}^{mn}(\Phi) = \left[g^{mn}\mathcal{F}_0(\Phi) + \Re^{mn}(\Phi)\right] . \tag{11}$$

These quantities are linear in the curvature and are arbitrary functions of the dilaton field. They can be used for the definition of dilatonically modified color permittivities, as well as color and acoustic metrics along the lines described in [10].

Non-minimal extension of the Yang-Mills equations

The variation of the action $S_{(\mathrm{NMEYMd})}$ with respect to the Yang-Mills potential $A_i^{(a)}$ yields

$$\hat{D}_k \mathcal{H}^{ik}_{(a)} = 0 \,, \quad \mathcal{H}^{ik}_{(a)} = C^{ikmn}_{(a)(b)} \, F^{(b)}_{mn} \,, \tag{12}$$

where the term $\mathcal{H}^{ik}_{(a)}$ with constitutive tensor from (11) describes a non-minimal color excitation in analogy with electrodynamics of continuous media (see, e.g., [19]). In this context the quantity $\mathcal{R}^{ikmn}(\Phi)G_{(a)(b)}$ can be indicated as a color-dilatonic susceptibility tensor.

Non-minimal extension of the scalar field equations

The variation of the action $S_{(\mathrm{NMEYMd})}$ with respect to the scalar Φ gives the master equation for the dilaton field

$$\nabla_m \left\{ \left[g^{mn} \mathcal{F}_0(\Phi) + \Re^{mn}(\Phi) \right] \nabla_n \Phi \right\} = -\frac{1}{2} \frac{d}{d\Phi} V(\Phi) - \frac{1}{2} R \, \frac{d}{d\Phi} \mathcal{F}(\Phi) + \frac{1}{2} \nabla_m \Phi \nabla^m \Phi \frac{d}{d\Phi} \mathcal{F}_0(\Phi) -$$

$$-\frac{1}{4} \left[F^{mn}_{(a)} F^{(a)}_{mn} \frac{d}{d\Phi} f_0(\Phi) + F_{ik(a)} F^{(a)}_{mn} \, \frac{d}{d\Phi} \mathcal{R}^{ikmn}(\Phi) \right] - \frac{1}{2} \nabla_m \Phi \nabla_n \Phi \, \frac{d}{d\Phi} \Re^{mn}(\Phi) \,. \tag{13}$$

Clearly, this equation is coupled to the non-minimally extended Yang-Mills equation (12), when the constitutive tensor (11) is dilatonically extended.

Master equations for the gravitational field

The variation of the action $S_{(\mathrm{NMEYMd})}$ with respect to the metric gives the non-minimally extended equations of the Einstein type

$$\left(R_{ik} - \frac{1}{2} R g_{ik} \right) \cdot \left[1 + \kappa \mathcal{F}(\Phi) \right] = \Lambda g_{ik} + \kappa \left(\nabla_i \nabla_k - g_{ik} \nabla_m \nabla^m \right) \mathcal{F}(\Phi) +$$

$$+ \kappa \left[T^{(YM)}_{ik} + T^{(\Phi)}_{ik} + T^{(\mathrm{NonMin})}_{ik} \right] \,. \tag{14}$$

Here the term $T^{(YM)}_{ik}$

$$T^{(YM)}_{ik} \equiv f_0(\Phi) \left[\frac{1}{4} g_{ik} F^{(a)}_{mn} F^{mn}_{(a)} - F^{(a)}_{in} F_{k\,(a)}^{\ n} \right] \,, \tag{15}$$

is a stress-energy tensor of Yang-Mills field extended by the dilatonic multiplier. The term $T^{(\Phi)}_{ik}$

$$T^{(\Phi)}_{ik} = \mathcal{F}_0(\Phi) \left[\nabla_i \Phi \nabla_k \Phi - \frac{1}{2} g_{ik} \nabla_m \Phi \nabla^m \Phi \right] + \frac{1}{2} V(\Phi) g_{ik} \,, \tag{16}$$

is a dilatonically extended stress-energy tensor of the scalar field. The term $T^{(\mathrm{NonMin})}_{ik}$ can be decomposed into five parts

$$T^{(\mathrm{NonMin})}_{ik} = T^{(I)}_{ik} + T^{(II)}_{ik} + T^{(III)}_{ik} + T^{(IV)}_{ik} + T^{(V)}_{ik} \,, \tag{17}$$

which are enumerated corresponding to the functions f_1, f_2, ...f_5:

$$T^{(I)}_{ik}{=}f_1(\Phi)R\left[\frac{1}{4}g_{ik}F^{(a)}_{mn}F^{mn}_{(a)}{-}F^{(a)}_{in}F_{k\,(a)}^{\ n}\right]-\frac{1}{2}f_1(\Phi)R_{ik}F^{(a)}_{mn}F^{mn}_{(a)}+$$
$$+\frac{1}{2}\left[\hat{D}_i\hat{D}_k{-}g_{ik}\hat{D}^l\hat{D}_l\right]\left[f_1(\Phi)F^{(a)}_{mn}F^{mn}_{(a)}\right], \tag{18}$$

$$T^{(II)}_{ik}=-\frac{1}{2}g_{ik}\left\{\hat{D}_m\hat{D}_l\left[f_2(\Phi)F^{mn(a)}F^l_{\ n(a)}\right]-f_2(\Phi)R_{lm}F^{mn(a)}F^l_{\ n(a)}\right\}$$
$$-f_2(\Phi)F^{ln(a)}\left[R_{il}F_{kn(a)}+R_{kl}F_{in(a)}\right]-f_2(\Phi)R^{mn}F^{(a)}_{im}F_{kn(a)}-\frac{1}{2}\hat{D}^m\hat{D}_m\left[f_2(\Phi)F^{(a)}_{in}F_{k\,(a)}^{\ n}\right]$$
$$+\frac{1}{2}\hat{D}_l\left\{\hat{D}_i\left[f_2(\Phi)F^{(a)}_{kn}F^{ln}_{(a)}\right]+\hat{D}_k\left[f_2(\Phi)F^{(a)}_{in}F^{ln}_{(a)}\right]\right\}, \tag{19}$$

$$T^{(III)}_{ik}=\frac{1}{4}f_3(\Phi)g_{ik}R^{mnls}F^{(a)}_{mn}F_{ls(a)}-\frac{3}{4}f_3(\Phi)F^{ls(a)}\left[F_{i\,(a)}^{\ n}R_{knls}+F_{k\,(a)}^{\ n}R_{inls}\right]$$
$$-\frac{1}{2}\hat{D}_m\hat{D}_n\left\{f_3(\Phi)\ \left[F_i^{\ n(a)}F_{k\,(a)}^{\ m}+F_k^{\ n(a)}F_{i\,(a)}^{\ m}\right]\right\}, \tag{20}$$

$$T^{(IV)}_{ik}=f_4(\Phi)\ \left[\left(R_{ik}-\frac{1}{2}Rg_{ik}\right)\nabla_m\Phi\nabla^m\Phi+R\,\nabla_i\Phi\nabla_k\Phi\right]$$
$$+\left(g_{ik}\nabla_n\nabla^n-\nabla_i\nabla_k\right)\left[f_4(\Phi)\ \nabla_m\Phi\nabla^m\Phi\right], \tag{21}$$

$$T^{(V)}_{ik}=f_5(\Phi)\ \nabla_m\Phi\left[R^m_i\nabla_k\Phi+R^m_k\nabla_i\Phi\right]+$$
$$+\frac{1}{2}g_{ik}\left[\nabla_m\nabla_n-R_{mn}\right]\left[f_5(\Phi)\ \nabla^m\Phi\nabla^n\Phi\right]$$
$$-\frac{1}{2}\nabla^m\left\{\nabla_i\left[f_5(\Phi)\ \nabla_m\Phi\nabla_k\Phi\right]+\nabla_k\left[f_5(\Phi)\ \nabla_m\Phi\nabla_i\Phi\right]-\nabla_m\left[f_5(\Phi)\ \nabla_i\Phi\nabla_k\Phi\right]\right\}. \tag{22}$$

Straightforward calculations, based on the Bianchi identities and on the properties of the Riemann tensor, show that the equality

$$\nabla^k\left\{\frac{\left(\nabla_i\nabla_k-g_{ik}\nabla_m\nabla^m\right)\mathcal{F}(\Phi)+T^{(YM)}_{ik}+T^{(\Phi)}_{ik}+T^{(\mathrm{NonMin})}_{ik}}{1+\kappa\mathcal{F}(\Phi)}\right\}=0 \tag{23}$$

is satisfied identically, when $F^{(a)}_{ik}$ is a solution of the Yang-Mills equations (12), and Φ is the solution of (13). Thus, we deal with self-consistent system of master equations (12), (13) and (14)- (22), which form the non-minimally extended Einstein-Yang-Mills-dilaton model with eight arbitrary functions $\mathcal{F}_0(\Phi)$, $\mathcal{F}(\Phi)$, $f_0(\Phi)$, $f_1(\Phi)$, ... $f_5(\Phi)$.

3 Application of the non-minimal EYMd theory to the model with pp-wave symmetry

3.1 Reduction of master equations

Let us consider now a plane-symmetric space-time associated usually with a gravitational radiation. We assume the metric to be of the form

$$ds^2=2dudv-L^2(u)\left[e^{2\beta(u)}(dx^2)^2+e^{-2\beta(u)}(dx^3)^2\right], \tag{24}$$

where $u = (t - x^1)/\sqrt{2}$ and $v = (t + x^1)/\sqrt{2}$ are the retarded and advanced time, respectively. This space-time is known to admit the G_5 group of isometries [20], and three Killing four-vectors, ξ^k, $\xi^k_{(2)}$ and $\xi^k_{(3)}$ form three-dimensional Abelian subgroup G_3. The four-vector ξ^k is the null one and covariantly constant, i.e.,

$$\xi^k = \delta^k_v \,, \quad g_{kl}\, \xi^k \xi^l = 0 \,, \quad \nabla_l \, \xi^k = 0 \,. \tag{25}$$

The four-vectors $\xi^k_{(\alpha)}$ $(\alpha = 2, 3)$ are space-like and orthogonal to ξ^k and to each other, i.e.,

$$\xi^k_{(\alpha)} = \delta^k_\alpha \,, \quad g_{kl}\, \xi^k_{(2)} \xi^l_{(3)} = 0 \,, \quad g_{kl}\, \xi^k \xi^l_{(\alpha)} = 0 \,. \tag{26}$$

The non-vanishing components of the Ricci and Riemann tensors are, respectively

$$R_{uu} = R^2{}_{u2u} + R^3{}_{u3u} \,, \quad R^2{}_{u2u} = - \left[\frac{L''}{L} + (\beta')^2\right] - \left[2\beta' \, \frac{L'}{L} + \beta''\right] , \tag{27}$$

$$R^3{}_{u3u} = - \left[\frac{L''}{L} + (\beta')^2\right] + \left[2\beta' \, \frac{L'}{L} + \beta''\right] , \quad R = 0 \,. \tag{28}$$

We consider a *toy-model*, which satisfies the following requirements. *First*, the potentials of the Yang-Mills field are parallel in the group space [21], i.e.,

$$A^{(a)}_k = q^{(a)} A_k \,, \quad G_{(a)(b)} q^{(a)} q^{(b)} = 1 \,, \quad q^{(a)} = const \,. \tag{29}$$

Second, the vector field A_k and scalar field Φ inherit the symmetry of the space-time, i.e., the Lie derivatives of these quantities along generators of the group G_3, $\{\xi\} \equiv \{\xi^k, \xi^k_{(2)}, \xi^k_{(3)}\}$, are equal to zero:

$$\pounds_{\{\xi\}} A_k = 0 \,, \quad \pounds_{\{\xi\}} \Phi = 0 \,. \tag{30}$$

Third, the vector field A^k is transverse, i.e.,

$$A^k = - \left[A_2 \xi^k_{(2)} + A_3 \xi^k_{(3)}\right] , \quad \xi^k A_k = 0 \,. \tag{31}$$

Fourth, the potential of the scalar field is equal to zero, $V(\Phi) = 0$. *Fifth*, the cosmological constant is absent, $\Lambda = 0$. These five requirements lead to the following simplifications.
(i) The fields A_k and Φ depend on the retarded time u only; there are two non-vanishing components of the field strength tensor

$$F^{(a)ik} = q^{(a)} \left[\left(\xi^i \xi^k_{(2)} - \xi^k \xi^i_{(2)}\right) A'_2(u) + \left(\xi^i \xi^k_{(3)} - \xi^k \xi^i_{(3)}\right) A'_3(u)\right] , \tag{32}$$

the invariant $F^{(a)}_{ik} F^{ik}_{(a)}$ as well as the terms $\mathcal{R}^{ikmn} F^{(a)}_{mn}$ are equal to zero.
(ii) The equations (12) and (13) are satisfied identically.
(iii) The non-minimal terms $T^{(I)}_{ik}$, ..., $T^{(V)}_{ik}$ (18)-(22) disappear, and the functions $f_1(\Phi)$, $f_2(\Phi)$,..., $f_5(\Phi)$, being non-vanishing, happen to be hidden, i.e., they do not enter the equations for the gravity field.
After such simplifications the non-minimal equations for the gravity field (14)-(22) reduce to one equation

$$\frac{L''}{L} + (\beta')^2 = -\frac{1}{2} \kappa T(u) \,, \tag{33}$$

where

$$T(u) = \left\{\frac{f_0(\Phi)}{L^2} \left[e^{-2\beta} \left(A'_2\right)^2 + e^{2\beta} \left(A'_3\right)^2\right] + (\Phi')^2 \left[\mathcal{F}_0(\Phi) + \frac{d^2 \mathcal{F}(\Phi)}{d\Phi^2}\right] + \Phi'' \frac{d\mathcal{F}(\Phi)}{d\Phi}\right\} [1 + \kappa \mathcal{F}(\Phi)]^{-1} \,. \tag{34}$$

For this model $A_2(u)$, $A_3(u)$, $\Phi(u)$ are arbitrary functions of the retarded time, and the prime denotes the derivative with respect to u.

3.2 Regular solutions

Let us consider a model with $L(u) \equiv 1$. It can be indicated as a regular model, since $det(g_{ik}) = -L^4 \equiv -1$ and can not vanish, contrary to the standard situation with gravitational pp-waves [5]. When $\mathcal{F} = 0$, and $\mathcal{F}_0(\Phi)$, $f_0(\Phi)$ are positive functions, there are no real solutions of the equation (33) with $L = 1$, since the right-hand side is negative for arbitrary moment of the retarded time (see (34)). Nevertheless, such a possibility appears in the non-minimal case. Below we consider two explicit examples of the exact regular solutions.

First explicit example

Let the scalar field take a constant value $\Phi_0 \neq 0$, then the functions

$$A_2(u) = A_2(0)\ e^{\beta(u)}\,, \quad A_3(u) = A_3(0)\ e^{-\beta(u)}\,, \tag{35}$$

give the exact solution of (33) for arbitrary $\beta'(u)$, when

$$-\kappa\mathcal{F}(\Phi_0) = 1 + \frac{\kappa}{2} f_0(\Phi_0)\left[A_2^2(0) + A_3^2(0)\right]\,. \tag{36}$$

For a given negative function $\mathcal{F}(\Phi)$ and positive function $f_0(\Phi)$, this equality predetermines some special value of the dilaton field, Φ_0. The function $\beta(u)$ is now arbitrary, and we can use, for instance, the periodic finite function

$$\beta(u) = \frac{1}{2}h\ (1 - \cos 2\lambda u)\,, \quad \beta(0) = 0\,, \quad \beta'(0) = 0\,. \tag{37}$$

The metric for this non-minimal model is periodic and regular

$$ds^2 = 2dudv - \left[e^{2h\sin^2\lambda u}(dx^2)^2 + e^{-2h\sin^2\lambda u}(dx^3)^2\right]\,, \tag{38}$$

the potentials of the Yang-Mills field (B.5) are also periodic and regular.

Second explicit example

Let the dilaton field be periodic, i.e.,

$$\Phi(u) = \Phi_0 \sin\lambda u\,, \tag{39}$$

and the color wave be circularly polarized, i.e.,

$$A_2'(u) = E_0\ e^{\beta(u)}\cos\omega u\,, \quad A_3'(u) = E_0\ e^{-\beta(u)}\sin\omega u\,. \tag{40}$$

The wave is called circularly polarized in analogy with electrodynamics, since the function

$$E^2(u) \equiv -g^{ik}A_i'A_k' = e^{-2\beta}\left(A_2'\right)^2 + e^{2\beta}\left(A_3'\right)^2 = E_0^2 \tag{41}$$

is constant for such a field. Let us consider the functions $\mathcal{F}_0(\Phi)$, $f_0(\Phi)$ and $\mathcal{F}(\Phi)$ to be of the form

$$\mathcal{F}_0(\Phi) = 1\,, \quad f_0(\Phi) = 1 + \left(\frac{\Phi(u)}{\Phi_0}\right)^2\,, \quad \mathcal{F}(\Phi) = -\left(1 + \frac{2}{\kappa\Phi_0^2}\right)\Phi^2(u)\,, \tag{42}$$

and assume for simplicity, that $E_0 = \Phi_0\lambda$. Then the equations (33) and (34) yield

$$\left(\beta'(u)\right)^2 = 2\lambda^2\,, \qquad \beta(u) = \pm\sqrt{2}\lambda u\,. \tag{43}$$

Thus, the metric takes the form

$$ds^2 = 2dudv - \left[e^{\pm 2\sqrt{2}\lambda u}(dx^2)^2 + e^{\mp 2\sqrt{2}\lambda u}(dx^3)^2 \right] , \tag{44}$$

and the space-time is symmetric [20], i.e., all the non-vanishing components of the Riemann tensor are constant

$$R^2{}_{u2u} = R^3{}_{u3u} = -(\beta')^2 = -2\lambda^2 = \frac{1}{2}R_{uu} \,. \tag{45}$$

The Yang-Mills potentials are quasi-periodic functions

$$A_2(u) = \frac{E}{(\omega^2 + 2\lambda^2)} \left[e^{\pm\sqrt{2}\lambda u}(\sqrt{2}\lambda \cos\omega u + \omega \sin\omega u) - \sqrt{2}\lambda \right] , \quad A_3(0) = 0 \,, \tag{46}$$

$$A_3(u) = \frac{E}{(\omega^2 + 2\lambda^2)} \left[e^{\mp\sqrt{2}\lambda u}(\sqrt{2}\lambda \sin\omega u - \omega \cos\omega u) + \omega \right] , \quad A_3(0) = 0 \,. \tag{47}$$

This solution is also free of singularity at the finite moments of the retarded time.

4 Conclusions

The Lagrangian of the presented non-minimal Einstein-Yang-Mills-dilaton model includes eight arbitrary functions depending on the dilaton field, thus, we deal with a wide freedom of modeling. The first (simplest) example of the application shows, that for the model with pp-wave symmetry five of the eight arbitrary functions happen to be hidden, i.e., they do not enter the master equations. Nevertheless, the presence of three remained functions of the dilatonic field allows us to find exact explicit solutions of the whole self-consistent system of master equations, which can be indicated as regular (quasi)periodic solutions of the pp-wave type. This means, that the dilatonic extension of the non-minimal field theory seems to be a promising instrument for its modification.

We assume, that this non-minimal EYMd model might be fruitful for cosmological applications, especially, for the explanation of the accelerated expansion of the Universe and dark energy phenomenon. We also believe, that the mentioned wide freedom of modeling in the framework of the non-minimal EYMd theory can be used in searching for new regular solutions, describing colored static spherically symmetric objects. We intend to discuss these problems in detail in further papers.

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Cosmology in non-minimal Yang-Mills/Maxwell theory

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Abstract

We review cosmology in non-minimal Yang-Mills/Maxwell theory, in which the Yang-Mills/electromagnetic field couples to a function of the scalar curvature. We show that power-law inflation can be realized due to the non-minimal gravitational coupling of the Yang-Mills field which may be caused by quantum corrections. Moreover, we study non-minimal vector model in the framework of modified gravity and demonstrate that both inflation and the late-time accelerated expansion of the universe can be realized. We also discuss the cosmological reconstruction of the Yang-Mills theory. Furthermore, we investigate late-time cosmology in non-minimal Maxwell-Einstein theory. We explore the forms of the non-minimal gravitational coupling which generate the finite-time future singularities and the general conditions for this coupling in order that the finite-time future singularities cannot appear.

1 Introduction

It is observationally confirmed that the current expansion of the universe is accelerating, as well as that there existed the inflationary stage in the early universe [1, 2]. The former phenomenon is called "dark energy" problem (for recent reviews, see [3, 4]). The studies of the dark energy problem can be categorized into the following two approaches. One is to introduce some (unknown) matter which is responsible for dark energy in the framework of general relativity. Another is to modify the gravitational theory, e.g., to study the action described by an arbitrary function $F(R)$ of the scalar curvature R, which is called "$F(R)$ gravity" (for a review, see [4, 5]). Such a modified gravity must pass cosmological bounds and solar system tests because it is considered as an alternative gravitational theory.

A very realistic modified gravitational theory which evades solar-system tests has recently been proposed by Hu and Sawicki [6] (for related studies, see [7]). In this theory, an effective epoch described by the cold dark matter model with cosmological constant (ΛCDM), which accounts for high-precision observational data, is realized as in general relativity with

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cosmological constant. Although this theory can successfully explain the late-time acceleration of the universe, the possibility of the realization of inflation has not been discussed. In Refs. [8, 9, 10], following the previous inflation-acceleration unification proposal [2], modified gravities in which both inflation and the late-time acceleration of the universe can occur have been investigated. The classification of viable $F(R)$ gravities has also been suggested in Ref. [9].

As another gravitational source of inflation and the late-time acceleration of the universe, there is a coupling between the scalar curvature and matter Lagrangian [12, 13] (see also [14]). Such a coupling may be applied for the realization of the dynamical cancellation of cosmological constant [15]. In Refs. [16, 17, 18], the criteria for the viability of such theories have been examined. As a simple case, a coupling between a function of the scalar curvature and the kinetic term of a massless scalar field in a viable modified gravity has been considered in Ref. [19].

It is known that the coupling between the scalar curvature and the Lagrangian of the electromagnetic field arises in curved spacetime due to one-loop vacuum-polarization effects in Quantum Electrodynamics (QED) [20]. In the present article, following the considerations in Ref. [21], we review cosmology in non-minimal non-Abelian gauge theory (Yang-Mills (YM) theory), in which the non-Abelian gauge field (the YM field) couples to a function of the scalar curvature. In particular, it is shown that power-law inflation can be realized due to the non-minimal gravitational coupling of the Yang-Mills field. The consequences presented correspond to the generalization of the results for non-minimal Maxwell theory with the coupling of the electromagnetic field to a function of the scalar curvature [22]. Furthermore, we consider a non-minimal vector model in the framework of modified gravity. It is demonstrated that both inflation and the late-time accelerated expansion of the universe can be realized. We also study the reconstruction of the YM theory. In the past studies, inflation driven by a vector filed [23] and its instability [24] and gravitational-electromagnetic inflation from a 5-dimensional vacuum state [25] have been discussed. As a candidate for dark energy, the effective YM condensate [26], the Born-Infeld quantum condensate [27] and a vector field [28] have been proposed. Models of vector curvaton have also been constructed [29].

In addition, following the investigations in Ref. [47], we review late-time cosmology in the non-minimal Maxwell-Einstein theory. We investigate the forms of the non-minimal gravitational coupling of the electromagnetic field generating the finite-time future singularities and the general conditions for the non-minimal gravitational coupling of the electromagnetic field in order that the finite-time future singularities cannot appear. In Ref. [31], $F(R)$ gravity coupled to non-linear electrodynamics has been examined.

This article is organized as follows. In Sec. II we investigate a non-minimal gravitational coupling of the $SU(N)$ YM field in general relativity. In Sec. III we consider non-minimal vector model in the framework of modified gravity. In Sec. IV we discuss the reconstruction of the YM theory. In Sec. V we study non-minimal Maxwell-Einstein theory with a general gravitational coupling. We explore the cosmological effects of the non-minimal gravitational coupling of the electromagnetic field to a function of the scalar curvature on the finite-time future singularities. Finally, summary is given in Sec. VI. We use units in which $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by κ^2, so that $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2$, where $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$GeV is the Planck mass. Moreover, in terms of electromagnetism we adopt Heaviside-Lorentz units.

2 Inflation in general relativity

In this section, we study non-minimal YM theory in general relativity. The model action is given as follows [21]:

$$
\begin{aligned}
S_{\mathrm{GR}} &= \int d^4x\sqrt{-g}\left(\mathcal{L}_{\mathrm{EH}}+\mathcal{L}_{\mathrm{YM}}\right), && (1)\\
\mathcal{L}_{\mathrm{EH}} &= \frac{1}{2\kappa^2}R, && (2)\\
\mathcal{L}_{\mathrm{YM}} &= -\frac{1}{4}I(R)F_{\mu\nu}^{a}F^{a\mu\nu}\left[1+b\tilde{g}^2\ln\left|\frac{-\left(1/2\right)F_{\mu\nu}^{a}F^{a\mu\nu}}{\mu^4}\right|\right], && (3)\\
I(R) &= 1+f(R)\,, \quad b=\frac{1}{4}\frac{1}{8\pi^2}\frac{11}{3}N\,, \quad F_{\mu\nu}^{a}=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+f^{abc}A_\mu^b A_\nu^c\,, && (4)
\end{aligned}
$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, R is the scalar curvature arising from the spacetime metric tensor $g_{\mu\nu}$, and $\mathcal{L}_{\mathrm{EH}}$ is the Einstein-Hilbert action. Moreover, $\mathcal{L}_{\mathrm{YM}}$ with $I(R)=1$ is the effective Lagrangian of the $SU(N)$ YM theory up to one-loop order [32, 33], $f(R)$ is an arbitrary function of R, b is the asymptotic freedom constant, $F_{\mu\nu}^{a}$ is the field strength tensor, A_μ^a is the $SU(N)$ YM field with the internal symmetry index a (Roman indices, a, b, c, run over $1,2,\ldots,N^2-1$, and in $F_{\mu\nu}^{a}F^{a\mu\nu}$ the summation in terms of the index a is also made), and f^{abc} is a set of numbers called structure constants and completely antisymmetric [34]. Furthermore, μ is the mass scale of the renormalization point, and a field-strength-dependent running coupling constant is given by [33]

$$
\tilde{g}^2(X)=\frac{\tilde{g}^2}{1+b\tilde{g}^2\ln\left|X/\mu^4\right|}\,, \quad X\equiv-\frac{1}{2}F_{\mu\nu}^{a}F^{a\mu\nu}\,, \tag{5}
$$

where $\tilde{g}$ is the value of the running coupling constant when $X=\mu^4$.

Taking the variations of the action (1) with respect to the metric $g_{\mu\nu}$ and the $SU(N)$ YM field A_μ^a, we obtain the gravitational field equation and the equation of motion of A_μ^a as

$$
R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa^2 T_{\mu\nu}^{(\mathrm{YM})}\,, \tag{6}
$$

$$
\begin{aligned}
T_{\mu\nu}^{(\mathrm{YM})} &= I(R)\left(g^{\alpha\beta}F_{\mu\beta}^{a}F_{\nu\alpha}^{a}\varepsilon-\frac{1}{4}g_{\mu\nu}\mathcal{F}\right)+\\
&\quad \frac{1}{2}\left[f'(R)\mathcal{F}R_{\mu\nu}+g_{\mu\nu}\Box\left(f'(R)\mathcal{F}\right)-\nabla_\mu\nabla_\nu\left(f'(R)\mathcal{F}\right)\right], && (7)\\
\varepsilon &= 1+b\tilde{g}^2\ln\left|e\left[\frac{-\left(1/2\right)F_{\mu\nu}^{a}F^{a\mu\nu}}{\mu^4}\right]\right|=1+b\tilde{g}^2\ln\left|e\left(\frac{X}{\mu^4}\right)\right|, && (8)\\
\mathcal{F} &= F_{\mu\nu}^{a}F^{a\mu\nu}\left[1+b\tilde{g}^2\ln\left|\frac{-\left(1/2\right)F_{\mu\nu}^{a}F^{a\mu\nu}}{\mu^4}\right|\right]=-2X\left(1+b\tilde{g}^2\ln\left|\frac{X}{\mu^4}\right|\right), && (9)
\end{aligned}
$$

and
$$
\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}I(R)\varepsilon F^{a\mu\nu}\right)-I(R)\varepsilon f^{abc}A_\mu^b F^{c\mu\nu}=0\,, \tag{10}
$$

respectively, where $T_{\mu\nu}^{(\mathrm{YM})}$ is the contribution to the energy-momentum tensor from the $SU(N)$ YM field, the prime denotes differentiation with respect to R, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian for a scalar field, and $R_{\mu\nu}$ is the Ricci curvature tensor. Moreover, ε is a field-strength-dependent effective dielectric constant [33], and $e\approx 2.72$ is the Napierian number. In deriving the second equalities in Eqs. (8) and (9), we have used $X=-\left(1/2\right)F_{\mu\nu}^{a}F^{a\mu\nu}$.

We assume the flat Friedmann-Robertson-Walker (FRW) spacetime with the metric $ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2 = a^2(\eta)(-d\eta^2 + d\boldsymbol{x}^2)$, where a is the scale factor and η is the conformal time.

In the search of exact solutions for non-minimal YM (electromagnetic)-gravity theory, the problem of off-diagonal components of YM (electromagnetic) stress tensor being non-zero while the right-hand side of Einstein equations is zero (for the argument about the problem of off-diagonal components of electromagnetic energy-momentum tensor in non-minimal Maxwell-gravity theory, see [35]). As a simple case, we can consider the following case in which the off-diagonal components of $T^{(\mathrm{YM})}_{\mu\nu}$ in Eq. (7) vanishes: (i) Only (YM) magnetic fields exist and hence (YM) electric fields are negligible. (ii) $\boldsymbol{B^a} = (B^a_1, B^a_2, B^a_3)$, where $B^a_1 = B^a_2 = 0, B^a_3 \neq 0$, namely, we consider the case in which only one component of $\boldsymbol{B}^a$ is non-zero and other two components are zero. In such a case, it follows from $\mathrm{div}\boldsymbol{B}^a = 0$ that the off-diagonal components of the last term on the right-hand side of $T^{(\mathrm{YM})}_{\mu\nu}$, i.e., $\nabla_\mu \nabla_\nu \left(f'(R)\mathcal{F}\right)$ are zero. Thus, all of the off-diagonal components of $T^{(\mathrm{YM})}_{\mu\nu}$ are zero. Throughout this article, we consider the above case.

In Eq. (10), we can neglect the higher order than or equal to the quadratic terms in A^a_μ because the amplitude of A^a_μ is small, and investigate the linearized equation of Eq. (10) in terms of A^a_μ in the Coulomb gauge, $\partial^j A^a_j(t, \boldsymbol{x}) = 0$, and the case of $A^a_0(t, \boldsymbol{x}) = 0$. Replacing the independent variable t by η, we find that the Fourier mode $A^a_i(k, \eta)$ satisfies the equation $\left(\partial^2 A^a_i(k, \eta)\right) / \left(\partial \eta^2\right) + \left(1/I(\eta)\right) \left[dI(\eta)/\left(d\eta\right)\right] \left[\partial A^a_i(k, \eta)/\left(\partial \eta\right)\right] + k^2 A^a_i(k, \eta) = 0$. By using the WKB approximation on subhorizon scales and the long-wavelength approximation on superhorizon scales, and matching these solutions at the horizon crossing [36], we find that an approximate solution is given by

$$|A^a_i(k, \eta)|^2 = |C(k)|^2 = \frac{1}{2kI(\eta_k)} \left| 1 - \left(\frac{1}{2kI(\eta_k)} \frac{dI(\eta_k)}{d\eta} + i \right) k \int_{\eta_k}^{\eta_{\mathrm{f}}} \frac{I(\eta_k)}{I(\tilde{\eta})} d\tilde{\eta} \right|^2 , \tag{11}$$

where η_k and η_{f} are the conformal time at the horizon-crossing and at the end of inflation, respectively. It follows from $B_i{}^{\mathrm{proper}}(t, \boldsymbol{x}) = a^{-2} \epsilon_{ijk} \partial_j A_k(t, \boldsymbol{x})$, where ϵ_{ijk} is the totally antisymmetric tensor ($\epsilon_{123} = 1$), that the amplitude of the proper YM magnetic fields on a comoving scale $L = 2\pi/k$ in the position space is given by

$$|B^{a\,(\mathrm{proper})}_i(t)|^2 = \frac{k|C(k)|^2}{\pi^2} \frac{k^4}{a^4} \left[1 + \frac{1}{2} f^{abc} u^b u^c \frac{k|C(k)|^2}{2\pi^2} \right] , \tag{12}$$

where $u^b(= 1)$ and $u^c(= 1)$ are the quantities denoting the dependence on the indices b and c, respectively. From Eq. (12), we see that the YM magnetic fields evolves as $|B^{a\,(\mathrm{proper})}_i(t)|^2 = |\bar{B}^a|^2/a^4$, where $|\bar{B}^a|$ is a constant.

In this case, using the $(\mu, \nu) = (0, 0)$ component and the trace part of the $(\mu, \nu) = (i, j)$ component of Eq. (6), where i and j run from 1 to 3, and eliminating $I(R)$ from these equations, we obtain

$$\begin{aligned} \dot{H} + \frac{\varepsilon}{\varepsilon - b\tilde{g}^2} H^2 \;\; = \;\; & \kappa^2 \Bigg(f'(R) \left\{ -(2\varepsilon + b\tilde{g}^2)\dot{H} + \left[\left(\frac{3\varepsilon - 7b\tilde{g}^2}{\varepsilon - b\tilde{g}^2} \right) \varepsilon + 8b\tilde{g}^2 \right] H^2 \right\} + \\ & 3f''(R) \left[(\varepsilon - b\tilde{g}^2)\dddot{H} - 2\left(\varepsilon + 2b\tilde{g}^2\right) H \ddot{H} + 4(\varepsilon - b\tilde{g}^2)\dot{H}^2 - 24\varepsilon H^2 \dot{H} \right] + \\ & 18 f'''(R)(\varepsilon - b\tilde{g}^2) \left(\ddot{H} + 4H\dot{H} \right)^2 \Bigg) \frac{|\bar{B}^a|^2}{a^4} , \end{aligned} \tag{13}$$

where $H = \dot{a}/a$ is the Hubble parameter and a dot denotes a time derivative, $\dot{} = \partial/\partial t$. From Eq. (8), we see that the value of ε depends on the field strength, in other words, it varies in time. The change in time of ε, however, is smaller than that of other quantities because the dependence of ε on the field strength is logarithmic, so that we can approximately regard ε as constant in Eq. (13). In what follows, we regard ε as constant.

We consider the case in which $f(R)$ is given by the Hu-Sawicki form [6, 9],

$$f(R) = f_{\mathrm{HS}}(R) \equiv \frac{c_1 \left(R/m^2\right)^n}{c_2 \left(R/m^2\right)^n + 1}\,, \tag{14}$$

which satisfies the conditions: $\lim_{R\to\infty} f_{\mathrm{HS}}(R) = c_1/c_2 = \mathrm{const}$ and $\lim_{R\to 0} f_{\mathrm{HS}}(R) = 0$. Here, c_1 and c_2 are dimensionless constants, n is a positive constant, and m denotes a mass scale. The above second condition means that there could exist a flat spacetime solution. Hence, because in the late time universe the value of the scalar curvature becomes zero, the YM coupling I becomes unity, so that the standard YM theory can be naturally recovered. To show that power-law inflation can be realized, we consider the case in which the scale factor is given by $a(t) = \bar{a}\,(t/\bar{t})^p$, where $\bar{t}$ is some fiducial time during inflation, $\bar{a}$ is the value of $a(t)$ at $t = \bar{t}$, and p is a positive constant. In this case, $H = p/t$ and $R = 6\left(\dot{H} + 2H^2\right) = 6p(2p-1)/t^2$. At the inflationary stage, because $R/m^2 \gg 1$, we can use the approximate relation $f_{\mathrm{HS}}(R) \approx (c_1/c_2)\left[1 - (1/c_2)\left(R/m^2\right)^{-n}\right]$. Substituting the above relations in terms of a, H and R, and the approximate expressions of $f'_{\mathrm{HS}}(R)$, $f''_{\mathrm{HS}}(R)$ and $f'''_{\mathrm{HS}}(R)$ derived from the above approximate expression of $f_{\mathrm{HS}}(R)$ for $R/m^2 \gg 1$ into Eq. (13), we find $p = (n+1)/2$. If $n \gg 1$, p becomes much larger than unity, so that power-law inflation can be realized. Consequently, the YM field with a non-minimal gravitational coupling in Eq. (3) can be a source of inflation.

The constraint on a non-minimal gravitational coupling of matter from the observational data of the central temperature of the Sun has been proposed [18]. Furthermore, the existence of the non-minimal gravitational coupling of the electromagnetic field changes the value of the fine structure constant, i.e., the strength of the electromagnetic coupling. Hence, the deviation of the non-minimal electromagnetism from the ordinary Maxwell theory can be constrained from the observations of radio and optical quasar absorption lines [37], those of the anisotropy of the cosmic microwave background (CMB) radiation [38, 39], those of the absorption of the CMB radiation at 21 cm hyperfine transition of the neutral atomic hydrogen [40], and big bang nucleosynthesis (BBN) [41, 42] as well as solar-system experiments [43] (for a recent review, see [44]). On the other hand, because the energy scale of the YM theory is higher than the electroweak scale, the existence of the non-minimal gravitational coupling of the YM field might influence on models of the grand unified theories (GUT).

3 Non-minimal vector model

In this section, we investigate cosmology in non-abelian non-minimal vector model in the framework of $F(R)$ gravity. The model action is given as follows [21]:

$$\bar{S}_{\mathrm{MG}} = \int d^4x\sqrt{-g}\left(\mathcal{L}_{\mathrm{MG}} + \mathcal{L}_{\mathrm{V}}\right)\,, \tag{15}$$

$$\mathcal{L}_{\mathrm{MG}} = \frac{1}{2\kappa^2}\left(R + F(R)\right)\,, \tag{16}$$

$$\mathcal{L}_{\mathrm{V}} = I(R)\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - V\left[A^{a2}\right]\right)\,, \tag{17}$$

where $F(R)$ is an arbitrary function of R, $F^a_{\mu\nu}$ is given by Eq. (4), and $A^{a2} = g^{\mu\nu}A^a_\mu A^a_\nu$. ($F(R)$ is the modified part of gravity, and hence $F(R)$ is completely different from the non-minimal gravitational coupling of the YM field $f(R)$ in (4).)

We should note that the last term $V\left[A^{a2}\right]$ in the action (17) is not gauge invariant but can be rewritten in a gauge invariant way. For example, if the gauge group is a unitary group, we may introduce a σ-model like field U, which satisfies $U^\dagger U = 1$. The last term could be rewritten in the gauge invariant form:

$$V\left[A^{a2}\right] \to V\left[-\bar{c}\mathrm{tr}\left(U^\dagger D_\mu U\right)\left(U^\dagger D^\mu U\right)\right], \tag{18}$$

where $\bar{c}$ is a constant for the normalization and D_μ is a covariant derivative $D_\mu = \partial_\mu + iA^a_\mu T^a$ (T^a's are the generators of the gauge algebra). If we choose the unitary gauge $U = 1$, the term in (18) reduces to the original one: $V\left[A^{a2}\right]$. This may tells that the action (17) described the theory where the gauge group is spontaneously broken.

Taking the variations of the action (15) with respect to the metric $g_{\mu\nu}$ and the vector field A^a_μ, we obtain the gravitational field equation and the equation of motion of A^a_μ as

$$\left(1 + F'(R)\right)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R + F(R)\right) + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R) = \kappa^2 T^{(\mathrm{V})}_{\mu\nu}, \tag{19}$$

$$\begin{aligned} T^{(\mathrm{V})}_{\mu\nu} &= I(R)\left(g^{\alpha\beta}F^a_{\mu\beta}F^a_{\nu\alpha} + 2A^a_\mu A^a_\nu \frac{dV\left[A^{a2}\right]}{dA^{a2}} - \frac{1}{4}g_{\mu\nu}\bar{\mathcal{F}}\right) \\ &\quad + \frac{1}{2}\left\{f'(R)\bar{\mathcal{F}}R_{\mu\nu} + g_{\mu\nu}\Box\left[f'(R)\bar{\mathcal{F}}\right] - \nabla_\mu\nabla_\nu\left[f'(R)\bar{\mathcal{F}}\right]\right\}, \end{aligned} \tag{20}$$

$$\bar{\mathcal{F}} = F^a_{\mu\nu}F^{a\mu\nu} + 4V\left[A^{a2}\right], \tag{21}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}I(R)F^{a\mu\nu}\right) - I(R)\left(f^{abc}A^b_\mu F^{c\mu\nu} + 2\frac{dV\left[A^{a2}\right]}{dA^{a2}}A^{a\nu}\right) = 0, \tag{22}$$

respectively, where $T^{(\mathrm{V})}_{\mu\nu}$ is the contribution to the energy-momentum tensor from A^a_μ.

We consider the case in which $V\left[A^{a2}\right]$ is given by a class of the following power-law potential: $V\left[A^{a2}\right] = \bar{V}\left(A^{a2}/\bar{m}^2\right)^{\bar{n}}$, where $\bar{V}$ is a constant, $\bar{m}$ denotes a mass scale, and $\bar{n}(> 1)$ is a positive integer. Similarly to the preceding section, we consider the linearized equation of Eq. (22) in terms of A^a_μ. For the above power-law potential, the form of the linearized equation of motion under the ansatz $\partial^j A^a_j(t, \boldsymbol{x}) = 0$ and $A^a_0(t, \boldsymbol{x}) = 0$ is the same as in the preceding section. (This is similar to the Coulomb gauge but since the action (17) is not gauge invariant, or gauge symmetry is completely fixed by the unitary gauge as in after (18), this condition is only a working hypothesis.)

From $A^a_0 = 0$, we have $\left(1/a^2\right)A^a_i A^a_i dV\left[A^{a2}\right]/\left(dA^{a2}\right) = \bar{n}V\left[A^{a2}\right]$. Using the $(\mu, \nu) = (0, 0)$ component and the trace part of the $(\mu, \nu) = (i, j)$ component of Eq. (19), and eliminating $I(R)$ from these equations, we obtain

$$
\begin{aligned}
&\dot{H} + H^2 + \left\{ \frac{1}{6}F(R) - F'(R)H^2 + 3F''(R)\left[\dddot{H} + 4\left(\dot{H}^2 + H\ddot{H}\right)\right] + 18F'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2 \right\} \\
&\quad = \kappa^2 \Biggl(\biggl[f'(R)\left(-2\dot{H} + 3H^2\right) + 3f''(R)\left(\dddot{H} - 2H\ddot{H} + 4\dot{H}^2 - 24H^2\dot{H}\right) \\
&\quad + 18f'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2 \biggr] \frac{|\bar{B}^a|^2}{a^4} + 2\Bigl\{ -f'(R)\left[\bar{n}\dot{H} + \left(1 + 2\bar{n} - 2\bar{n}^2\right)H^2\right] \\
&\quad + 3f''(R)\left[\dddot{H} + 2\left(3 - 2\bar{n}\right)H\ddot{H} + 4\dot{H}^2 + 8\left(1 - 2\bar{n}\right)H^2\dot{H}\right] \\
&\quad + 18f'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2 \Bigr\} V\left[A^{a2}\right] \Biggr) . \qquad (23)
\end{aligned}
$$

In the case in which $|B_i^{a\,(\mathrm{proper})}(t)|^2 = |\bar{B}^a|^2/a^4$, $V\left[A^{a2}\right] \propto a^{-2\bar{n}}$. If $\bar{n} = 2$, the time evolution of $V\left[A^{a2}\right]$ is the same as that of $|B_i^{a\,(\mathrm{proper})}(t)|^2$. On the other hand, if $\bar{n} \geq 2$, $V\left[A^{a2}\right]$ decreases much more rapidly than $|B_i^{a\,(\mathrm{proper})}(t)|^2$ during inflation. Hence, in the latter case we can neglect the terms proportional to $V\left[A^{a2}\right]$ on the right-hand side of Eq. (23).

We examine the following case. $F(R)$ is given by [10]

$$
F(R) \;=\; -M^2 \frac{\left[\left(R/M^2\right) - \left(R_0/M^2\right)\right]^{2l+1} + \left(R_0/M^2\right)^{2l+1}}{c_3 + c_4\left\{\left[\left(R/M^2\right) - \left(R_0/M^2\right)\right]^{2l+1} + \left(R_0/M^2\right)^{2l+1}\right\}} , \qquad (24)
$$

which satisfies the conditions: $\lim_{R\to\infty} F(R) = -M^2/c_4 = \mathrm{const}$, $\lim_{R\to 0} F(R) = 0$. Here, c_3 and c_4 are dimensionless constants, l is a positive integer, and M denotes a mass scale. We consider that in the limit $R \to \infty$, i.e., at the very early stage of the universe, $F(R)$ becomes an effective cosmological constant, $\lim_{R\to\infty} F(R) = -M^2/c_4 = -2\Lambda_{\mathrm{i}}$, where $\Lambda_{\mathrm{i}}\left(\gg {H_0}^2\right)$ is an effective cosmological constant in the very early universe, and that at the present time $F(R)$ becomes a small constant, $F(R_0) = -M^2\left(R_0/M^2\right)^{2l+1} / \left[c_3 + c_4\left(R_0/M^2\right)^{2l+1}\right] = -2R_0$, where $R_0\left(\approx {H_0}^2\right)$ is current curvature. Here, H_0 is the Hubble constant at the present time: $H_0 = 100h\ \mathrm{km\ s^{-1}\ Mpc^{-1}} = 2.1h \times 10^{-42}\mathrm{GeV} \approx 1.5 \times 10^{-33}\mathrm{eV}$ [45], where we have used $h = 0.70$ [46]. Moreover, $f(R)$ is given by [10]:

$$
f(R) = f_{\mathrm{NO}}(R) \equiv \frac{\left[\left(R/M^2\right) - \left(R_0/M^2\right)\right]^{2q+1} + \left(R_0/M^2\right)^{2q+1}}{c_5 + c_6\left\{\left[\left(R/M^2\right) - \left(R_0/M^2\right)\right]^{2q+1} + \left(R_0/M^2\right)^{2q+1}\right\}} , \qquad (25)
$$

which satisfies the following conditions: $\lim_{R\to\infty} f_{\mathrm{NO}}(R) = 1/c_6 = \mathrm{const}$, $\lim_{R\to 0} f_{\mathrm{NO}}(R) = 0$. Here, c_5 and c_6 are dimensionless constants, and q is a positive integer. The form of $F(R)$ in Eq. (24) and that of $f_{\mathrm{NO}}(R)$ in Eq. (25) correspond to the extension of that of $f_{\mathrm{HS}}(R)$ in Eq. (14). It has been shown in Ref. [10] that modified gravitational theories described by the action (16) with $F(R)$ in Eq. (24) successfully pass the solar-system tests as well as cosmological bounds, and that they are free of instabilities.

Using Eqs. (23), (24) and (25), we find that at the very early stage of the universe ($R/M^2 \gg 1$ and $R/M^2 \gg R_0/M^2$), $a(t) \propto \exp\left(\sqrt{\Lambda_{\mathrm{i}}/3}t\right)$, so that exponential inflation can be realized, and that at the present time ($F(R) = F(R_0) = -2R_0$), $a(t) \propto \exp\left(\sqrt{R_0/3}t\right)$, so that the late-time acceleration of the universe can also be realized.

4 Reconstruction of the YM theory

In this section, we indicate how to reconstruct the YM theory from the known evolution of the universe (for a review, see [47]). We consider the following action:

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2}+\tilde{\mathcal{F}}\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right)\right)\ . \tag{26}$$

By introducing an auxiliary scalar field ϕ, we may rewrite the action (26) in the following form:

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2}+\frac{1}{4}P(\phi)F^a_{\mu\nu}F^{a\,\mu\nu}+\frac{1}{4}Q(\phi)\right)\ . \tag{27}$$

Taking the variations of the action (27) with respect to ϕ, we obtain

$$0=\frac{dP(\phi)}{d\phi}F^a_{\mu\nu}F^{a\,\mu\nu}+\frac{dQ(\phi)}{d\phi}\ , \tag{28}$$

which could be solved with respect ϕ as $\phi=\phi\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right)$. Here, the prime denotes differentiation with respect to ϕ. Substituting the expression into the action (27), we obtain the action (26) with

$$\tilde{\mathcal{F}}\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right)=\frac{1}{4}\left(P\left(\phi\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right)\right)F^a_{\mu\nu}F^{a\,\mu\nu}+Q\left(\phi\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right)\right)\right)\ . \tag{29}$$

Taking the variations of the action (27) with respect to the metric tensor $g_{\mu\nu}$, we obtain the Einstein equation:

$$\frac{1}{2\kappa^2}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=-\frac{1}{2}P(\phi)F^a_{\mu\rho}F^{a\ \rho}_{\ \nu}+\frac{1}{8}g_{\mu\nu}\left(P(\phi)F^a_{\rho\sigma}F^{a\,\rho\sigma}+Q(\phi)\right)\ . \tag{30}$$

On the other hand, taking the variations of the action (27) with respect to A^a_μ, we obtain

$$0=\partial_\nu\left(\sqrt{-g}P(\phi)F^{a\,\nu\mu}\right)-\sqrt{-g}P(\phi)f^{abc}A^b_\nu F^{c\,\nu\mu}\ . \tag{31}$$

For simplicity, we only consider the case in which the gauge algebra is $SU(2)$, where $f^{abc}=\epsilon^{abc}$, and we assume that the gauge fields are given in the following form:

$$A^a_\mu=\left\{\begin{array}{cc}\bar{\alpha}e^{\lambda(t)}\delta^a_{\ i} & (\mu=i=1,2,3)\\ 0 & (\mu=0)\end{array}\right.\ , \tag{32}$$

where $\bar{\alpha}$ is a constant with mass dimension and λ is a proper function of t. In general, if the vector field is condensed, the rotational invariance of the universe could be broken. In case of (32), the direction of the vector field is gauge variant. Hence, all the gauge invariant quantities given by (32) do not break the rotational invariance.

By the assumption, Eq. (28) has the following form:

$$0=6\left(-\bar{\alpha}^2\dot{\lambda}^2e^{2\lambda}a^{-2}+\bar{\alpha}^4e^{4\lambda}a^{-4}\right)\frac{dP(\phi)}{d\phi}+\frac{dQ(\phi)}{d\phi}\ , \tag{33}$$

and (t,t)-component of Eq. (30) is given by

$$0=\frac{3}{\kappa^2}H^2-\frac{3}{2}\left(\bar{\alpha}^2\dot{\lambda}^2e^{2\lambda}a^{-2}+\bar{\alpha}^4e^{4\lambda}a^{-4}\right)-\frac{1}{4}Q(\phi)\ . \tag{34}$$

The $\mu = 0$ component of Eq. (31) becomes identity and $\mu = i$ component gives

$$0 = \partial_t \left(aP(\phi)\dot{\lambda}e^{\lambda} \right) - 2\bar{\alpha}^2 a^{-1} P(\phi) e^{3\lambda} \,. \tag{35}$$

Since we can always take the scalar field ϕ properly, we may identify the scalar field with the time coordinate $\phi = t$. By differentiating Eq. (34) with respect to t and eliminating $\dot{Q} = dQ(\phi)/(d\phi)$, we obtain

$$0 = \frac{2}{\kappa^2} H\dot{H} + \bar{\alpha}^2 \dot{\lambda}^2 e^{2\lambda} a^{-2} \dot{P} - P \left[\bar{\alpha}^2 \left(\dot{\lambda}\ddot{\lambda} + \dot{\lambda}^3 - \lambda^2 H \right) e^{2\lambda} a^{-2} + 2\bar{\alpha}^4 \left(\dot{\lambda} - H \right) e^{4\lambda} a^{-4} \right] \,. \tag{36}$$

Furthermore, eliminating $\dot{P}$ by using Eq. (35), we find

$$P = \frac{2H\dot{H}}{\kappa^2 \left[2\bar{\alpha}^2 a^{-2} e^{2\lambda} \left(\dot{\lambda}^2 + \dot{\lambda}\ddot{\lambda} \right) - \bar{\alpha}^4 e^{4\lambda} a^{-4} H \right]} \,. \tag{37}$$

Using Eq. (37), we can eliminate P (and $\dot{P}$) in Eq. (35) and obtain

$$\begin{aligned} 0 \;=\;& 2\left(\dot{\lambda}\dddot{\lambda} + \ddot{\lambda}^2 + 3\dot{\lambda}^2\ddot{\lambda} \right) - \bar{\alpha}^2 e^{2\lambda} a^{-2} \dot{H} + 4\left(\dot{\lambda}^3 + \dot{\lambda}\ddot{\lambda} - \bar{\alpha}^2 e^{2\lambda} a^{-2} H \right) \left(\dot{\lambda} - H \right) \\ & + \left[2\left(\dot{\lambda}^3 + \dot{\lambda}\ddot{\lambda} \right) - \bar{\alpha}^2 e^{2\lambda} a^{-2} H \right] \left(\frac{\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}} + H + \frac{\ddot{\lambda}}{\dot{\lambda}} + \dot{\lambda} - \frac{2\bar{\alpha}^2 a^{-2} e^{2\lambda}}{\dot{\lambda}} \right) \,. \end{aligned} \tag{38}$$

If we give a proper $a = a(t)$ and therefore $H = H(t)$, Eq. (38) can be regarded as a third order differential equation with respect to λ. If we find the solution of λ with three constants of the integration, we find the explicit form of $P(\phi) = P(t)$ by using Eq. (37) and further obtain $Q(\phi)$ by using Eq. (34). Thus, we find the explicit form of three parameter families of the action (27). This tells that almost arbitrary time development of the university could be realized by the action (27) or (26).

As an example, we may consider the case of the power law expansion: $a = (t/t_1)^{h_1}$, $H = h_1/t$, where t_1 and h_1 are constants. By assuming $\lambda = (h_1 - 1)\ln(t/t_1) + \lambda_1$, where λ_1 is a constant, Eq. (28) is reduced to the algebraic equation: $0 = [2h_1/(h_1-1)]\bar{X}^2 + \left(-4h_1^2 + 13h_1 + 2\right)\bar{X} + (h_1-1)^2 (h_1-2)(4H_1 - 20)$, where $\bar{X} = \bar{\alpha}^2 t_1^2 e^{2\lambda}$. If this equation has a real positive solution with respect to $\bar{X}$, we obtain λ_1 and therefore the exact form of λ. Consequently, we can reconstruct a model to give the above power expansion. Similarly, any other evolutional history of the universe may be reproduced by the specific form of the action under consideration.

5 Finite-time future singularities in non-minimal Maxwell-Einstein theory

In this section, we consider non-minimal Maxwell-Einstein theory with a general gravitational coupling. The model action is as follows [47]:

$$S_{\text{GR}} = \int d^4x \sqrt{-g} \left(\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{EM}} \right) \,, \tag{39}$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} I(R) F_{\mu\nu} F^{\mu\nu} \,, \tag{40}$$

where $\mathcal{L}_{\mathrm{EH}}$ is the Einstein-Hilbert action in (2), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor, A_μ is the $U(1)$ gauge field, and $\tilde{I}(R)$ is an arbitrary function of R.

Taking the variations of the action Eq. (39) with respect to the metric $g_{\mu\nu}$ and the $U(1)$ gauge field A_μ, we obtain the gravitational field equation and the equation of motion of A_μ as [22]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T^{(\mathrm{EM})}_{\mu\nu} , \tag{41}$$

$$\begin{aligned} T^{(\mathrm{EM})}_{\mu\nu} &= I(R) \left(g^{\alpha\beta} F_{\mu\beta} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \\ &+ \frac{1}{2} \left[I'(R) F_{\alpha\beta} F^{\alpha\beta} R_{\mu\nu} + g_{\mu\nu} \Box \left(I'(R) F_{\alpha\beta} F^{\alpha\beta} \right) - \nabla_\mu \nabla_\nu \left(I'(R) F_{\alpha\beta} F^{\alpha\beta} \right) \right] , \end{aligned} \tag{42}$$

and

$$- \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} I(R) F^{\mu\nu} \right) = 0 , \tag{43}$$

respectively, where $T^{(\mathrm{EM})}_{\mu\nu}$ is the contribution to the energy-momentum tensor from the electromagnetic field.

It follows from Eq. (43) that in the flat FRW background, the equation of motion for the $U(1)$ gauge field in the Coulomb gauge, $\partial^j A_j(t, \boldsymbol{x}) = 0$, and the case of $A_0(t, \boldsymbol{x}) = 0$, is the same form as in Sec. II. We therefore obtain the approximate solution as $\left| A_i(k, \eta) \right|^2 = |C(k)|^2$, where $|C(k)|^2$ is given by Eq. (11). Using this solution, we find that the amplitude of the proper magnetic fields in the position space is given by $|{B_i}^{(\mathrm{proper})}(t)|^2 = \left(k |C(k)|^2 / \pi^2 \right) \left(k^4 / a^4 \right)$ on a comoving scale $L = 2\pi/k$. From this equation, we see that the proper magnetic fields evolve as $|{B_i}^{(\mathrm{proper})}(t)|^2 = |\bar{B}|^2 / a^4$, where $|\bar{B}|$ is a constant. The conductivity of the universe σ_{c} is negligibly small during inflation, because there are few charged particles at that time. After the reheating stage, a number of charged particles are produced, so that the conductivity immediately jumps to a large value: $\sigma_{\mathrm{c}} \gg H$. Consequently, for a large enough conductivity at the reheating stage, the proper magnetic fields behave in proportion to $a^{-2}(t)$ in the radiation-dominated stage and the subsequent matter-dominated stage [48].

In this case, it follows from Eq. (42) that the quantities corresponding to the effective energy density of the universe ρ_{eff} and the effective pressure p_{eff} are given by

$$\rho_{\mathrm{eff}} = \left\{ \frac{I(R)}{2} + 3 \left[- \left(5H^2 + \dot{H} \right) I'(R) + 6H \left(4H\dot{H} + \ddot{H} \right) I''(R) \right] \right\} \frac{|\bar{B}|^2}{a^4} , \tag{44}$$

$$\begin{aligned} p_{\mathrm{eff}} = \Bigg[&- \frac{I(R)}{6} - \left(H^2 - 5\dot{H} \right) I'(R) + 6 \left(20H^2\dot{H} - 4\dot{H}^2 + H\ddot{H} - \dddot{H} \right) I''(R) \\ &- 36 \left(4H\dot{H} + \ddot{H} \right)^2 I'''(R) \Bigg] \frac{|\bar{B}|^2}{a^4} . \end{aligned} \tag{45}$$

We now suppose that $I(R)$ is (almost) constant at the present time, and that for the small curvature, $I(R)$ behaves as $I(R) \sim I_0 R^\alpha$, where I_0 and α are constants. We consider the case $\alpha < 0$. The energy density of the magnetic fields is given by $\rho_B = (1/2) \, |{B_i}^{(\mathrm{proper})}(t)|^2 I(R) = \left[|\bar{B}|^2 / \left(2a^4 \right) \right] I(R)$. We take de Sitter background as the future universe. In such a case, when R becomes much smaller in the future, the energy density of the magnetic field becomes larger and larger in comparison with its current value. Thus, the strength of current magnetic fields of the universe may evolve to very large values in the future universe.

In the flat FRW background, the Einstein equations are given by $3H^2/\kappa^2 = \rho_{\mathrm{eff}}$ and $-\left(2\dot{H} + 3H^2 \right) / \kappa^2 = p_{\mathrm{eff}}$, where ρ_{eff} and p_{eff} are given by Eqs. (44) and (45), respectively.

We examine the form of $I(R)$ producing the Big Rip singularity $H \sim h_0/(t_0 - t)$, where h_0 is a positive constant, and H diverges at $t = t_0$. In this case, the scalar curvature and the scale factor are given by $R \sim 6h_0 (2h_0 + 1) / (t_0 - t)^2$ and $a \sim a_0 (t_0 - t)^{-h_0}$, respectively, where a_0 is a constant. We now assume that for the large curvature, $I(R)$ behaves as $I(R) \sim I_0 R^\alpha$. Hence $\rho_{\rm eff}$ in Eq. (44) behaves as $(t_0 - t)^{-2\alpha + 4h_0}$, but the left-hand side (l.h.s.) on the Friedmann equation $3H^2/\kappa^2 = \rho_{\rm eff}$ evolves as $(t_0 - t)^{-2}$. The consistency gives $-2 = -2\alpha + 4h_0$, i.e., $h_0 = (\alpha - 1)/2$ or $\alpha = 1 + 2h_0$. The Friedmann equation $3H^2/\kappa^2 = \rho_{\rm eff}$ also shows $3h_0^2/\kappa^2 = -\left[I_0 h_0 \left(12h_0^2 + 6h_0\right)^\alpha |\bar{B}|^2\right] / \left(2a_0^4\right)$, where we have used $\alpha = 1 + 2h_0$. This equation requires that I_0 should be negative. As a result, it follows from $I(R) \sim I_0 R^\alpha$ and $\alpha = 1 + 2h_0$ that the Big Rip singularity can appear only when for the large curvature, $I(R)$ behaves as R^{1+2h_0}. If the form of $I(R)$ is given by other terms, the Big Rip singularity cannot emerge. We note that if exactly $I(R) = I_0 R^\alpha$, $H = h_0/(t_0 - t)$ is an exact solution.

Next, we investigate the form of $I(R)$ giving a more general singularity $H \sim h_0 (t_0 - t)^{-\beta}$. In this case, the scalar curvature and the scale factor are given by

$$R \sim 6h_0 \left[\beta + 2h_0 (t_0 - t)^{-(\beta-1)}\right] (t_0 - t)^{-(\beta+1)}$$

and $a \sim a_0 \exp\left[h_0 (\beta - 1)^{-1} (t_0 - t)^{-(\beta-1)}\right]$, respectively.

In Ref. [45], the finite-time future singularities has been classified in the following way:

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$. This also includes the case of ρ, p being finite at t_s.
- Type II ("sudden") [50] : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$
- Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$ and higher derivatives of H diverge. This also includes the case in which p (ρ) or both of p and ρ tend to some finite values, while higher derivatives of H diverge.

Here, t_s, $a_s (\neq 0)$ and ρ_s are constants. We now identify t_s with t_0. The Type I corresponds to $\beta > 1$ or $\beta = 1$ case, Type II to $-1 < \beta < 0$ case, Type III to $0 < \beta < 1$ case, and Type IV to $\beta < -1$ but β is not any integer number. We note that if only higher derivatives of the Hubble rate diverge, then some combination of curvature invariants also diverges and it leads to singularity.

We assume that for the large curvature, $I(R)$ behaves as $I(R) \sim I_0 R^\alpha$. If $\beta < -1$, in the limit $t \to t_0$, $R \to 0$. We therefore consider this case later. If $\beta > 1$, $a \to \infty$ and $\rho_{\rm eff} \to 0$ and $p_{\rm eff} \to 0$ because $\rho_{\rm eff} \propto a^{-4}$ and $p_{\rm eff} \propto a^{-4}$. On the other hand, $H \to \infty$. Hence the Einstein equations cannot be satisfied.

If $\alpha > 0$ and $0 < \beta < 1$, $\rho_{\rm eff}$ in Eq. (44) evolves as $(t_0 - t)^{-\alpha(\beta+1)}$, but the l.h.s. on $3H^2/\kappa^2 = \rho_{\rm eff}$ behaves as $(t_0 - t)^{-2\beta}$. The consistency gives $-2\beta = -\alpha(\beta + 1)$, i.e., $\beta = \alpha/(2 - \alpha)$ or $\alpha = 2\beta/(\beta + 1)$. From $3H^2/\kappa^2 = \rho_{\rm eff}$, we also find

$$3h_0^2/\kappa^2 = -\left[I_0 (6h_0\beta)^\alpha (1 - \beta) |\bar{B}|^2\right] / \left[2a_0^4 (\beta + 1)\right],$$

where we have used $\alpha = 2\beta/(\beta + 1)$ and on the l.h.s. we have taken only the leading term. This equation requires that I_0 should be negative. If $\alpha > 0$ and $0 < \beta < 1$, in the limit $t \to t_0$, $a \to a_0$, $R \to \infty$, $\rho_{\rm eff} \to \infty$ and $|p_{\rm eff}| \to \infty$. Hence the Type III singularity appears. If $\alpha > 0$ and $-1 < \beta < 0$, $\rho_{\rm eff} \to \infty$, but $H \to 0$. Thus $3H^2/\kappa^2 = \rho_{\rm eff}$ cannot be satisfied.

If $(\beta-1)/(\beta+1) < \alpha < 0$ and $-1 < \beta < 0$, in the limit $t \to t_0$, $a \to a_0$, $R \to \infty$, $\rho_{\mathrm{eff}} \to 0$ and $|p_{\mathrm{eff}}| \to \infty$. Although the final value of ρ_{eff} is not finite but vanishes, this singularity can be considered to the Type II. The reason is as follows. In this case, when I and H are given by $I = 1 + I_0 R^\alpha$ and $H = H_0 + h_0 (t_0 - t)^{-\beta}$, respectively, where H_0 is a constant, in the above limit $\rho_{\mathrm{eff}} \to \rho_0$. From Eq. (44) and $3H^2/\kappa^2 = \rho_{\mathrm{eff}}$, we find $\rho_0 = 3H_0^2/\kappa^2 = |\bar{B}|^2/\left(2a_0^4\right)$. Hence ρ_0 is a finite value.

If $\alpha \leq (\beta-1)/(\beta+1)$ and $-1 < \beta < 0$, in the limit $t \to t_0$, $a \to a_0$, $R \to \infty$, $\rho_{\mathrm{eff}} \to 0$ and $|p_{\mathrm{eff}}| \to 0$, but $\dot{H} \to \infty$. Hence $-\left(2\dot{H} + 3H^2\right)/\kappa^2 = p_{\mathrm{eff}}$ cannot be satisfied. If $\alpha < 0$ and $0 < \beta < 1$, $\rho_{\mathrm{eff}} \to 0$, but $H \to \infty$. Thus $3H^2/\kappa^2 = \rho_{\mathrm{eff}}$ cannot be satisfied.

In addition, we investigate the case in which $\beta < -1$. In this case, in the limit $t \to t_0$, $a \to a_0$ and $R \to 0$. We assume that for the small curvature, $I(R)$ behaves as $I(R) \sim I_0 R^\alpha$. If $\alpha \geq (\beta-1)/(\beta+1)$, in the limit $t \to t_0$, $\rho_{\mathrm{eff}} \to 0$, $|p_{\mathrm{eff}}| \to 0$, and higher derivatives of H diverge. Hence the Type IV singularity appears. If $0 < \alpha < (\beta-1)/(\beta+1)$, $\rho_{\mathrm{eff}} \to 0$ and $|p_{\mathrm{eff}}| \to \infty$. However, $H \to 0$ and $\dot{H} \to 0$. Thus $-\left(2\dot{H} + 3H^2\right)/\kappa^2 = p_{\mathrm{eff}}$ cannot be satisfied.

We remark that if $I(R)$ is a constant (the case in which $I(R) = 1$ corresponds to the ordinary Maxwell theory), any singularity cannot appear. We also mention the case in which $I(R)$ is given by the Hu-Sawicki form in Eq. (14) as $I(R) = f_{\mathrm{HS}}(R)$ or expressed by Eq. (25) as $I(R) = f_{\mathrm{NO}}(R)$. If $\beta < -1$ and $I(R) = f_{\mathrm{HS}}(R)$ or $I(R) = f_{\mathrm{NO}}(R)$, in the limit $t \to t_0$, $a \to a_0$, $R \to 0$, $\rho_{\mathrm{eff}} \to 0$ and $|p_{\mathrm{eff}}| \to 0$. In addition, higher derivatives of H diverge. Thus the Type IV singularity appears.

As a consequence, it has been demonstrated that the Maxwell theory coupled non-minimally with the Einstein gravity may produce finite-time singularities in future, depending on the form of the non-minimal gravitational coupling. The general conditions for $I(R)$ in order that the finite-time future singularities cannot appear are that in the limit $t \to t_0$, $I(R) \to \bar{I}$, where $\bar{I}(\neq 0)$ is a finite constant, $I'(R) \to 0$, $I''(R) \to 0$ and $I'''(R) \to 0$.

6 Conclusion

In the present article, we have reviewed cosmology in non-minimal YM/Maxwell theory, in which the YM (electromagnetic) field couples to a function of the scalar curvature. It has been shown that power-law inflation can be realized due to the non-minimal gravitational coupling of the YM field which may be caused by quantum corrections. Furthermore, we have considered non-minimal vector model in the framework of modified gravity. It has been demonstrated that both inflation and the late-time accelerated expansion of the universe can be realized. We have also discussed the cosmological reconstruction of the YM theory. In addition, we have studied late-time cosmology in the non-minimal Maxwell-Einstein theory. We have investigated the forms of the non-minimal gravitational coupling which generates the finite-time future singularities and the general conditions for this coupling in order that the finite-time future singularities cannot appear. Finally, it is interesting to mention that using the reconstruction method developed in Refs. [47, 51], one can present models with the crossing of the phantom divide in non-minimal YM-modified gravity generalizing the models suggested in Ref. [11].

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Relation between first and second order formalism F(R) Theories

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Abstract

In this work we show, for the cases of lagrangian f(R), the relations between the solutions of the first order formalism theories and the solutions of the second order formalism

Present data from high redshift surveys of type Ia supernovae [1, 2], and from the anisotropy power spectrum of the cosmic microwave background CMB [3, 4] combined with Large Scale Structure LSS, apparently show a late accelerated expansion of the universe.

Up to day there are two ways to explain this behavior, there exist an ether called dark energy which is a new source of gravitation and it explains the cosmological acceleration or we have to modify the gravity equation. Maybe the first one is the most accepted explanation, the Universe has been dominated by some form of unobserved (dark) energy for a long time(cosmological constant, quintessence, quantum effect etc). A simple candidate as dark energy source is a scalar field slowly-rolling under a potential V, in which case

$$T_{ab} = -V(\phi)g_{ab} \tag{1}$$

.

However, none of the existing dark energy models are completely satisfactory. As it was said by T Padmanabhan "Dark Energy: the Cosmological Challenge of the Millennium" [5]

Another way to explain this acceleration of the universe is by changing the field equation in a way that they reproduce this behavior.

From the 1920 ´s alternative theories of gravitation have been studied which arise from different Lagrangian than the Hilbert-Einstein one, which contains other invariants as well as R

$$L = \sqrt{-g}f(R, R_{cd}\ R^{cd}, R^{ca}_{bc}\ R^{bc}_{ca}) \tag{2}$$

The quadratic terms in the Riemann tensor can be eliminated using the Gauss-Bonnet identity. The equations are obtained with second order formalism.

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In 2003, Sean Carroll and Capozziello [6, 7, 8] proposed a Lagrangian of this type $L = \sqrt{-g}(R + \alpha/R)$ to explain the acceleration. Since then, some authors have proposed adding a R^{-n} term in the Einstein-Hilbert action to modify general relativity. All of them obtained the field equations using second-order formalism, varying only the metric field, and thus obtained the so-called fourth-order field equations. Although the models were obtained using corrections of the Einstein-Hilbert Lagrangian of type R^n, where n can take a positive or negative value to explain both the inflation at an early time and the expansion at the present time [6, 9], at the beginning some of them were criticized, since they suffered from violent instabilities [10], or they did not satisfy the data of the solar system, or did not have a good newtonian limit [11]. The stability problem of the $1/R$ term was deleted by the addition of a R^2-like term to the Lagrangian [9], or by account of quantum effects [12].

Moreover, these works have been generalized using different kinds of $f(R)$ Lagrangian which are singular in $R = 0$ and which present the same advantages and disadvantages as the above mentioned modifications of the Hilbert Lagrangian (for review [13].

On the other hand, during the last year a group of authors has been working improving our knowledge about theses alternatives theories an looking for a theory that fit all the cosmological and astronomical data [14]. Thanks to them, today we have some alternatives theories that apparently satisfies the observational data and the stability requirement[15].A big review on various modified gravities and its applications to cosmology was analyzed in [16]

Among the possible modifications of the Einstein equations, we can consider all those theories that are obtained from a Lagrangian density $\mathcal{L}_T(R) = f(R)\sqrt{-g} + \mathcal{L}_M$, which depends on the scalar curvature and a matter Lagrangian and that does not depend on the connection, and then apply Palatini's method to obtain the field equations [17, 18]. In Refs. [17, 18], we showed the universality of the Einstein equation using a cosmological constant. For these theories we have studied the conserved quantities [21], the spherically symmetric solutions [22], the Newtonian limit [17, 23], and the cosmology described by the Friedmann-Robertson-Walker (FRW) metric [24].

Moreover, it is well known [23, 20, 19, 22] that in a vacuum, or in the case of $T = const$, the solutions of these theories are the same for general relativity with a cosmological constant, even when $f(R)$ is not analytical at $R = 0$. On the other hand, solutions corresponding to different cosmological constants are allowed by some of these theories. Therefore, the homogeneous and isotropic vacuums solution for these theories is the de Sitter space-time with different cosmological constants, except when one of the allowed cosmological constants is $\Lambda = 0$, which corresponds to flat space-time. For $f(R)$ analytical is very difficult to test these models in the (post-) Newtonian approximation [23]. The reason for this is that the departures from Newtonian behavior are both very small and are masked by other effects, due to the fact that these departures have to be measured when the body is moving "through" a matter-filled region.

Some time ago Vollick [25] used the corrected Lagrangian of the works of Carroll *et al.*, and Capozziello *et al.* [6, 8], $f(R) = R - \alpha^2/R$, and the Palatini variational principle –which is a particular case of the above theories but is not analytical at $R = 0$– to prove that the solutions of the field equations approach de Sitter universe at a late time. This result was obtained using the above well known property of the vacuum solutions which is satisfied even when $f(R)$ is not analytical [22]. Thus, the inclusion of the $1/R$ curvature term in the gravitational Lagrangean provides us with an alternative explanation for the cosmological acceleration. On the other hand, solutions corresponding to different cosmological constants are allowed by some of these theories. The generalization to the case of scalar tensorial theories was analyzed in [16]. Moreover, Meng and Wang [26, 27] have studied the modified Friedmann equation with the Palatini variational principle and its first, second, and third-order approximated equations.

To have a good theory we need not only that it fits the data respect to the cosmological acceleration which is observed today, but also that the theory must be in agreement with the Newtonian limit, and its solution corresponding to physical situation must be stable. Also we must ask that the solution would be continues respect the variations of the sources parameter and initial data. Actually without these last condition little variations in the sources or in the initial data could produces a strong change in the solution and a such solution would not have good physical condition and would depend on a fine tuning of the matter parameter or on the initial data.Therefore, the Newtonian limit [28][29] and the stability problems are very important area of research and today there exist plentiful literature about them.

As we can see, we have two classes of $f(R)$ alternatives theories, one to use the second order formalism to obtain the field equation and the other to use the first order formalism. There are some literature about these connection between these two formalism [30] and there exist literature about the conformal connection between theses theories and general relativity [31, 32]. In this work we will study $f(R) = \beta R + \alpha h(R)$, and tray to show the connection between the solutions of both formalism in the first order approximation of the parameter α.

In the first secction we review the field equations in the first order formalism. In the second section we show that the solutions of the Palatini formalism satisfy the field equation of the second order formalism when we take the lineal approximation in the parameter α. In the third section we show, as an example, the relation between the newtonian limit of both formalism.

1 REVIEW OF THE FIELD EQUATIONS IN THE FIRST ORDER FORMALISM

We review in this section the structure of the theory as presented elsewhere in previous works [17, 23]. Let $\mathcal{M}$ be a manifold with metric g_{ab} and a torsion-free derivative operator ∇_a, both considered as independent variables. Consider a Lagrangian density $\mathcal{L} = f(R)\sqrt{-g}+\mathcal{L}_M$, where the matter Lagrangian $\mathcal{L}_M$ does not depend on the connection. Suppose we have a smooth one-parameter λ family of field configurations starting from given fields g^{ab}, ∇_a, and ψ (the matter fields), with appropriate boundary conditions. Let δg^{ab}, $\delta\Gamma^c_{ab}$, $\delta\psi$ be the corresponding variations of those fields, i.e., $\delta g^{ab} = (dg^{ab}_\lambda/d\lambda)\mid_{\lambda=0}$, etc. Then, if we vary with respect to the metric, the field equations are

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} = \alpha_M T_{ab}. \tag{3}$$

where $f'(R) = (df/dR)$, $(\delta S_M/\delta g^{ab}) \equiv -T_{ab}\sqrt{-g}$ and $\alpha_M = -8\pi$. The variation with respect to the connection, recalling that this is fixed at the boundary, gives

$$\nabla_c[\sqrt{-g}g^{ab}f'(R)] = 0. \tag{4}$$

Now, we choose $f(R)$ with $f'(R)$ derivable. Then, the last equation becomes

$$\nabla_c g_{ab} = b_c g_{ab}, \tag{5}$$

where

$$b_c = -[\ln f'(R)]_{,c}. \tag{6}$$

Thus, we have a Weyl conformal geometry with a Weyl field given by Eq. (4). From Eq. (3) we obtain

$$f'(R)R - 2f(R) = \alpha_M T, \tag{7}$$

which defines $R(T)$, and we suppose that the function $f(R)$ is such that $R(T)$ is derivable with respect to the variable T. Therefore, b_c is determined by T and its derivatives except in the case $f(R) = \omega R^2$, for which $Rf' - 2f \equiv 0$, so we must consistently have $T \equiv 0$. It is important to note that b_c has a solution only if T is differentiable in $\mathcal{M}$; this condition on T, for the existence of a solution, is not necessary in other theories such as general relativity (GR) or fourth-order theories. The connection solution to Eq. (3) is

$$\Gamma^a_{bc} = C^a_{bc} - \frac{1}{2}(\delta^a_b b_c + \delta^a_c b_b - g_{bc} b^a), \tag{8}$$

where $C^a{}_{bc}$ are the Christoffel symbols (metric connection). Then, we have to solve only Eq. (3). The Riemann tensor can be defined in the usual way, and then, the Ricci tensor and scalar curvature are

$$R_{ab} = R^m_{ab} - D_a b_b - \frac{1}{2} g_{ab} D \cdot b - \frac{1}{2} b_a b_b + \frac{1}{2} g_{ab} b^2 \tag{9}$$

$$R = R^m - 3D \cdot b + \frac{3}{2} b^2, \tag{10}$$

where R^m_{ab}, R^m, and D_c are the Ricci tensor, scalar curvature, and covariant derivative defined from the metric connection, respectively. Also we have take into consideration that $D_a b_b$ is a symmetric tensor due to eq.(4).

Because the matter action must be invariant under diffeomorphisms and the matter fields satisfy the matter field equations, T_{ab} is conserved [23]

$$D^a T_{ab} = 0. \tag{11}$$

Therefore, a test particle will follow the geodesics of the metric connection. When we consider a FRW spacetime and the source is a perfect fluid with

$$p = (\gamma - 1)\rho \ , \tag{12}$$

we find

$$\rho(t) = \frac{\rho_0}{a^{3\gamma}}(t) \ . \tag{13}$$

Using Eqs.(4) and(7) we have

$$b_c = -\frac{f'' \alpha_M \nabla_c T}{f'(Rf'' - f')}. \tag{14}$$

Except for the case of GR, where $f'' \equiv 0$, the Weyl field is nonzero wherever the trace of the energy-momentum tensor varies with respect to the coordinates. If T is constant, then R is also constant, $b_c = 0$ and (7) takes the form

$$G_{ab} - \frac{1}{2}\Lambda g_{ab} = K T_{ab}, \tag{15}$$

where Λ and K are two functions of R. All those cases with constant trace of the energy-momentum tensor are equivalent to GR for a given cosmological constant. This is the so-called universality of the Einstein equations for matter for which T is constant [17, 18]. In the case $T = 0$, the scalar R is any of the roots, R_i, of the equation $f'(R)R - 2f(R) = 0$. For each root the solutions of the field equations are the solutions of GR with cosmological constant $\Lambda = -R_i/4$. Therefore, the maximal symmetric vacuum solution of these theories is the de Sitter space-time.

2 Relation between both formalisms

We consider those lagrangian which are perturbations of the Einstein-Hilbert. This assumption is in a accordance with almost all models $f(R)$ which were consider in the pass in order to explain the cosmological acceleration or the dark mater behavior. We write this class of lagrangians in the following way:

$$f(R) = \beta R + \alpha h(R) + \Lambda \ , \tag{16}$$

where $h(R)$ has not linear terms. Therefore, the field equation are

$$(\beta + \alpha\, h'(R))R_{ab} - \frac{1}{2}(\beta\, R + \alpha\, h(R))g_{ab} = \alpha_M T_{ab} \ . \tag{17}$$

In equations (6,8) we have the metric part of the Ricci tensor, the metric part of the scalar tensor, and the parts which depend on the vector b_a. The vector b_a, In the case we are considering, is :

$$b_c = -(ln(\beta + \alpha\, h'))_{,c} \ . \tag{18}$$

In the first order of approximation, in the parameter α, the last equation can be written as

$$b_c = -\frac{\alpha}{\beta} h'_c + O(\alpha^2) \ . \tag{19}$$

Then, in this order of approximation, equation (6) is

$$R_{ab} = R^m_{ab} - D_a b_b - \frac{1}{2} g_{ab} D.b + O(\alpha^2) \ , \tag{20}$$

and the scalar curvature is $R = R^m - 3D.b + O(\alpha^2)$.

Therefore, the field equation is:

$$\beta\left(R^m_{ab} - \frac{1}{2}R^m g_{ab}\right) - \frac{1}{2}\Lambda g_{ab} - \alpha\left(\Box h' g_{ab} - D_a h'_{,b}\right) + \alpha(h' R^m_{ab} - \frac{1}{2}h(R)g_{ab}) = \alpha_M Tab \ . \tag{21}$$

In first order approximation

$$\alpha h(R) = \alpha h(R^m) \ , \tag{22}$$

and when h is analytical $\alpha h'(R) = \alpha h'(R^m)$

On the other hand, in the second order formalism, the field equations are obtained by varying the Lagrangian $\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_M$ only respect to the metric, and they are:

$$f' R_{ab} - \frac{1}{2} f g_{ab} - f'; ab + g_{ab}\Box f' = \alpha_M T_{ab} \ . \tag{23}$$

Using the above lagrangian (16) we obtain:

$$\beta\left(R_{ab} - \frac{1}{2}R g_{ab}\right) - \frac{1}{2}\Lambda g_{ab} - \alpha\left(\Box h' g_{ab} - D_a h'_{,b}\right) + \alpha(h' R_{ab} - \frac{1}{2}h(R)g_{ab}) = \alpha_M Tab \ . \tag{24}$$

But R in the second order formalism is R^m.Then, in the first order approximation in the parameter α and with h analytical, the Palatini $f(R)$ theory has the same field equation of the second order formalism.

Therefore, when h is analytical, the solutions of the Palatini $f(R)$ theories have a first order approximation in the parameter α which satisfies the second order formalism field equations. Then, the second order formalism solutions, which have first order approximations in the parameter α, must coincide in this order of approximation with the first order formalism solution.

3 Application to the Newtonian Limit

When $f(R)$ is analytical around R_0 and $f(R) = f_1 R + f_2 h(R)$ with $h(R) = R^2 + O(2)$, the newtonian point particle limit in second order formalism is [28]

$$g^1_{tt}/2 = \frac{1}{8\pi f_1}\Big(\frac{1 + (1/3)e^{-r\sqrt{f_1/6f_2}}}{r}\Big) \ , \tag{25}$$

where $f(R) = f_1 R + f_2 h(r)$, and we have put $f_0 = 0$ due to the zero order of the equations.

In the cases of first order formalism this limit is [17]

$$g^1_{tt}/2 = \frac{1}{8\pi f_1 r} + \frac{f_2}{4\pi r^2 f_1^2}\delta(r) \ . \tag{26}$$

This potential is lineal in the parameter f_2 because the parameter always appears along T in the first order formalism. Therefore, when we take the newtonian limit, first order in ρ, we also take the parameter first order.

The potential (15) is not lineal in f_2 but we can see what happens when we take the first order in the parameter

$$g^1_{tt}/2 = \frac{1}{8\pi f_1 r} + \frac{1}{8\pi r^2 f_1}(2f_2/f_1)\lim_{f_2\to 0}\left(\frac{re^{-r\sqrt{f_1/6f_2}}}{6f_2/f_1}\right) \ . \tag{27}$$

The last limit is well know:

$$\lim_{\omega\to 0}\frac{re^{-r/\omega}}{\omega^2} = \delta(r) \ . \tag{28}$$

Therefore, the result (27) is

$$g^1_{tt}/2 = -\frac{1}{8\pi f_1 r} + \frac{f_2}{4\pi r^2 f_1^2}\delta(r) \ . \tag{29}$$

4 Conclusions

In this work we have shown, for the lagrangian $f(R) = \beta R + \alpha h(R)$, the relations between the first order formalism- Palatini- solutions and the second order formalism solutions in the linear approximation of the parameter α.

We have also shown how this relations works in the particular case of the newtonian limit of both theories.

We think that this relations could be useful to study both theories because the parameter α, if it exists, it would be very small.

We expect, in the future, to study this kind of relations between both theories for singular lagrangian.

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Non-minimal curvature-matter couplings in modified gravity

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Abstract

Recently, in the context of $f(R)$ modified theories of gravity, it was shown that a curvature-matter coupling induces a non-vanishing covariant derivative of the energy-momentum, implying non-geodesic motion and, under appropriate conditions, leading to the appearance of an extra force. We study the implications of this proposal and discuss some directions for future research.

1 Introduction

Current experimental evidence indicates that gravitational physics is in agreement with Einstein's theory of General Relativity (GR) to considerable accuracy (for thorough discussions see [1]); however, quite fundamental questions suggest that it is unlikely that GR stands as the ultimate description of gravity. Actually, difficulties arise from various corners, most particularly in connection to the strong gravitational field regime and the existence of spacetime singularities. Quantization is a possible way to circumvent these problems, nevertheless, despite the success of gauge field theories in describing the electromagnetic, weak, and strong interactions, the description of gravity at the quantum level is still missing, despite outstanding progress achieved, for instance, in the context of superstring/M-theory.

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Furthermore, in fundamental theories that attempt to include gravity, new long-range forces often arise in addition to the Newtonian inverse-square law. Even if one assumes the validity of the Equivalence Principle, Einstein's theory does not provide the most general way to establish the spacetime metric. There are also important reasons to consider additional fields, especially scalar fields. Although the latter appear in unification theories, their inclusion predicts a non-Einsteinian behaviour of gravitating systems. These deviations from GR include violations of the Equivalence Principle, modification of large-scale gravitational phenomena, and variation of the fundamental couplings.

On large scales, recent cosmological observations lead one to conclude that our understanding of the origin and evolution of the Universe based on GR requires that most of the energy content of the Universe is in the form of currently unknown dark matter and dark energy components that may permeate much, if not all spacetime. Indeed, recent Cosmic Microwave Background Radiation (CMBR) data indicate that our Universe is well described, within the framework of GR, by a nearly flat Robertson-Walker metric. Moreover, combination of CMBR, supernovae, baryon acoustic oscillation and large scale structure data are consistent with each other only if, in the cosmic budget of energy, dark energy corresponds to about 73% of the critical density, while dark matter to about 23% and baryonic matter to only about 4%. Several models have been suggested to address issues related to these new dark states. For dark energy, one usually considers the so-called "quintessence" models, which involves the slow-roll down of a scalar field along a smooth potential, thus inducing the observed accelerated expansion (see [2] for a review). For dark matter, several weak-interacting particles (WIMPs) have been suggested, many arising from extensions to the Standard Model (*e.g.* axions, neutralinos). A scalar field can also account for an unified model of dark energy and dark matter [3]. Alternatively, one can implement this unification through an exotic equation of state, such as the generalized Chaplygin gas [4].

However, recently a different approach has attracted some attention, namely the one where one considers a generalization of the action functional. The most straight forward approach consists in replacing the linear scalar curvature term in the Einstein-Hilbert action by a function of the scalar curvature, $f(R)$. In this context, a renaissance of $f(R)$ modified theories of gravity has recently been verified in an attempt to explain the late-time accelerated expansion of the Universe (see for instance Refs. [5, 6] for recent reviews). One could alternatively, resort to other scalar invariants of the theory and necessarily analyze the observational signatures and the parameterized post-Newtonian (PPN) metric coefficients arising from these extensions of GR. In the context of dark matter, the possibility that the galactic dynamics of massive test particles may be understood without the need for dark matter was also considered in the framework of $f(R)$ gravity models [7]. Despite the extensive literature on these $f(R)$ models, an interesting possibility has passed unnoticed till quite recently. It includes not only a non-minimal scalar curvature term in the Einstein-Hilbert Lagrangian density, but also a non-minimal coupling between the scalar curvature and the matter Lagrangian density [8] (see also Ref. [9] for related discussions). It is interesting to note that nonlinear couplings of matter with gravity were analyzed in the context of the accelerated expansion of the Universe [10], and in the study of the cosmological constant problem [11]. In this contribution we discuss various aspects of this proposal.

This work is organized as follows: in the following Section, the main features of this novel model are presented. In Section III, the issue of the degeneracy of Lagrangian densities, actually a feature well known in GR [12, 13, 14], is addressed in the context of the new non-minimally coupled model [15]. In Section IV, the scalar-tensor representation of the model is presented, with particular emphasis on the new features and difficulties encountered in the new model. These issues are quite relevant, as they allow one to properly obtain the PPN parameters β and γ and show that they are consistent with the observations [16]. Section V, addresses the compatibility of the model with the astrophysical condition for stellar equilibrium [17]. In Section VI, a further generalization of the model is discussed and an upper bound on the extra acceleration introduced by the new non-minimal coupling is obtained [18]. Finally, in Section VI our conclusions are presented and objectives for further research are discussed.

Throughout this work, the convention $8\pi G = 1$ and the metric signature $(-,+,+,+)$ are used.

2 Linear curvature-matter couplings

The action for curvature-matter couplings, in $f(R)$ modified theories of gravity [8], takes the following form

$$S = \int \left[\frac{1}{2}f_1(R) + [1+\lambda f_2(R)]\,\mathcal{L}_m\right]\sqrt{-g}\;d^4x\,, \tag{1}$$

where $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the curvature scalar R and $\mathcal{L}_m$ is the Lagrangian density corresponding to matter and λ is a constant. Since the matter Lagrangian is not modified in the total action, these may be called modified gravity models with a non-minimal coupling between matter and geometry.

Varying the action with respect to the metric $g_{\mu\nu}$ yields the field equations, given by

$$\begin{aligned} F_1 R_{\mu\nu} - \frac{1}{2}f_1 g_{\mu\nu} - \nabla_\mu\nabla_\nu F_1 + g_{\mu\nu}\Box F_1 = (1+\lambda f_2)T_{\mu\nu} - 2\lambda F_2 \mathcal{L}_m R_{\mu\nu} + \\ 2\lambda(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\mathcal{L}_m F_2\,, \end{aligned} \tag{2}$$

where one denotes $F_i(R) = f_i'(R)$, and the prime denotes differentiation with respect to the scalar curvature. The matter energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(g^{\mu\nu})}\,. \tag{3}$$

Now, taking into account the generalized Bianchi identities, one deduces the following generalized covariant conservation equation

$$\nabla^\mu T_{\mu\nu} = \frac{\lambda F_2}{1+\lambda f_2}\left[g_{\mu\nu}\mathcal{L}_m - T_{\mu\nu}\right]\nabla^\mu R\,. \tag{4}$$

It is clear that the non-minimal coupling between curvature and matter yields a non-trivial exchange of energy and momentum between the geometry and matter fields [16].

Considering, for instance, the energy-momentum tensor for a perfect fluid,

$$T_{\mu\nu} = (\rho + p)\,U_\mu U_\nu + p g_{\mu\nu}\,, \tag{5}$$

where ρ is the energy density and p is the pressure, respectively. The four-velocity, U_μ, satisfies the conditions $U_\mu U^\mu = -1$ and $U^\mu U_{\mu;\nu} = 0$. Introducing the projection operator $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, one can show that the motion is non-geodesic, and governed by the following equation of motion for a fluid element

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = f^\mu\,, \tag{6}$$

where the extra force, f^μ, appears and is given by

$$f^\mu = \frac{1}{\rho+p}\left[\frac{\lambda F_2}{1+\lambda f_2}\left(\mathcal{L}_m - p\right)\nabla_\nu R + \nabla_\nu p\right] h^{\mu\nu}\,. \tag{7}$$

One verifies that the first term vanishes for the specific choice of $\mathcal{L}_m = p$, as noted in [19]. However, as pointed out in [15], this is not the unique choice for the Lagrangian density of a perfect fluid, as will be outlined below.

3 Perfect fluid Lagrangian description

The novel coupling in action (2.7) has attracted some attention and, in a recent paper [19], this possibility has been applied to distinct matter contents. Regarding the latter, it was argued that a "natural choice" for the matter Lagrangian density for perfect fluids is $\mathcal{L}_m = p$, based on [12, 13], where p is the pressure. This specific choice implies the vanishing of the extra force. However, although $\mathcal{L}_m = p$ does indeed reproduce the perfect fluid equation of state, it is not unique: other choices include, for instance, $\mathcal{L}_m = -\rho$ [13, 14], where ρ is the energy density, or $\mathcal{L}_m = -na$, where n is the particle number density, and a is the physical free energy defined as $a = \rho/n - Ts$, with T being the fluid temperature and s the entropy per particle.

In this section, following [13, 15], the Lagrangian formulation of a perfect fluid in the context of GR is reviewed. The action is presented in terms of Lagrange multipliers along the Lagrange coordinates α^A in order to enforce specific constraints, and is given by

$$S_m = \int d^4x\left[-\sqrt{-g}\,\rho(n,s) + J^\mu\left(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A\alpha^A_{,\mu}\right)\right]\,. \tag{8}$$

Note that the action $S_m = S(g_{\mu\nu}, J^\mu, \varphi, \theta, s, \alpha^A, \beta_A)$ is a functional of the spacetime metric $g_{\mu\nu}$, the entropy per particle s, the Lagrangian coordinates α^A, and spacetime scalars denoted by φ, θ, and β_A, where the index A takes the values 1, 2, 3 (see [13] for details).

The vector density J^μ is interpreted as the flux vector of the particle number density, and defined as $J^\mu = \sqrt{-g}\,nU^\mu$. The particle number density is given by $n = |J|/\sqrt{-g}$, so that the energy density is a function $\rho = \rho(|J|/\sqrt{-g}, s)$. The scalar field φ is interpreted as a potential for the chemical free energy f, and is a Lagrange multiplier for $J^\mu_{,\mu}$, the particle number conservation. The scalar fields β_A are interpreted as the Lagrange multipliers for $\alpha^A_{,\mu}J^\mu = 0$, restricting the fluid 4−velocity to be directed along the flow lines of constant α^A.

The variation of the action with respect to J^μ, φ, θ, s, α^A and β_A, provides the equations of motion, which are not written here (we refer the reader to Ref. [15] for details). Varying the action with respect to the metric, and using the definition given by Eq. (3), provides the stress-energy tensor for a perfect fluid

$$T^{\mu\nu} = \rho\, U^\mu U^\nu + \left(n\frac{\partial\rho}{\partial n} - \rho\right)\left(g^{\mu\nu} + U^\mu U^\nu\right)\,, \tag{9}$$

with the pressure defined as

$$p = n\frac{\partial\rho}{\partial n} - \rho\,. \tag{10}$$

This definition of pressure is in agreement with the First Law of Thermodynamics, $d\rho = \mu\,dn + nTds$. The latter shows that the equation of state can be specified by the energy density $\rho(n,s)$, written as a function of the number density and entropy per particle. The quantity $\mu = \partial\rho/\partial n = (\rho + p)/n$ is defined as the chemical potential, which is the energy gained by the system per particle injected into the fluid, maintaining a constant sample volume and entropy per particle s.

Taking into account the equations of motions and the definitions $J^\mu = \sqrt{-g}\, nU^\mu$ and $\mu = (\rho+p)/n$, the action Eq. (8) reduces to the on-shell Lagrangian density $\mathcal{L}_{m(1)} = p$, with the action given by [15]

$$S_m = \int d^4x\sqrt{-g}\, p\,, \tag{11}$$

which is the form considered in Ref. [12]. It was a Lagrangian density given by $\mathcal{L}_m = p$ that the authors of [19] use to obtain a vanishing extra-force due to the non-trivial coupling of matter to the scalar curvature R. For concreteness, replacing $\mathcal{L}_m = p$ in Eq. (7), one arrives at the general relativistic expression

$$f^\mu = \frac{h^{\mu\nu}\nabla_\nu p}{\rho + p}\,. \tag{12}$$

However, an on-shell degeneracy of the Lagrangian densities arises from adding up surface integrals to the action. For instance, consider the following surface integrals added to the action Eq. (8),

$$-\int d^4x(\varphi J^\mu)_{,\mu}\,, \quad -\int d^4x(\theta s J^\mu)_{,\mu}\,, \quad -\int d^4x(J^\mu\beta_A\alpha^A)_{,\mu}\,,$$

so that the resulting action takes the form

$$S = \int d^4x\Big[-\sqrt{-g}\,\rho(n,s) - \varphi J^\mu_{,\mu} - \theta(sJ^\mu)_{,\mu} - \alpha^A(\beta_A J^\mu)_{,\mu}\Big]\,. \tag{13}$$

This action reproduces the equations of motion, and taking into account the latter, the action reduces to [15]

$$S_m = -\int d^4x\sqrt{-g}\,\rho\,, \tag{14}$$

i.e., the on-shell matter Lagrangian density takes the following form $\mathcal{L}_m = -\rho$. This choice is also considered for isentropic fluids, where the entropy per particle is constant $s = \text{const.}$ [13, 14]. For the latter, the First Law of Thermodynamics indicates that isentropic fluids are described by an equation of state of the form $a(n,T) = \rho(n)/n - sT$ [13] (see Ref. [20] for a bulk-brane discussion of this choice).

For this specific choice of $\mathcal{L}_{m(2)} = -\rho$ the extra force takes the following form:

$$f^\mu = \left(-\frac{\lambda F_2}{1+\lambda f_2}\nabla_\nu R + \frac{1}{\rho+p}\nabla_\nu p\right) h^{\mu\nu}\,. \tag{15}$$

An interesting feature of Eq. (15) is that the term related to the specific curvature-matter coupling is independent of the energy-matter distribution.

The above discussion confirms that if one adopts a particular on-shell Lagrangian density as a suitable functional for describing a perfect fluid, then this leads to the issue of distinguishing between different predictions for the extra force. It is therefore clear that no straightforward conclusion may be extracted regarding the additional force imposed by the non-minimal coupling of curvature to matter, given the different available choices for the Lagrangian density. One could even doubt the validity of a conclusion that allows for different physical predictions arising from these apparently equivalent Lagrangian densities.

Despite the fact that the above Lagrangian densities $\mathcal{L}_{m(i)}$ are indeed obtainable from the original action, it turns out that they are not equivalent to the original Lagrangian density $\mathcal{L}_m$. Indeed, this equivalence demands that not only the equations of motion of the fields describing the perfect fluid remain invariant, but also that the gravitational field equations do not change. Indeed, the guiding principle behind the proposal first put forward in Ref. [8] is to allow for a non-minimal coupling between curvature and matter.

The modification of the perfect fluid action Eq. (8) should only affect the terms that show a minimal coupling between curvature and matter, *i.e.*, those multiplied by $\sqrt{-g}$ [15]. Thus, the current density term, which is not coupled to curvature, should not be altered. Writing $\mathcal{L}_c = -\rho(n,s)$, $V_\mu \equiv \varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu}$, for simplicity, the modified action reads

$$S'_m = \int d^4x \left[\sqrt{-g}\left[1+\lambda f_2(R)\right]\mathcal{L}_c + J^\mu V_\mu + B^\mu_{;\mu}\right] , \tag{16}$$

and one can see that only the non-minimal coupled term $\mathcal{L}_c$ appears in the field equations, as variations with respect to $g^{\mu\nu}$ of the remaining terms vanish:

$$\begin{aligned} F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \nabla_\mu \nabla_\nu F_1 + g_{\mu\nu} \Box F_1 = (1+\lambda f_2) T_{\mu\nu} - 2\lambda F_2 \mathcal{L}_c R_{\mu\nu} \\ +2\lambda(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)\mathcal{L}_c F_2 \,. \end{aligned} \tag{17}$$

Thus, quite logically, one finds that different predictions for non-geodesic motion are due to different forms of the gravitational field equations. Therefore, the equivalence between different on-shell Lagrangian densities $\mathcal{L}_{m(i)}$ and the original quantity $\mathcal{L}_m$ is broken, so that one can no longer freely choose between the available forms. For the same reason, the additional extra force is unique, and obtained by replacing $\mathcal{L}_c = -\rho$ into Eq. (7), yielding expression (15).

Indeed, in a recent paper [29], a generalization of the above approach is considered, by using a systematic method that is not tied up to a specific choice of matter Lagrangians. In particular, the propagation equations for pole-dipole particles for a gravity theory with a very general coupling between the curvature scalar and the matter fields is examined, and it is shown that, in general, the extra-force does not vanish.

4 Scalar-tensor representation

The connection between $f(R)$ theories of gravity and scalar-tensor models with a "physical" metric coupled to the scalar field is well known. In this section, one pursues the equivalence between the model described by Eq. (2.7) and an adequate scalar-tensor theory. In close analogy with the equivalence of standard $f(R)$ models [21], this equivalence allows for the calculation of the PPN parameters β and γ [22].

One may first approach this equivalence by introducing two auxiliary scalars ψ and ϕ [19], and considering the following action

$$S_1 = \int \left[\frac{1}{2} f_1(\phi) + \left[1+\lambda f_2(\phi)\right]\mathcal{L}_m + \psi(R-\phi)\right]\sqrt{-g}\, d^4x \,. \tag{18}$$

Now, varying the action with respect to ψ gives $\phi = R$ and, consequently, action (2.7) is recovered. Varying the action with respect to ϕ, yields

$$\psi = \frac{1}{2} F_1 + \lambda F_2 \mathcal{L}_m \,. \tag{19}$$

Substituting this relationship back in (18), and assuming that at least one of the functions f_i is nonlinear in R, one arrives at the following modified action

$$S_1 = \int \left[\frac{f_1(\phi)}{2} + [1 + \lambda f_2(\phi)] \mathcal{L}_m + \left[\frac{1}{2} F_1(\phi) + \lambda F_2(\phi) \mathcal{L}_m \right] (R - \phi) \right] \sqrt{-g}\, d^4x \,, \tag{20}$$

where one still verifies the presence of the curvature-matter coupling. Note that this is not an ordinary scalar-tensor theory, due to the presence of the third and last terms. The former represents a scalar-matter coupling, and the latter a novel scalar-curvature-matter coupling. One may also use alternative field definitions to cast the action (18) into a Bran-Dicke theory with $\omega = 0$, *i.e.* no kinetic energy term for the scalar field, but with the addition of a R-matter coupling [19]. In conclusion, despite the fact that the introduction of the scalar fields helps in avoiding the presence of the nonlinear functions of R, the curvature-matter couplings are still present and, consequently, these actions cannot be cast into the form of a familiar scalar-tensor gravity [19].

However, one may instead pursue an equivalence with a theory with not just one, but two scalar fields [16]. This is physically well motivated, since the non-minimal coupling of matter and geometry embodied in Eq. (2.7) gives rise to an extra degree of freedom (notice that the case of a minimal coupling $f_2 = 0$ yields $\psi = F_1(\phi)/2$, so that this degree of freedom is lost). Indeed, action of Eq. (18) may be rewritten as a Jordan-Brans-Dicke theory with a suitable potential,

$$S_1 = \int \left[\psi R - V(\phi, \psi) + [1 + \lambda f_2(\phi)] \mathcal{L}_m \right] \sqrt{-g}\, d^4x \,, \tag{21}$$

with $V(\phi, \psi) = \phi\psi - f_1(\phi)/2$.

Variation of this action yields the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \frac{1 + \lambda f_2(\phi)}{\psi} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \frac{V(\phi, \psi)}{\psi} + \frac{1}{\psi} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) \psi \tag{22}$$

which, after the substitutions $\phi = R$ and $\psi = F_1/2 + \lambda F_2 \mathcal{L}_m$, collapses back to Eqs. (2). Likewise, the Bianchi identities yield the generalized covariant conservation equation

$$\nabla^\mu T_{\mu\nu} = \frac{1}{1 + \lambda f_2} \left[(\phi - R) \nabla_\nu \psi + \left[\left(\psi - \frac{1}{2} F_1 \right) g_{\mu\nu} - \lambda F_2 T_{\mu\nu} \right] \nabla^\mu \phi \right], \tag{23}$$

also equivalent to Eq. (4).

Through a conformal transformation $g_{\mu\nu} \rightarrow g^*_{\mu\nu} = \psi g_{\mu\nu}$ (see *e.g.* [23]), the scalar curvature can decouple from the scalar fields, so that the action is written in the so-called Einstein frame). A further redefinition of the scalar fields,

$$\varphi^1 = \frac{\sqrt{3}}{2} \log \psi \quad , \quad \varphi^2 = \phi \,, \tag{24}$$

allows the theory to be written canonically, that is,

$$S_1 = \int \left[R^* - 2 g^{*\mu\nu} \sigma_{ij} \varphi^i_{,\mu} \varphi^j_{,\nu} - 4U(\varphi^1, \varphi^2) + \left[1 + \lambda f_2(\varphi^2) \right] \mathcal{L}^*_m \right] \sqrt{-g^*}\, d^4x \,, \tag{25}$$

with $\mathcal{L}^*_m = \mathcal{L}_m/\psi^2$, the redefined potential

$$U(\varphi^1, \varphi^2) = \frac{1}{4} \exp\left(-\frac{2\sqrt{3}}{3} \varphi^1 \right) \left[\varphi^2 - \frac{1}{2} f_1(\varphi^2) \exp\left(-\frac{2\sqrt{3}}{3} \varphi^1 \right) \right], \tag{26}$$

and the metric in the field space (φ^1, φ^2),

$$\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{27}$$

which, after a suitable addition of an anti-symmetric part, will be used to raise and lower Latin indexes.

Variation of action Eq. (25) with respect to the metric $g^*_{\mu\nu}$ yields the field equations

$$R^*_{\mu\nu} - \frac{1}{2} g^*_{\mu\nu} R^* = 8\pi G \left(1 + \lambda f_2\right) T^*_{\mu\nu} + \sigma_{ij} \left(2\varphi^i_{,\mu} \varphi^j_{,\nu} - g^*_{\mu\nu} g^{*\alpha\beta} \varphi^i_{,\alpha} \varphi^j_{,\beta} \right) - 2 g^*_{\mu\nu} U \, , \tag{28}$$

while variation with respect to φ^i gives the Euler-Lagrange equations for each field:

$$\Box^* \varphi^i = B^i + 4\pi G \left[\alpha^i \left(1 + \lambda f_2\right) T^* - \lambda \sigma^{i2} F_2 \mathcal{L}^* \right] \tag{29}$$

where one defines $B_i = \partial U / \partial \varphi^i$ and

$$\alpha_i = -\frac{1}{2} \frac{\partial \log \psi}{\partial \varphi^i} \quad \rightarrow \quad \alpha_1 = -\frac{\sqrt{3}}{3} \quad , \quad \alpha_2 = 0 \, , \tag{30}$$

Eqs. (28), together with the Bianchi identities, result in the generalized conservation law

$$\nabla^{*\mu} T^*_{\mu\nu} = \frac{\sqrt{3}}{3} T^* \nabla^*_\nu \varphi^1 + \frac{\lambda F_2}{1 + \lambda f_2} \left(g^*_{\mu\nu} \mathcal{L}^* - T^*_{\mu\nu} \right) \nabla^{*\mu} \varphi^2 \, . \tag{31}$$

From current bounds on the Equivalence Principle, it is reasonable to assume that the effect of the non-minimum coupling of curvature to matter is weak, $\lambda f_2 \ll 1$. Substituting this into (31) one gets, at zeroth-order in λ,

$$\nabla^{*\mu} T^*_{\mu\nu} \simeq -\alpha_j T^* \varphi^j_{,\nu} \, , \tag{32}$$

so that one may disregard the $f_2(\varphi^2)$ factor in the action (25) and consider only through the coupling present in T^* (stemming from the definition of $\mathcal{L}^*_m$) and the derivative of φ^1 (since $\varphi^1 \propto \log \psi$ and $\psi = F_1 + F_2 \mathcal{L}$).

If both scalar fields are light, leading to long range interactions, one may calculate the PPN parameters β and γ [22], given by

$$\beta - 1 = \frac{1}{2} \left[\frac{\alpha^i \alpha^j \alpha_{j,i}}{\left(1 + \alpha^2\right)^2} \right]_0 \quad , \quad \gamma - 1 = -2 \left[\frac{\alpha^2}{1 + \alpha^2} \right]_0 \, , \tag{33}$$

where $\alpha_{j,i} = \partial \alpha_j / \partial \varphi^i$ and $\alpha^2 = \alpha_i \alpha^i = \sigma^{ij} \alpha_i \alpha_j$; the subscript $_0$ refers to the asymptotic value of the related quantities, which is connected to the cosmological values of the curvature and matter Lagrangian density. From the values found in Eq. (30), one concludes that $\beta = \gamma = 1$, as obtained in GR. However, it should be expected that small deviations of order $O(\lambda)$ arise when one considers the full impact of Eq. (31).

Furthermore, it should be empathized that the added degree of freedom embodied in the non-minimal $f_2 \neq 0$ coupling is paramount in obtaining values for the PPN parameters β and γ within the current experimental bounds (or, conversely, allowing for future constraints of the magnitude of λ and the form of f_2); indeed, in the case where only the curvature term is non-trivial, $f_1 \neq R$ and $f_2 = 0$, one degree of freedom is lost and the parameter $\alpha \neq 0$ defined in Eq. (30) is no longer a vector, but a scalar quantity: as a result, $\alpha^2 \neq 0$ and one gets $\gamma = 1/2$. In the discussed model, the vector α_i has $\alpha^2 = 0$, thus solving this pathology (see [16] and references therein for a thorough discussion).

Finally, notice that these results should be independent of the particular scheme chosen for the equivalence between the original model and a scalar-tensor theory; this may be clearly seen by opting for a more "natural" choice for the two scalar fields (in the Jordan frame), such that $\phi = R$ and $\psi = \mathcal{L}$. Although more physically motivated, this choice of fields is less pedagogical and mathematically more taxing [16].

5 Implications for stellar equilibrium

In this section, one studies the impact of the non-minimally coupled gravity model embodied in action Eq. (2.7) in what may be viewed as its natural proving ground: regions where curvature effects may be high enough, to evidence some deviation from GR, although moderate enough so these are still perturbative – a star [17] (see also [24] for other physical examples of the adopted methodology). As will be shown, the purpose of this exercise is to calculate deviations to the central temperature of the Sun (known with an accuracy of 6%), due to the perturbative effect of the non-minimal coupling of geometry to matter.

Clearly, a full treatment of the equations of motion (2) is exceedingly demanding, unless a specific form for $f_1(R)$ and $f_2(R)$ is considered. Furthermore, since one is mainly interested in the ascertaining the effects of the non-minimal coupling within a high curvature and pressure medium, the modifications due to the pure curvature term f_1 should be overwhelmed by the effect of f_2; under such circumstances, one may discard the former term, as thus take the trivial $f_1 = R$ case. A thorough discussion on the validity of this approximation with regard to representative, physically viable candidates for the function $f_1(R)$ is found in Ref. [17].

One now deals with the particular form of the coupling function f_2. One considers the simplest form, which might arise from the first order expansion of a more general function in the weak field environ of the Sun, $f_2 = R$ (this implies that $[\lambda] = M^{-2}$). Also, one assumes that stellar matter is described by an ideal fluid characterized by a Lagrangian density $\mathcal{L}_m = p$, [12, 13]. Adopting $f_1 = f_2 = R$, the field equations become

$$(1 + 2\lambda p)\, R_{\mu\nu} - \frac{1}{2} R\,(g_{\mu\nu} + 2\lambda T_{\mu\nu}) = 2\lambda(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)p + \frac{1}{2} T_{\mu\nu} \quad , \tag{34}$$

Notice that both λp and $\lambda\rho$ are dimensionless quantities: the perturbative condition $\lambda f_2 \ll 1$ translates to $\lambda p \ll 1$ and $\lambda\rho \ll 1$.

Taking the trace of the above equation yields

$$R = \frac{3p - \rho + 6\lambda\Box p}{2\,[1 + \lambda(\rho - 5p)]} \quad , \tag{35}$$

inserting $T = T^\mu_\mu = \rho - 3p$. Substituting this into Eq. (34) and keeping only first order terms in λ, one obtains

$$2[1 + \lambda(\rho - 3p)]R_{\mu\nu} = (3p - \rho)g_{\mu\nu} + 2(1 - 2\lambda p)T_{\mu\nu} + 2\lambda(4\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)p \quad , \tag{36}$$

Since temporal variations are assumed to occur at the cosmological scale H_0^{-1}, and are thus negligible at an astrophysical time scale, one considers an ideal, spherically symmetric system, with a line element derived from the Birkhoff metric (in its anisotropic form)

$$ds^2 = e^{\nu(r)} dt^2 - \left(e^{\sigma(r)} dr^2 + d\Omega^2 \right) \quad , \tag{37}$$

with $d\Omega = r^2(d\theta^2 + sin^2\theta\ d\phi^2)$. Following the usual treatment, one defines the effective mass m_e through $e^{-\sigma} = 1 - 2Gm_e/r$ which, replacing in Eq. (36) yields, to first order in λ,

$$m_e' \approx 4\pi r^2 \rho \left[1 + 2\lambda \left(p - \frac{\rho}{2} - \frac{3}{2}\frac{p^2}{\rho}\right)\right] + \frac{\lambda r^2}{4G}\left(5e^{-\nu}\nabla_0\nabla_0 + 3e^{-\sigma}\nabla_r\nabla_r + 2\frac{\nabla_\theta\nabla_\theta}{r^2}\right)p \ , \quad (38)$$

which clearly shows the perturbation to the gravitational mass, defined by $m_g' = 4\pi r^2\rho$ (in here, the prime denotes differentiation with respect to r).

Taking the Newtonian limit

$$r \gg 2Gm_e(r) \ , \ \rho(r) \gg p(r) \ , \ m_e(r) \gg 4\pi p(r) r^3 \ , \quad (39)$$

and going through a few algebraic steps (depicted in [17]), one eventually obtains the non-relativistic hydrostatic equilibrium equation

$$p' + \frac{Gm_e\rho}{r^2} = 2\lambda \left[\left(\left[\frac{5}{8}p'' - 4\pi G p\rho\right] r - \frac{p'}{4}\right)\rho + p\rho'\right] \ . \quad (40)$$

where the perturbation introduced by the non-minimal coupling is clearly visible.

In order to scrutinize the profile of pressure and density inside the Sun, one requires a suitable equation of state. Instead of pursuing a realistic representation of the various layers of the solar structure, one resorts to a very simplistic assumption, the so-called polytropic equation of state. This is commonly given by $p = K\rho^{(n+1)/n}$, where K is the polytropic constant, ρ is the mass density and n is the polytropic index. A polytropic equation of state with $n = 3$ was used by Eddington in his first solar model, and will be adopted here due.

Given this equation of state, one may write $\rho = \rho_c\theta^n(\xi)$ and $p = p_c\theta^{n+1}(\xi)$, with $\xi = r/r_0$ a dimensionless variable and $r_0^2 \equiv (n+1)p_c/4\pi G\rho_c^2$; $\rho_c = 1.622 \times 10^5\ \mathrm{kg/m}^3$ is the central density, and $p_c = 2.48 \times 10^{16}$ Pa is the central pressure. One obtains the perturbed Lane-Emden equation for the function $\theta(\xi)$:

$$\frac{1}{\xi^2}\left[\xi^2\theta'\left(1 + A_c\theta^n \times \left[\left[\frac{5}{8}\left(\theta'' + n\frac{\theta'^2}{\theta}\right) - N_c\theta^{n+1}\right]\frac{\xi}{\theta'} + \frac{3n-1}{4(n+1)}\right]\right)\right]' = -\theta^n\left[1 + A_c\left(\frac{3}{8}\left[\theta'' + n\frac{\theta'^2}{\theta}\right] + \frac{\theta'}{4\xi} - \frac{\theta^n}{2}\right)\right] \ , \quad (41)$$

where the prime now denotes derivation with respect to the dimensionless radial coordinate ξ, and one defines the dimensionless parameters $A_c \equiv \lambda\rho_c$ and $N_c \equiv p_c/\rho_c = 1.7 \times 10^{-6}$, for convenience. Clearly, setting $A_c = 0$ one recovers the unperturbed Lane-Emden equation [25].

Notice that the perturbed Lane-Emden equation is a third-degree differential equation; its numerical resolution is computationally intensive and displays some complex behaviour; conveniently, the assumed perturbative regime prompts for the expansion of the function $\theta(\xi) = \theta_0(1 + A_c\delta)$ around the unperturbed solution $\theta_0(\xi)$. Inserting this into Eq. (41) and expanding to first-order in A_c, one obtains

$$\delta'' + 2\left(\frac{\theta_0'}{\theta_0} + \frac{1}{\xi}\right)\delta' + (n-1)\theta_0^{n-1}\delta = \frac{5n}{2}\,\xi\,\theta_0^{2n-2}\theta_0' + (2n+1)N_c\xi\theta_0^{2n-1}\theta_0' + \frac{9n+5}{4(n+1)}\theta_0^{2n-1} + 3N_c\theta_0^{2n} - \frac{5n(n-1)}{8}\xi\theta_0^{n-3}\theta_0'^3 + \frac{n(3n+7)}{4(n+1)}\theta_0^{n-2}\theta_0'^2 + \frac{1}{2}\frac{\theta_0^{n-1}\theta_0'}{\xi} \ . \quad (42)$$

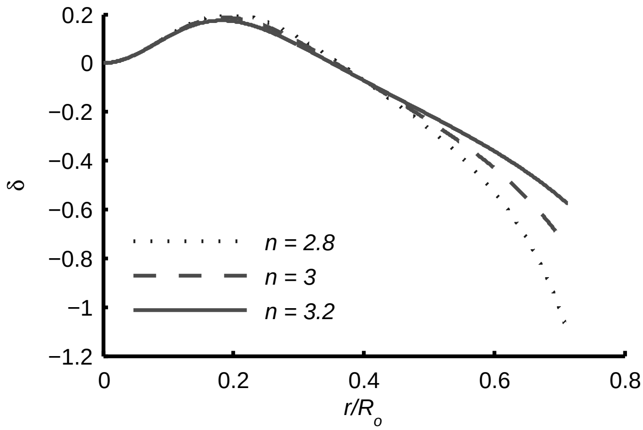

Figure 1: Relative perturbation δ for $2.8 \leq n \leq 3.2$.

supplemented by the initial conditions $\delta(0) = \delta'(0) = 0$. Notice that the choice for the perturbative expansion leads to a solution δ independent from the parameter A_c.

After dealing with the issue of exterior matching conditions and bypassing a troublesome divergence of δ near the boundary of the star [17], one may obtain the numerical solution for Eq. (42) for a polytropic index in the vicinity of $n = 3$, as depicted in Fig. 1.

Finally, one turns to the issue of calculating one of the observables under scrutiny, that is, the central temperature of the Sun. The polytropic equation of state indicates that $\rho \propto T^{n+1}$, which yields

$$1 - \left(\frac{T_{c0}}{T_c}\right)^{n+1} = \frac{A_c}{\xi_r^2 \theta'_{0r}} \int_0^{\xi_r} \xi^2 \theta_0^n \left[n\delta + \frac{3n}{8}\frac{\theta_0'^2}{\theta_0} - \frac{\theta_0'}{2\xi} - \frac{7}{8}\theta_0^n \right] d\xi \ . \tag{43}$$

where $\xi_r = R_r/r_0$ and $R_r = 0.713 R_\odot$ marks the onset of the convection zone (where the chosen equation of state fails) and T_{c0} is the central temperature derived from the $A_c = 0$ unperturbed scenario. One may derive a parameter plot in the (n, A_c) parameter space, shown in Fig. 2. As can be seen, no relative deviation of the central temperature occurs above the experimentally determined level of 6%. However, since the values found are of the order of 1%, one may hope that any future refinement of the experimental error of T_c could yield a direct bound on the parameter A_c. Furthermore, the perturbative condition $\lambda \ll \kappa\rho_c$ is confirmed (reintroducing the factor κ, for clarity), which translates to $|\lambda| \ll 4.24 \times 10^{33}$ eV^{-2}.

6 Models with arbitrary couplings between matter and geometry

The discussed gravity models with linear coupling between matter and geometry, given by Eq. (2.7), can be further generalized by assuming that the supplementary coupling between matter and geometry takes place via an arbitrary function of the matter Lagrangian $\mathcal{L}_m$, so that the action is given by [18]

$$S = \int \left\{ \frac{1}{2} f_1(R) + \mathcal{G}\left(\mathcal{L}_m\right)\left[1 + \lambda f_2\left(R\right)\right] \right\} \sqrt{-g} d^4x, \tag{44}$$

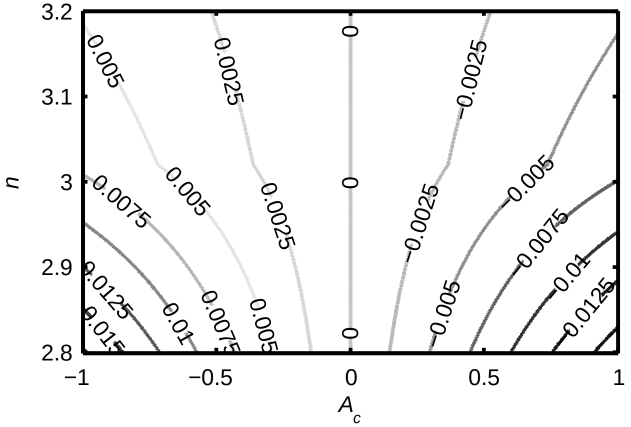

Figure 2: Relative deviation of the central temperature $T_c/T_{c0} - 1$, with contour lines of step 0.1%.

where $\mathcal{G}(\mathcal{L}_m)$ is an arbitrary function of the matter Lagrangian density $\mathcal{L}_m$. The action given by Eq. (44) represents the most general extension of the Einstein-Hilbert action for GR, $S = \int [R/2 + \mathcal{L}_m] \sqrt{-g} d^4x$. For $f_1(R) = R$, $f_2(R) = 0$ and $\mathcal{G}(\mathcal{L}_m) = \mathcal{L}_m$, one recovers GR. With $f_2(R) = 0$ and $\mathcal{G}(\mathcal{L}_m) = \mathcal{L}_m$ one obtains the $f(R)$ generalized gravity models. The case $\mathcal{G}(\mathcal{L}_m) = \mathcal{L}_m$ corresponds to the linear coupling between matter and geometry, given by Eq. (2.7). The only requirement for f_i, $i = 1, 2$ and $\mathcal{G}$ is that they are analytical functions of the Ricci scalar R and $\mathcal{L}_m$, respectively – that is, they can be expressed as a Taylor series expansion about any point.

The field equations corresponding to action (44) are

$$F_1(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) F_1(R) = -2\lambda\mathcal{G}(\mathcal{L}_m) F_2(R)R_{\mu\nu}$$
$$- 2\lambda (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) \mathcal{G}(\mathcal{L}_m) F_2(R)$$
$$- [1 + \lambda f_2(R)] [K(\mathcal{L}_m) \mathcal{L}_m - \mathcal{G}(\mathcal{L}_m)] g_{\mu\nu} - [1 + \lambda f_2(R)] K(\mathcal{L}_m) T_{\mu\nu}, \tag{45}$$

where $F_i(R) = df_i(R)/dR$, $i = 1, 2$, and $K(\mathcal{L}_m) = d\mathcal{G}(\mathcal{L}_m)/d\mathcal{L}_m$, respectively.

By taking the covariant divergence of Eq. (45), with the use of the mathematical identity $\nabla^\mu [a'(R)R_{\mu\nu} - a(R)g_{\mu\nu}/2 + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) a(R)] = 0$ [26], where $a(R)$ is an arbitrary function of the Ricci scalar and $a'(R) = da/dR$, we obtain

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu \ln \{[1 + \lambda f_2(R)] K(\mathcal{L}_m)\} (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) =$$
$$2\nabla^\mu \ln \{[1 + \lambda f_2(R)] K(\mathcal{L}_m)\} \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}. \tag{46}$$

For $\mathcal{G}(\mathcal{L}_m) = \mathcal{L}_m$, one recovers the equation of motion of massive test particles in the linear theory, Eq. (7). As a specific model of generalized gravity models with arbitrary matter-geometry coupling, one considers the case in which the matter Lagrangian density is an arbitrary function of the energy density of the matter ρ only, so that $\mathcal{L}_m = \mathcal{L}_m(\rho)$. One assumes that during the hydrodynamic evolution the energy density current is conserved, $\nabla_\nu(\rho U^\nu) = 0$. Then, the energy-momentum tensor of matter is given by

$$T^{\mu\nu} = \rho \frac{d\mathcal{L}_m}{d\rho} U^\mu U^\nu + \left(\mathcal{L}_m - \rho \frac{d\mathcal{L}_m}{d\rho}\right) g^{\mu\nu}, \tag{47}$$

where we have used the relation $\delta\rho = (1/2)\,\rho\,(g_{\mu\nu} - U_\mu U_\nu)\,\delta g^{\mu\nu}$, a direct consequence of the conservation of the energy density current.

The energy-momentum tensor given by Eq. (47) can be written in a form similar to the perfect fluid case if one assumes that the thermodynamic pressure p obeys a barotropic equation of state, $p = p(\rho)$. In this case the matter Lagrangian density and the energy-momentum tensor can be written as

$$\mathcal{L}_m(\rho) = \rho\,[1 + \Pi(\rho)] = \rho\left(1 + \int_0^p \frac{dp}{\rho}\right) - p(\rho), \tag{48}$$

and

$$T^{\mu\nu} = \{\rho\,[1 + \Pi(\rho)] + p(\rho)\}\,U^\mu U^\nu + p(\rho)\,g^{\mu\nu}, \tag{49}$$

respectively, where

$$\Pi(\rho) = \int_0^p \frac{dp}{\rho} - \frac{p(\rho)}{\rho}. \tag{50}$$

Physically, $\Pi(\rho)$ can be interpreted as the elastic (deformation) potential energy of the body, and therefore Eq. (49) corresponds to the energy-momentum tensor of a compressible elastic isotropic system. From Eq. (46), one obtains the equation of motion of a test particle in the modified gravity model with the matter Lagrangian an arbitrary function of the energy density of matter as Eq. (6), where the extra force is now given by

$$f^\mu = \nabla_\nu \ln\left\{[1 + \lambda f_2(R)]\,K\,[\mathcal{L}_m(\rho)]\,\frac{d\mathcal{L}_m(\rho)}{d\rho}\right\} h^{\mu\nu}. \tag{51}$$

It is easy to see that the extra-force f^μ, generated due to the presence of the coupling between matter and geometry, is perpendicular to the four-velocity, $f^\mu U_\mu = 0$. The equation of motion, Eq. (6), can be obtained from the variational principle

$$\delta S_p = \delta\int \mathcal{L}_p ds = \delta\int \sqrt{Q}\sqrt{g_{\mu\nu}U^\mu U^\nu}ds = 0, \tag{52}$$

where S_p and $\mathcal{L}_p = \sqrt{Q}\sqrt{g_{\mu\nu}U^\mu U^\nu}$ are the action and Lagrangian density, respectively, and

$$\sqrt{Q} = [1 + \lambda f_2(R)]\,K\,[\mathcal{L}_m(\rho)]\,\frac{d\mathcal{L}_m(\rho)}{d\rho}. \tag{53}$$

The variational principle Eq. (52) can be used to study the Newtonian limit of the model. In the weak gravitational field limit, $ds \approx \sqrt{1 + 2\phi - \vec{v}^2}dt \approx \left(1 + \phi - \vec{v}^2/2\right)dt$, where ϕ is the Newtonian potential and $\vec{v}$ is the usual tridimensional velocity of the particle. By representing the function $\sqrt{Q}$ as

$$\sqrt{Q} = [1 + \lambda f_2(R)]\,K\,[\mathcal{L}_m(\rho)]\,\frac{d\mathcal{L}_m(\rho)}{d\rho} = 1 + \Phi\left(R, \mathcal{L}_m(\rho), \frac{d\mathcal{L}_m(\rho)}{d\rho}\right), \tag{54}$$

where $|\Phi| << 1$, the equation of motion of a test particle can be obtained from the variational principle

$$\delta\int\left[\Phi\left(R, \mathcal{L}_m(\rho), \frac{d\mathcal{L}_m(\rho)}{d\rho}\right) + \phi - \frac{\vec{v}^2}{2}\right]dt = 0, \tag{55}$$

and is given by

$$\vec{a} = -\nabla\phi - \nabla\Phi = \vec{a}_N + \vec{a}_E, \tag{56}$$

where $\vec{a}_N = -\nabla\phi$ is the usual Newtonian gravitational acceleration and $\vec{a}_E = -\nabla\Phi$ a supplementary effect induced by the coupling between matter and geometry.

An estimative of the effect of the extra-force generated by the coupling between matter and geometry on the orbital parameters of planetary motion around the Sun can be obtained by using the properties of the Runge-Lenz vector, defined as $\vec{A} = \vec{v} \times \vec{L} - \alpha\vec{e}_r$, where $\vec{v}$ is the velocity relative to the Sun, with mass $M_\odot$, of a planet of mass m, $\vec{r} = r\vec{e}_r$ is the two-body position vector, $\vec{p} = \mu\vec{v}$ is the relative momentum, $\mu = mM_\odot/(m + M_\odot)$ is the reduced mass, $\vec{L} = \vec{r} \times \vec{p} = \mu r^2\dot{\theta}\vec{k}$ is the angular momentum, and $\alpha = GmM_\odot$ [27]. For an elliptical orbit of eccentricity e, major semi-axis a, and period T, the equation of the orbit is given by $\left(L^2/\mu\alpha\right) r^{-1} = 1 + e\cos\theta$. The Runge-Lenz vector and its derivative can be expressed as

$$\vec{A} = \left(\frac{\vec{L}^2}{\mu r} - \alpha\right)\vec{e}_r - \dot{r}L\vec{e}_\theta, \tag{57}$$

and

$$\frac{d\vec{A}}{d\theta} = r^2\left[\frac{dV(r)}{dr} - \frac{\alpha}{r^2}\right]\vec{e}_\theta, \tag{58}$$

respectively, where $V(r)$ is the potential of the central force [27]. The potential term consists of the Post-Newtonian potential,

$$V_{PN}(r) = -\frac{\alpha}{r} - \frac{3\alpha^2}{mr^2}, \tag{59}$$

plus the contribution from the general coupling between matter and geometry. Thus, one has

$$\frac{d\vec{A}}{d\theta} = r^2\left[\frac{6\alpha^2}{mr^3} + m\vec{a}_E(r)\right]\vec{e}_\theta, \tag{60}$$

where it is also assumed that $\mu \approx m$. The change in direction $\Delta\phi$ of the perihelion for a variation of θ of 2π is obtained as

$$\Delta\phi = \frac{1}{\alpha e}\int_0^{2\pi}\left|\dot{\vec{L}} \times d\frac{\vec{A}}{d\theta}\right| d\theta, \tag{61}$$

and is given by

$$\Delta\phi = 24\pi^3\left(\frac{a}{T}\right)^2\frac{1}{1-e^2} + \frac{L}{8\pi^3 me}\frac{\left(1-e^2\right)^{3/2}}{(a/T)^3}\int_0^{2\pi}\frac{a_E\left[L^2\left(1+e\cos\theta\right)^{-1}/m\alpha\right]}{\left(1+e\cos\theta\right)^2}\cos\theta d\theta, \tag{62}$$

where the relation $\alpha/L = 2\pi\left(a/T\right)/\sqrt{1-e^2}$ is used. The first term of this equation corresponds to the GR prediction for the precession of the perihelion of planets, while the second gives the contribution to the perihelion precession due to the presence of the new coupling between matter and geometry.

As an example of the application of Eq. (62), one considers the case for which the extra-force a_E may be considered constant — an approximation that might be valid for small regions of the space-time. Thus, through Eq. (62), one obtains the perihelion precession

$$\Delta\phi = \frac{6\pi GM_\odot}{a\left(1-e^2\right)} + \frac{2\pi a^2\sqrt{1-e^2}}{GM_\odot}a_E, \tag{63}$$

resorting to Kepler's third law, $T^2 = 4\pi^2 a^3/GM_\odot$.

For Mercury, $a = 57.91 \times 10^9$ m and $e = 0.205615$, respectively, while $M_\odot = 1.989 \times 10^{30}$ kg: the first term in Eq. (63) gives the GR value for the precession angle, $(\Delta\phi)_{GR} = 42.962$ arcsec per century, while the observed value is $(\Delta\phi)_{obs} = 43.11 \pm 0.21$ arcsec per century [28]. Therefore, the difference $(\Delta\phi)_E = (\Delta\phi)_{obs} - (\Delta\phi)_{GR} = 0.17$ arcsec per century can be attributed to other physical effects. Hence, the observational constraints requires that the value of the constant extra acceleration a_E must satisfy the condition

$$a_E \leq 1.28 \times 10^{-11} \text{ m/s}^2. \tag{64}$$

This value of a_E, obtained from the solar system observations, is somewhat smaller than the value of the extra-acceleration $a_0 \approx 10^{-10}$ m/s^2, necessary to account for the Pioneer anomaly However, it does not rule out the possibility of the presence of some extra gravitational effects acting at both solar system and galactic scale, since the assumption of a constant extra-force may not be correct on large astronomical scales.

7 Conclusions and Outlook

In this contribution we have discussed a wide range of implications of the gravity model action, Eq. (2.7), whose main feature is the non-minimal coupling between curvature and the Lagrangian density of matter (or a function of it, in Section VI). This exhibits an extra force with respect to the GR motion, as well as the non-conservation of the matter energy-momentum tensor. The prevalence of these features for different choices for the matter Lagrangian density was discussed in Section III. In Section IV, the specific features of the associated scalar-tensor theory were discussed — and it was shown that the model is consistent with the observational values of the PPN parameters, namely $\beta = \gamma = 1$, to zeroth-order in λ. In Section V, we consider the impact of the novel coupling on the issue of stellar equilibrium. It is shown that, for the simplest model of the Sun, the effect of the new coupling on the central temperature is smaller than 1 %, which is consistent with the uncertainty of current estimates. Finally, in Section VI, a general function of the matter Lagrangian density has been introduced, and the value of the resulting extra force obtained, $a_E \leq 10^{-11}$ m/s^2.

Of course, further work is still required in order to quantify the violation of the Equivalence Principle introduced by the model under realistic physical conditions. A low bound for the coupling λ, would justify the results discussed in this work, which are first order in λ. Implications of the discussed model in what concerns the issue of singularities are still to be addressed, as well as the impact that the new coupling term might have on the early Universe cosmology.

We would like to close this contribution with our best wishes to our colleague Sergei Odintsov, on the occasion of his 50th birthday.

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From Dark Energy to Dark Matter via Non-Minimal Coupling

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Abstract

Toy cosmological models based on non-minimal coupling between gravity and scalar dilaton-like field are presented in the framework of Palatini formalism. They have the following property: preceding to a given cosmological epoch is a dark energy epoch with an accelerated expansion. The next (future) epoch becomes dominated by some kind of dark matter.

0.1 Preliminaries and notation

Modification of Eintein's General Relativity becomes viable candidate to address accelerated expansion, dark matter and dark energy problems in modern cosmology (see e.g. [1, 2] and references therein). This includes modified theories with non-trivial gravitational coupling [3, 4, 5, 6, 7]. Particularly, viable non-minimal models unifying early-time inflation with late-time acceleration have been discussed in [5].

Main object of our considerations in this note is cosmological applications of some non-minimally coupled scalar–tensor Lagrangians of the type

$$L = \sqrt{g}\left(f(R) + F(R)\, L_d\right) + L_{mat} \tag{1}$$

treated within Palatini approach as in [4]. Hereafter we set $L_d = -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ the Lagrangian for a scalar (massless) dilaton-like field ϕ and L_{mat} represents any matter Lagrangian. Because of Palatini formalism R is a scalar $R = R(g,\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ composed of the metric g and the Ricci tensor $R_{\mu\nu}(\Gamma)$ of the symmetric ($\equiv$ torsionless) connection Γ (for more details concerning Palatini formalism see e.g. [8, 9, 10]). Therefore (g, Γ) are dynamical variables. Particularly, the metric g will be used for raising and lowering indices.

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We began with recalling some general formulae already developed in [4]: both $f(R)$ and $F(R)$ are assumed to be analytical functions of R. Dynamics of the system (1) is controlled by the so-called master equation

$$2f(R) - f'(R)R + \tau = (F'(R)R - F(R))L_d \tag{2}$$

where prime denotes derivative with respect to R. We set $T^{mat} = g^{\mu\nu}T^{mat}_{\mu\nu}$ and $T^d = g^{\mu\nu}T^d_{\mu\nu} = L_d$ for traces of the stress-energy tensors: matter $T^{mat}_{\mu\nu} = \frac{\delta L_{mat}}{\delta g_{\mu\nu}}$ and dilaton $T^d_{\mu\nu} = -\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi$.

Equations of motion for gravitational fields (Γ, g) can be recast [4] into the form of generalized Einstein equations

$$R_{\mu\nu}(h) \equiv R_{\mu\nu}(bg) = g_{\mu\alpha}P^\alpha_\nu \tag{3}$$

(see also [9, 10]), where $R_{\mu\nu}(\Gamma)$ is now the Ricci tensor of the new conformally related metric $h = bg$. The conformal factor b is specified below and a (1, 1) tensor P^μ_ν is defined by:

$$P^\mu_\nu = \frac{c}{b}\delta^\mu_\nu - \frac{F(R)}{b}T^{d\ \mu}_{\ \ \nu} + \frac{1}{b}T^{mat\ \mu}_{\ \ \ \ \nu}$$

Here one respectively has:

$$\begin{cases} c = \frac{1}{2}(f(R) + F(R)L_d) = (L - L_{mat})/2\sqrt{g} \\ b = f'(R) + F'(R)L_d \end{cases} \tag{4}$$

Field equations for the scalar (dilaton-like) field ϕ is

$$\partial_\nu\left(\sqrt{g}F(R)g^{\mu\nu}\partial_\mu\phi\right) = 0 \tag{5}$$

which reproduces the same field equations as treated in [4].

0.2 Cosmology from the generalized Einstein equations

We assume the physical metric g to be a standard Friedmann-Robertson-Walker (FRW) metric g:

$$g = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{6}$$

where $a(t)$ is a scale factor. We also suppose the Cosmological Principle to hold. The matter content $T^{mat}_{\mu\nu}$ of the universe is described by a non-interacting mixture of perfect fluids. We denote by w_i the barotropic coefficients. Each species is represented by the stress-energy tensor $T^{(i)}_{\mu\nu} = (\rho_i + p_i)u_\mu u_\nu + p_i g_{\mu\nu}$ satisfying a metric (with the Christoffel connection of g) conservation equation $\nabla^{(g)\,\mu}T^{(i)}_{\mu\nu} = 0$ (see [11]). This gives rise to the standard relations between pressure and density (equation of state) $p_i = w_i\rho_i$ and $\rho_i = \eta_i a^{-3(1+w_i)}$.

It follows thence that the generalized Einstein equations lead to the generalized Friedmann equation under the form:

$$\left(\frac{\dot a}{a} + \frac{\dot b}{2b}\right)^2 = \frac{F(R)L_d}{6b} + \frac{c}{3b} + \sum_i \frac{(1+3w_i)\eta_i}{6b}a^{-3(1+w_i)} \tag{7}$$

where $(1+3w_i)\eta_i a^{-3(1+w_i)}$ represents a perfect fluid component with an equation of state (EoS) parameter w_i.

Let us observe that for standard cosmological model based on the standard Einstein-Hilbert variational principle

$$L_{EH} = \sqrt{g}R + L_{mat} \tag{8}$$

(considered both in purely metric as well as in Palatini formalism) the corresponding Friedmann equation takes the form

$$H^2 \equiv \frac{\dot{a}}{a} = \frac{1}{3}\sum_i \eta_i \, a^{-3(1+w_i)} \tag{9}$$

when coupled to (non-interacting) multi-component perfect fluid. This is due to the fact that geometry contributes to the r.h.s. of the Friedmann equation through

$$R = -T^{mat} = \sum_i \frac{(1-3w_i)\eta_i}{6b} a^{-3(1+w_i)}$$

For example, the preferred ΛCDM model requires three fluid components: cosmological constant $w_\Lambda = -1$, dust $w_{dust} = 0$ and radiation $w_{rad} = \frac{1}{3}$ and can be obtained from (7) provided $\alpha = \beta = \gamma = 0$.

On the other hand we have that the field equation for the scalar field $\phi \equiv \phi(t)$ is $\frac{d}{dt}(\sqrt{g}F(R)\dot{\phi}) = 0$, so that $\sqrt{g}F(R)\dot{\phi} = const$ and consequently $gF(R)^2 L_d = A^2 = const.$ This simply implies that:

$$F(R)^2 L_d = A^2 a^{-6} \tag{10}$$

with an arbitrary positive integration constant A^2 (see (5)).

0.3 Toy cosmological models

Our objective here is to investigate a possible cosmological applications of the following subclass of gravitational Lagrangians (1)

$$L = \sqrt{g}\left(R + \alpha R^2 + \beta R^{1+\delta} + \gamma R^{1+2\delta} L_d\right) + L_{mat} \tag{11}$$

where $\alpha, \beta, \gamma, \delta$, are free parameters of the theory. It should to be observed that the gravitational part $f(R)$ contains Starobinsky term [11] with some $R^{1+\delta}$ contribution. In the limit $\alpha\,,\beta\,,\gamma \to 0$ our Lagrangians reproduce General Relativity. The numerical value for the constant γ (when $\neq 0$) is unessential since it can be always incorporated (by re-scaling) into the field ϕ. As matter contribution we assume two non-interacting most natural components: pressureless dust ($w_{dust} = 0$) and radiation $w_{rad} = \frac{1}{3}$.

Following common strategy particularly applicable within Palatini formalism (see [8, 9, 10]) one firstly finds out an exact solution of the master equation (2). In the case under consideration it can be chosen as

$$R = \left[\frac{\eta}{(1-\delta)\,\beta}\right]^{\frac{1}{1+\delta}} a^{-\frac{3}{1+\delta}} \equiv \xi a^{-\frac{3}{1+\delta}} \tag{12}$$

where $\xi \equiv \left[\frac{\eta}{(1-\delta)\beta}\right]^{\frac{1}{1+\delta}}$ provided the the integration constant A (see 10) takes the value

$$A^2 = \frac{\gamma}{2\delta}\left[\frac{\eta}{(1-\delta)\,\beta}\right]^2 \tag{13}$$

which can vanish only in the case $\gamma = 0$ (no dilaton) and/or $\eta = 0$ (no matter).

Then the conformal factor b reads:

$$b = \frac{1+4\delta}{2\delta} + 2\alpha\xi a^{-\frac{3}{1+\delta}} + \beta(1+\delta)\xi^\delta a^{-\frac{3\delta}{1+\delta}} \tag{14}$$

As a consequence, we have obtained the generalized Friedmann equations under the form:

$$\left(\frac{\dot a}{a} + \frac{\dot b}{2b}\right)^2 = (6b)^{-1}\Big[\frac{1+\delta}{\delta}\xi a^{-\frac{3}{1+\delta}} + \alpha\xi^2 a^{-\frac{6}{1+\delta}} + \\ + \frac{\delta}{1-\delta}\eta a^{-3} + 2\eta_{rad}a^{-4}\Big] \tag{15}$$

We are now in position to calculate the generalized Hubble factor as

$$\frac{\dot a}{a} + \frac{\dot b}{2b} \equiv \frac{\dot a}{a} B \equiv H\, B \tag{16}$$

where H denotes ordinary Hubble "constant" and

$$B = \frac{\frac{1+4\delta}{\delta} - 2\frac{1-2\delta}{1+\delta}\alpha\xi a^{-\frac{3}{1+\delta}} + (2-\delta)\beta\xi^\delta a^{-\frac{3\delta}{1+\delta}}}{\frac{1+4\delta}{\delta} + 4\alpha\xi a^{-\frac{3}{1+\delta}} + 2\beta(1+\delta)\xi^\delta a^{-\frac{3\delta}{1+\delta}}} \tag{17}$$

Before proceeding further let us observe that scaling properties of (7)

$$H^2\, B^2 = (6b)^{-1}\Big[\frac{1+\delta}{\delta}\xi a^{-\frac{3}{1+\delta}} + \alpha\xi^2 a^{-\frac{6}{1+\delta}} + \frac{\delta}{1-\delta}\eta a^{-3} + \\ + 2\eta_{rad}a^{-4}\Big] \tag{18}$$

are analogical to that in standard cosmology (9). More exactly, choosing some reference epoch $a_e \equiv a(t_e)$, e.g. the current cosmological epoch, one can rewrite the generalized Friedmann equation (18) as

$$H^2\, B^2 = (6b)^{-1}\Big[\frac{1+\delta}{\delta}\xi_e\left[\frac{a}{a_e}\right]^{-\frac{3}{1+\delta}} + \alpha\xi_e^2\left[\frac{a}{a_e}\right]^{-\frac{6}{1+\delta}} + \frac{\delta}{1-\delta}\eta_e\left[\frac{a}{a_e}\right]^{-3} + \\ + 2\eta_{rad,e}\left[\frac{a}{a_e}\right]^{-4}\Big] \tag{19}$$

where one has $\eta_e = \eta a_e^3$, $\eta_{rad,e} = \eta_{rad}a_e^4$, $\xi_e = \left[\frac{\eta_e}{(1-\delta)\beta}\right]^{\frac{1}{1+\delta}}$,

$$b = \frac{1+4\delta}{2\delta} + 2\alpha\xi_e\left[\frac{a}{a_e}\right]^{-\frac{3}{1+\delta}} + \beta(1+\delta)\xi_e^\delta\left[\frac{a}{a_e}\right]^{-\frac{3\delta}{1+\delta}}$$

etc..

Assume now that $0 < \delta < 1$.

Thus for $\frac{a}{a_e} \gg 1$, one can approximate (19) by the following Hubble law

$$H^2 \approx \frac{1}{3}\Big[\frac{1+\delta}{1+4\delta}\xi_e\left[\frac{a}{a_e}\right]^{-\frac{3}{1+\delta}} + \frac{\delta\alpha}{1+4\delta}\xi_e^2\left[\frac{a}{a_e}\right]^{-\frac{6}{1+\delta}} +$$

$$+\frac{\delta^2}{(1+4\delta)(1-\delta)}\eta_e\left[\frac{a}{a_e}\right]^{-3}+\frac{2\delta}{1+4\delta}\eta_{rad,e}\left[\frac{a}{a_e}\right]^{-4}\Big] \tag{20}$$

Due to the factor δ the "true matter" and radiation decay as $\delta\mapsto 0$. From the other hand the first term on the r.h.s of (20) plays a role of matter: it can be considered as "dark matter" which amount is controlled by the factor ξ_e. In the regime $\delta\mapsto 1$ the first term gives a bit of acceleration while the second mimics matter, etc..

For $\frac{a}{a_e}\ll 1$ (preceding epoch), Friedmann type approximation reads instead

$$H^2\approx\frac{1}{3}\Big[\frac{(1+\delta)^3}{\delta(1-2\delta)^2\alpha}+\frac{(1+\delta)^2}{(1-2\delta)^2}\xi_e\left[\frac{a}{a_e}\right]^{-\frac{3}{1+\delta}}+$$
$$+\frac{\delta(1+\delta)^2}{((1-2\delta)^2(1-\delta)}\frac{\eta_e}{\alpha\xi_e}\left[\frac{a}{a_e}\right]^{-\frac{3\delta}{1+\delta}}+\frac{2(1+\delta)^2}{(1-2\delta)^2}\frac{\eta_{rad,e}}{\alpha\xi_e}\left[\frac{a}{a_e}\right]^{-\frac{1+4\delta}{1+\delta}}\Big] \tag{21}$$

This epoch is dominated by dark energy in the form of cosmological constant which produces Starobinsky inflation. The universe described by such Freedmann equation undergoes two additional phases of accelerated expansion (power–law inflation) followed by the matter dominated era when $\delta\mapsto 0$. Similarly, when $\delta\mapsto 1$. In such scenario the evolution goes from dark energy to dark matter dominated eras. More detailed study of such models will be given elsewhere.

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Exploring the dark side of the Universe in a dilatonic brane-world scenario

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Abstract

We describe the late-time acceleration of the Universe within the paradigm of the brane-world scenario. More precisely, we show how a phantom-like behaviour or a crossing of the cosmological constant line can be achieved safely in a dilatonic brane-world model with an induced gravity term on the brane. The brane tension plays the role of dark energy which is coupled to the dilaton bulk scalar field. The phantom mimicry as well as the crossing of the cosmological constant line are achieved without invoking any phantom matter either on the brane or in the bulk.

1 Introduction

Understanding the recent acceleration of the universe is a challenge and a landmark problem in physics. Its resolution may affect in the short term our understanding of a fundamental interaction like gravity as well as enlarge the framework of particle physics. The smoking gun of the acceleration of the universe (if we assume it is homogeneous and isotropic on large scales) was provided by the analysis of the Hubble diagram of SNe Ia a decade ago [1]. This discovery, together with (i) the measurement of the fluctuations in the CMB which implied that the universe is (quasi) spatially flat and (ii) that the amount of matter which clusters gravitationally is much less than the critical energy density, implied the existence of a "dark energy component" that drives the late-time acceleration of the universe. Subsequent precision measurements of the CMB anisotropy by WMAP [2] and the power spectrum of galaxy clustering by the 2dFGRS and SDSS surveys [3, 4] have confirmed this discovery.

A plethora of different theoretical models have been so far proposed to explain this phenomenon [5], although unfortunately none of the models advanced so far is both completely convincing and well motivated. A cosmological constant corresponding to roughly two thirds of the total energy density of the universe is perhaps the simplest *phenomenological* way to explain the late-time speed up of the universe –and in addition match rather well the observational data. However, the expected theoretical value of the cosmological constant is about 120 orders of magnitude larger than the value needed to fit the data [6].

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Alternative approaches to explain the late-time acceleration invoke (i) a dark energy component in the universe which would provide a negative pressure or (ii) an infrared modification of general relativity on large scales (like in some brane-world scenarios [7] or f(R) models [8]) which, by weakening the gravitational interaction on those scales, allows the recent speed up of the universal expansion. The second approach is also motivated by the fact that we only have precise measurements of gravity from sub-millimiter scales up to solar system scales while the Hubble radius, which is the scale relevant for the cosmic acceleration, is many orders of magnitude larger.

A pioneering scheme in the second approach is the Dvali, Gabadadze and Porrati (DGP) model [7] which corresponds to a 5-dimensional (5D) induced gravity brane-world model [8, 10, 11, 12, 13], where a low-energy modification occurs with respect to general relativity; i.e. an infrared effect takes place, leading to two branches of solutions: (i) the self-accelerating branch and (ii) the normal branch.

The self-accelerating branch solution gives rise to an asymptotically de Sitter brane; i.e. a late-time accelerating brane universe. The acceleration of the brane expansion arises naturally, i.e. without invoking the presence of any dark energy on the brane to produce the speed-up. Most importantly, it has recently been shown that by embedding the DGP model in a higher dimensional space-time the ghost problem in the original model [14] may be cured [15] while preserving the existence of a self-accelerating solution [16].

The normal branch also constitutes in itself an extremely interesting physical setup of the DGP model however, as it can mimic a phantom behaviour on the brane by means of the ΛDGP scenario [11]. We would like to highlight that observational data do not seem incompatible with a phantom-like behaviour [6] and therefore we should keep an open mind about what is producing the recent inflationary era of our universe. Furthermore, this phantom-like behaviour may well be a property acquired only recently by dark energy. This leads to an interest in modelling the so called crossing of the phantom divide line $w = -1$; for example in the context of the brane-world scenario [12, 18, 19]. The most important aspect of the ΛDGP model is that the phantom-like mimicry is obtained without invoking any real phantom-matter [5] which is known to violate the null energy condition and induce quantum instabilities[2] [21]. In the DGP scenario it is as well possible to get a mimicry of the crossing of the phantom divide, however, at the cost of invoking a dynamical dark energy on the brane [12], for example modelled by a quiessence fluid or a (generalised) Chaplygin gas.

One aim of this paper is to show that a dilatonic brane-world model with an induced gravity term in the brane can mimic a phantom-like behaviour without including matter on the brane that violates the null energy condition. A second aim of this paper is to show that there is an alternative form (to the one introduced in [11, 12]) of mimicking the crossing the cosmological constant line $w = -1$ in the brane-world scenario. More precisely, we consider a 5D dilatonic bulk with a brane endowed with an induced gravity term, a brane matter content corresponding to cold dark matter, and a brane tension λ that depends on the minimally coupled bulk scalar field. We will show that in this set-up the vacuum generalised self-accelerating branch expands in a super-accelerating way and mimics a phantom-behaviour. On the other hand, the generalised normal branch expands in an accelerated way due to λ playing the role of *dark energy* –through its dependence on the bulk scalar field. Furthermore, in this case, it turns out that the brane tension grows with the brane scale factor until it reaches a maximum positive value and then starts decreasing. Therefore, in our model the brane tension mimics a crossing of the phantom divide. Most importantly no matter violating the null energy density is invoked in our model.

[2]We are referring here to a phantom energy component described through a minimally coupled scalar field with the wrong kinetic term.

The paper is organised as follows. In section 2, we present our dilatonic brane-world model with induced gravity. The bulk scalar field potential is an exponential potential. The matter content of the brane is coupled to the dilaton field. We deduce the modified Friedmann equation for both branches, the junction condition of the dilaton across the brane, which constrains the brane tension, and the energy balance on the brane. In section 3, we then analyse the vacuum (i.e., $\rho_m = 0$) solutions in both branches. We show that the brane tension has a phantom-like behaviour on the generalised self-accelerating branch in the sense that the brane tension grows as the brane expands. In this branch, the brane hits a singularity in its future evolution which may be interpreted as a "big rip" singularity pushed towards an infinite cosmic time. Then, in section 4, we show that, under some assumptions on the nature of the coupling parameters between λ and ϕ, $1+w_{\rm eff}$ changes sign as the normal brane evolves, with $w_{\rm eff}$ the effective equation of state for the brane tension. Our conclusions are presented in section 5.

2 The framework

We consider a brane, described by a 4D hyper-surface (h, metric g), embedded in a 5D bulk space-time ($\mathcal{B}$, metric $g^{(5)}$), whose action is given by

$$\mathcal{S} = \frac{1}{\kappa_5^2}\int_{\mathcal{B}} d^5X\sqrt{-g^{(5)}}\left\{\frac{1}{2}R[g^{(5)}] + \mathcal{L}_5\right\} + \int_h d^4X\sqrt{-g}\left\{\frac{1}{\kappa_5^2}K + \mathcal{L}_4\right\}, \tag{1}$$

where κ_5^2 is the 5D gravitational constant, $R[g^{(5)}]$ is the scalar curvature in the bulk and K the extrinsic curvature of the brane in the higher dimensional bulk, corresponding to the York-Gibbons-Hawking boundary term.

We consider a dilaton field ϕ living on the bulk [23, 22] and we choose ϕ to be dimensionless. Then, the 5D Lagrangian $\mathcal{L}_5$ can be written as

$$\mathcal{L}_5 = -\frac{1}{2}(\nabla\phi)^2 - V(\phi). \tag{2}$$

The 4D Lagrangian $\mathcal{L}_4$ corresponds to

$$\mathcal{L}_4 = \alpha R[g] - \lambda(\phi) + \Omega^4\mathcal{L}_m(\Omega^2 g_{\mu\nu}). \tag{3}$$

The first term on the right hand side (rhs) of the previous equation corresponds to an induced gravity term [7, 8, 10, 11], where $R[g]$ is the scalar curvature of the induced metric on the brane and α is a positive parameter which measures the strength of the induced gravity term and has dimensions of mass squared. The term $\mathcal{L}_m$ in Eq. (3) describes the matter content of the brane and $\lambda(\phi)$ is the brane tension, and we will restrict ourselves to the case where they are homogeneous and isotropic on the brane. We allow the brane matter content to be non-minimally coupled on the (5D) Einstein frame but to be minimally coupled respect to a conformal metric $\tilde{g}^{(5)}_{AB} = \Omega^2\, g^{(5)}_{AB}$, where $\Omega = \Omega(\phi)$ [22].

We are interested in the cosmology of this model. It is known that for an expanding FLRW brane the unique bulk space-time in Einstein gravity (in vacuum) is a 5D Schwarzschild-anti de Sitter space-time. This property as far as we know cannot be extended to a 5D dilatonic bulk. On the other hand, the presence of an induced gravity term in the brane-world scenario affects only the dynamics of the brane, through the junction conditions at the brane, and does not affect the bulk field equations. Therefore, in order to study the effect of an induced gravity term in a brane-world dilatonic model, it is possible to consider a bulk corresponding to a dilatonic 5D space-time and later on impose the junction conditions at the brane location. The junction conditions will then determine the cosmological evolution of the brane and constrain the brane tension. This is the approach we will follow.

From now on, we consider a 5D dilatonic solution obtained by Feinstein et al [24, 25] *without an induced gravity term on the brane*. The 5D dilatonic solution reads [25]

$$ds_5^2 = \frac{1}{\xi^2} r^{2/3(k^2-3)} dr^2 + r^2(-dt^2 + \gamma_{ij} dx^i dx^j), \tag{4}$$

where γ_{ij} is a 3D spatially flat metric. The bulk potential is

$$V(\phi) = \Lambda \exp[-(2/3)k\phi]. \tag{5}$$

The parameters k and ξ in Eq. (4) define the 5D cosmological constant Λ

$$\Lambda = \frac{1}{2}(k^2 - 12)\xi^2. \tag{6}$$

The 5D scalar field grows logarithmically with the radial coordinate r [25]

$$\phi = k \log(r). \tag{7}$$

Now, we consider a FLRW brane filled only with cold dark matter (CDM); i.e pressureless matter, and the brane tension $\lambda(\phi)$. On the other hand, the brane is considered to be embedded in the previous 5D dilatonic solution and its trajectory in the bulk is described by the following parametrisation

$$t = t(\tau), \quad r = a(\tau), \quad x_i = constant, \ i = 1 \ldots 3. \tag{8}$$

Here τ corresponds to the brane proper time. Then the brane metric reads

$$ds_4^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -d\tau^2 + a^2(\tau)\gamma_{ij} dx^i dx^j. \tag{9}$$

For an induced gravity brane-world model [8, 13], there are two physical ways of embedding the brane in the bulk when a $\mathbb{Z}_2$-symmetry across the brane is assumed: the generalised normal branch[3] and the generalised self-accelerating branch. For example, in the first case the brane is moving in the bulk away from the bulk naked singularity located at $r = 0$ [13].

For simplicity, we will consider that the matter content of the brane is minimally coupled respect to the conformal metric $\tilde{g}^{(5)}_{AB} = \exp(2b\phi)\ g^{(5)}_{AB}$; i.e. $\Omega = \exp(b\phi)$, where b is a constant. We will also consider only the case[4] $k > 0$; i.e. the scalar field is a growing function of the coordinate r. Then, the Israel junction condition at the brane [23] describes the cosmological evolution of the brane through the modified Friedmann equation, which in our case reads

$$\sqrt{\xi^2 a^{-\frac{2}{3}k^2} + H^2} = -\epsilon \frac{\kappa_5^2}{6} \left[\lambda(\phi) + \rho_m - 6\alpha H^2\right], \tag{10}$$

where $\epsilon = 1$ for the self-accelerating brane and $\epsilon = -1$ for the normal branch. The modified Friedmann equation can be more conveniently expressed as

$$H^2 = \frac{1}{6\alpha}\left\{\lambda + \rho_m + \frac{3}{\kappa_5^4\alpha}\left[1 + \epsilon\sqrt{1 + 4\kappa_5^4\alpha^2\xi^2 a^{-2k^2/3} + \frac{2}{3}\kappa_5^4\alpha(\lambda + \rho_m)}\right]\right\}, \tag{11}$$

[3] We will refer to the normal DGP branch also as the non-self-accelerating DGP branch.

[4] The main conclusions do not depend on the sign of k but on the sign of the parameter kb. Therefore, we can always describe the same physical situation on the brane for $k < 0$ by changing the sign of b.

where λ is the brane tension and ρ_m is the energy density of CDM.

On the other hand, as it is usual in a dilatonic brane-world scenario, matter on the brane –in this case CDM– is not conserved due to the coupling Ω (see Eq. (3)). In fact, we have

$$\dot{\rho}_m = -3H\left(1 - \frac{1}{3}kb\right)\rho_m, \tag{12}$$

where a dot stands for derivatives respect to the brane cosmic time τ. Therefore, CDM on the brane scales as

$$\rho_m = \rho_0 a^{-3+kb}. \tag{13}$$

Finally, the junction condition of the scalar field at the brane [23] constrains the brane tension $\lambda(\phi)$. In our model this is given by

$$a\frac{d\lambda}{da} = -kb\rho_m + \epsilon\frac{2k^2}{\kappa_5^2}\sqrt{\xi^2 a^{-\frac{2}{3}k^2} + H^2}, \tag{14}$$

where for convenience we have rewritten the scalar field (valued at the brane) in terms of the scale factor of the brane. At this respect we remind the reader that at the brane $\phi = k\log(a)$.

3 Vacuum solutions

The vacuum solutions, i.e. in absence of matter on the brane, depends crucially on the embedding of the brane in the bulk, therefore, which branch we are considering.

3.1 The self-accelerating branch

For the vacuum self-accelerating branch; i.e. $\epsilon = 1$ and $\rho_m = 0$, the brane tension is an increasing function of the scale factor of the brane[5] (see Eq. (6.1)). For small values of the scale factor, the brane tension reaches infinite negative values. On the other hand, for very large value of the scale factor the brane tension approaches infinite positive values. Therefore, when the brane tension acquires positive values, it mimics a phantom energy component in a standard FLRW universe. We remind at this respect that we have not included any matter that violates the null energy condition; i.e. any explicit phantom energy in the model.

The Hubble parameter is an increasing function of the scale factor, i.e the brane super-accelerates. In fact,

$$\dot{H} = -\frac{k^2H^2}{\kappa_5^4\alpha(\lambda - 6\alpha H^2) + 3}, \tag{15}$$

while the modified Friedmann equation (11) implies that the denominator of the previous equation has to be negative (see also footnote 5), consequently $\dot{H} > 0$. At small scale factors, H reaches a constant positive value. Therefore, in the vacuum self-accelerating brane there is no big bang singularity on the brane; indeed, the brane is asymptotically de Sitter. On the other hand, at very large values of the scale factor, the Hubble parameter diverges.

The divergence of the Hubble parameter for very large values of the scale factor might point out the existence of a big rip singularity in the future evolution of the brane; i.e. the scale factor and Hubble parameter blow up in a finite cosmic time in the future evolution of the brane. However, it can be shown that the divergence of H and a (and also of λ) occur in

[5]The constraint equation (6.1) (after substituting the Hubble rate given in Eq. (11)) can be solved analytically in this case [13] and it can be explicitly shown that the brane tension increases as the brane expands. In the same way a parametric expression can be found for the Hubble rate and its cosmic derivative.

an infinite cosmic time in the future evolution of this branch. This can be easily proven by noticing that the asymptotic behaviour of the Hubble parameter at large value of the scale factor is

$$H \sim \frac{k^2}{\kappa_5^2 \alpha} \ln(a). \tag{16}$$

Consequently, the Hubble rate does not grow as fast as in phantom energy models with a constant equation of state where a big rip singularity takes place on the future evolution of a homogeneous and isotropic universe [5].

In summary, we have proven that in the vacuum self-accelerating branch the brane tension mimics a phantom behaviour. On the other hand, there is a singularity in the future evolution of the brane. The singularity is such that for large value of the cosmic time, the scale factor and the Hubble parameter diverge. This kind of singularity can be interpreted as a "big rip" singularity pushed towards an infinite cosmic time of the brane.

3.2 The normal branch

For the vacuum normal branch; i.e. $\epsilon = -1$ and $\rho_m = 0$, the brane tension is a decreasing function of the scale factor of the brane[6] (see Eq. (6.1)). For small values of the scale factor, the brane tension reaches infinite positive values. On the other hand, for very large value of the scale factor the brane tension vanishes.

The Hubble parameter is a decreasing function of the scale factor, i.e the brane is never super-accelerating. In fact, Eq. (15) and the Israel junction condition (10) implies that the denominator of the previous equation has to be positive (see also footnote 6), therefore $\dot{H} < 0$. At high energy, H reaches a constant positive value. Consequently, in the vacuum brane there is no big bang singularity on the brane; indeed, the brane is asymptotically de Sitter. On the other hand, at very large values of the scale factor, the Hubble parameter vanishes (the brane is asymptotically Minkowski in the future). Although the brane never super-accelerates, the brane always undergoes an inflationary period.

The brane behaves in two different ways depending on the value taken by k^2 (cf. Fig. 1). Thus, for $k^2 \leq 3$ the brane is eternally inflating. A similar behaviour was found in [25]. On the other hand, for $k^2 > 3$ the brane undergoes an initial stage of inflation and later on it starts decelerating. This second behaviour contrasts with the results in [25] for a vacuum brane without an induced gravity term on the brane. Then, the inclusion of an induced gravity term on a dilatonic brane-world model with an exponential potential in the bulk allows for the normal branch to inflate in a region of parameter space where the vacuum dilatonic brane alone would not inflate. This behaviour has some similarity with steep inflation [26], where high energy corrections to the Friedmann equation in RS scenario [27] permit an inflationary evolution of the brane with potentials too steep to sustain it in the standard 4D case, although the inflationary scenario introduced by Copeland et al in [26] is supported by an inflaton confined in the brane while in our model inflation on the brane is induced by a dilaton field on the bulk.

4 Crossing the cosmological constant line

We now address the following question: is it possible to mimic a crossing of the phantom divide in particular in the model introduced in section 2? Unlike the vacuum case –which

[6]The constraint equation (6.1) (after substituting the Hubble rate given in Eq. (11)) can be solved analytically in this case [13] and it can be explicitly shown that the brane tension decreases as the brane expands. In the same way a parametric expression can be found for the Hubble rate and its cosmic derivative.

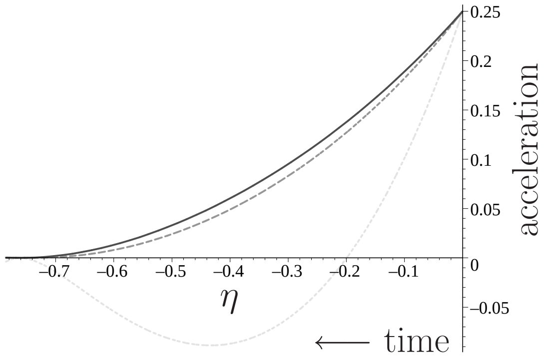

Figure 1: This figure shows the behaviour of the dimensionless acceleration parameter given by $\alpha^2\kappa_5^4\ddot{a}/a$ as a function of the time (see the left hand side arrow). The solid (darker grey), dotted and dashed-dotted (lighter grey) lines correspond to the acceleration parameter for $k^2 = 2, 3, 15$ respectively. For $k^2 = 2, 3$ the negative branch is eternally inflating. On the other hand, for $k^2 = 15$ the brane undergoes an initial transient inflationary epoch.

can be solved analytically [13]– in this case we cannot exactly solve the constraint (6.1). Nevertheless, we can answer the previous question based in some physical and reasonable assumptions and as well as on numerical methods. For simplicity, we will restrict to the normal branch.

In order to answer the previous question, it is useful to introduce the following dimensionless quantities

$$\bar{\lambda} \equiv \frac{2}{3}\kappa_5^4\alpha\lambda,\ x \equiv \frac{2}{3}k\phi - \ln d,\ d \equiv 4\alpha^2\kappa_5^4\xi^2,\ m \equiv 3 - kb,$$
$$\beta_0 \equiv \frac{9\beta_2}{2k^2},\ \beta_1 \equiv \frac{2\kappa_5^4\alpha}{m}\rho_0 d^{-\beta_0},\ \beta_2 \equiv \frac{m}{3}. \tag{17}$$

In terms of these new variables, the constraint given in Eq. (6.1) reads

$$\frac{d\bar{\lambda}}{dx} = 1 - \beta_0\beta_1(1-\beta_2)e^{-\beta_0 x} - \sqrt{1 + \bar{\lambda} + e^{-x} + \beta_1\beta_2 e^{-\beta_0 x}}. \tag{18}$$

The assumptions we make are the following:

(1). We assume that CDM dominates over the vacuum term ($a^{-2/3k^2}$) at early times on the brane. This implies that the parameter β_0 introduced in Eq. (17) must satisfy $\beta_0 > 1$. On the other hand, the brane tension will play the role of dark energy (through its dependence on the scalar field) in our model. This first assumption assumes that dark matter dominates over dark energy at high redshift which is a natural assumption to make. Indeed, at high redshift the brane tension would scale as

$$\bar{\lambda} \sim \beta_1(1-\beta_2)e^{-\beta_0 x} + \dots. \tag{19}$$

(2). We also assume that CDM redshifts away a bit faster than usual; i.e. $bk < 0$ or β_2 introduced in Eq. (17) is such that $\beta_2 > 1$. This lost energy will be used to increase the value of the scalar field $\phi(a)$ on the brane. That is, to push the brane to higher values of a.

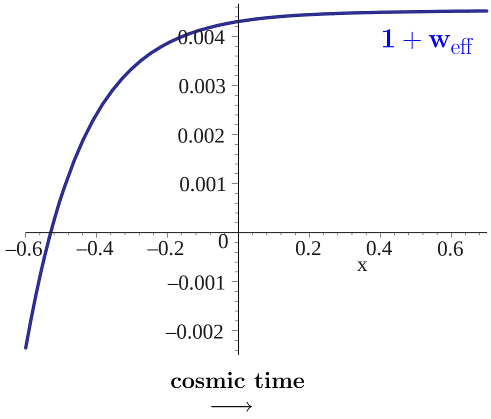

Figure 2: The figure shows the effective equation of state of the brane tension defined in Eq. (21) against the variable x defined in Eq. (17). Notice that x grows as the brane expands and therefore $dx/d\tau > 0$ where τ corresponds to the cosmic time of the brane. This illustrative numerical solution has been obtained for $b = -1$, $k = 1$ and $\beta_1 = 1$. The last parameter is defined in Eq. (17). In order to impose the right initial condition, we started the integration well in the past where CDM dominated over the scalar field on the brane and we took as a good approximated solution the dark matter solution given in Eq. (19).

(3). Finally, we also assume that $\beta_2 < 2\beta_0(\beta_2 - 1)$. This condition, together with $\beta_0, \beta_2 > 1$, is sufficient to prove the non existence of a local minimum of the brane tension during the cosmological evolution of the brane. In fact, we can show the existence of a unique maximum for an even larger set of parameters $\beta_0 > 1/2$, $\beta_2 > 1$ and $\beta_2 < 2\beta_0(\beta_2 - 1)$. Therefore, the set of allowed parameter k and b that fulfil the last three inequalities are such that

$$k < \min\left\{-3b, \frac{3}{2}\left[-b + \sqrt{b^2 + 4}\right]\right\} = -3b. \tag{20}$$

where b is positive.

Under these three assumptions, it can be shown that the brane tension has a local maximum which must be positive (we refer the reader to [19] for a detailed proof). In fact, what happens under these conditions is that the brane tension increases until it reaches its maximum positive value and then it starts decreasing. It is precisely at this maximum that the brane tension mimics crossing the phantom divide. Around the local maximum of the brane tension we can always define an effective equation of state in analogy with the standard 4D relativistic case:

$$1 + w_{\rm eff} = -\frac{1}{3H}\frac{1}{\lambda}\frac{d\lambda}{d\tau}. \tag{21}$$

As we mentioned earlier, the constraint equation (6.1) cannot be solved analytically and therefore we have to resort to numerical methods. We show in Fig. 2 an example of our numerical results where it can be seen clearly that $1 + w_{\rm eff}$ changes sign. It is precisely at that moment that the crossing takes place.

Another important question to address is whether the brane is accelerating at the time that the crossing takes place. We know that the vacuum term dominates at late times (see

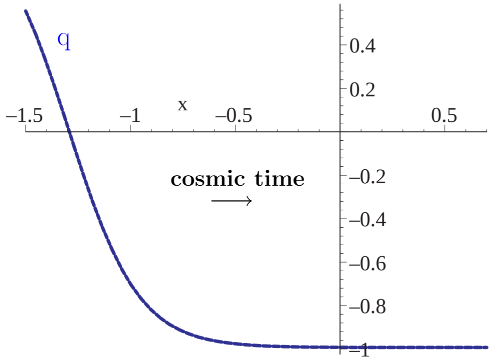

Figure 3: The figure shows the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$ against the variable x defined in Eq. (17). The brane is accelerating in the future when q is negative. Notice that x grows as the brane expands and therefore $dx/d\tau > 0$ where τ corresponds to the cosmic time of the brane. This numerical example has been obtained for $b = -1$, $k = 1$ and $\beta_1 = 1$. The last parameter is defined in Eq. (17). Again in order to impose the right initial condition, we started the integration well in the past where CDM dominated over the scalar field on the brane and we can take as a good approximated solution the dark matter solution given in Eq. (19).

the first assumption). Thus, at that point the brane tension will be adequately described by the vacuum solution; i.e.

$$\bar{\lambda} \sim C \exp(-x/2) + \ldots , \quad C = constant > 0. \tag{22}$$

The constant C is positive because for the vacuum solution the brane tension is always positive [13]. Now, from the results in the previous section, we can conclude that the brane will be speeding up at late times as long as $k^2 \leq 3$. On the other, hand it can be checked numerically that the brane can be accelerating at the crossing as Fig. 3 shows.

5 Conclusions

In this paper we analyse the behaviour of dilatonic brane-world models with an induced gravity term on the brane with a constant induced gravity parameter. We assume a $\mathbb{Z}_2$-symmetry across the brane. The dilatonic potential is an exponential function of the bulk scalar field and the matter content of the brane is coupled to the dilaton field. We deduce the modified Friedmann equation for the generalised self-accelerating and generalised normal branch (which specifies the way the brane is embedded in the bulk), the junction condition for the scalar field across the brane and the energy balance on the brane.

We describe the vacuum solutions; i.e. the matter content of the brane is specified by the brane tension, for a FLRW brane:

(1). In the vacuum self-accelerating branch, the brane tension is a growing function of the scale factor and, consequently, mimics the behaviour of a phantom energy component on the brane. This phantom-like behaviour is obtained without including a phantom fluid on the brane. In fact, the brane tension does not violate the null energy condition. The expansion of the brane is super-inflationary; i.e. the Hubble parameter is a growing function of the cosmic time. At high energy (small scale factors), the brane is asymptotically de Sitter. The brane faces a curvature singularity in its infinite future evolution, where the Hubble parameter, brane tension and scale factor diverge. The singularity happens in an infinite cosmic time. Therefore, the singularity can be interpreted as a "big rip" singularity pushed towards an infinite future cosmic time.

(2). On the other hand, in the vacuum normal branch, the brane tension is a decreasing function of the scale factor. Unlike the positive branch, the branch is not super-inflating. However, it always undergoes an inflationary expansion (see Fig. 1). The inflationary expansion can be eternal ($k^2 \leq 3$) or transient ($k^2 > 3$), where k is related to the slope of bulk scalar field. For large values of the scale factor, the negative branch is asymptotically Minkowski.

Furthermore, we have shown the existence of a mechanism that mimics the crossing of the cosmological constant line $w = -1$ in the brane-world scenario introduced in section 2, and which is different from the one introduced in Refs. [11, 12]. More precisely, we have shown that if we consider the 5D dilatonic bulk with an induced gravity term on the normal branch, a brane tension λ which depends on the minimally coupled bulk scalar field, and a brane matter content corresponding only to cold dark matter, then under certain conditions the brane tension grows with the brane scale factor until it reaches a maximum positive value at which it mimics crossing the phantom divide, and then starts decreasing. Most importantly no matter violating the null energy condition is invoked in our model. Despite the transitory phantom-like behaviour of the brane tension no big rip singularity is hit along the brane evolution (unlike the vacuum self-accelerating branch).

In this model for the normal branch or non-self-accelerating branch, the constraint equation fulfilled by the brane tension is too complicated to be solved analytically (see Eqs. (11) and (6.1)). However, we have shown that under certain physical and mathematical conditions -cold dark matter dominates at higher redshifts and it dilutes a bit faster than dust during the brane expansion as well as a mathematical condition that guarantees the non-existence of a local minimum of the brane tension- it is possible for the brane tension to cross the cosmological constant line. The analytical proof has been confirmed by numerical solutions. Furthermore, we have shown that for some values of the parameters the normal branch inflates eternally to the future due to the brane tension λ playing the role of dark energy through its dependence on the bulk scalar field.

In summary, in the models presented here the mimicry of a phantom-like behaviour and the phantom divide crossing is based on the interaction between the brane and the bulk through the brane tension that depends explicitly on the scalar field that lives in the bulk. We have also shown that in both cases the brane undergoes a late-time acceleration epoch.

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A brief review of the singularities in 4D and 5D viscous cosmologies near the future singularity

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Abstract

Analytic properties of physical quantities in the cosmic fluid such as energy density $\rho(t)$ and Hubble parameter $H(t)$ are investigated near the future singularity (Big Rip). Both 4D and 5D cosmologies are considered (the Randall-Sundrum II model in the 5D case), and the fluid is assumed to possess a bulk viscosity ζ. We consider both Einstein gravity and modified gravity, where in the latter case the Lagrangian contains a term R^α with α a constant. If ζ is proportional to the power $(2\alpha-1)$ of the scalar expansion, the fluid can pass from the quintessence region into the phantom region as a consequence of the viscosity. A property worth noticing is that the 4D singularity on the brane becomes carried over to the bulk region.

1 Introduction

The possibility of crossing the $w=-1$ barrier in dark energy cosmology has recently become a topic of considerable interest. One usually assumes that the equation of state for the cosmic fluid can be written in the form

$$p = w\rho, \tag{1}$$

where w is a constant. If $w=-1$ the fluid is called a "vacuum fluid", with peculiar thermodynamic properties such as negative entropy [1]. More general forms for the equation of state can be envisaged, such as

$$p = w(\rho)\rho = -\rho - f(\rho), \tag{2}$$

which is a form that we shall consider below. As is know, cosmological observations indicate that the present universe is accelerating. Recent discussions on the actual value of w can be

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found, for instance, in refs. [2, 3, 4]. Perhaps, w is even an oscillating function in time. For discussions on time-dependent values of w, one may consult Refs. [5, 6, 7]. The possibility of crossing from the quintessence region $(-1 < w < -1/3)$ into the phantom region $w < -1$, is obviously of physical interest. It may be noted that both quintessence and phantom fluids lead to the inequality $\rho + 3p \leq 0$, thus breaking the strong energy condition.

Once being in the phantom region, the cosmic fluid will inevitably be led into a future singularity, called the Big Rip [8, 9, 10, 11]. And this brings us to the main theme of the present paper, namely to give an overview of the behavior of central physical quantities near the future singularity. This is the case of main interest. We think that such an exposition should be useful, not least so because the situation is rather complex. Namely, there is a variety of different factors at play here: (i) the thermodynamic parameter $w(\rho)$, (ii) the possible time dependence of the bulk viscosity, $\zeta = \zeta(t)$, and (iii) the adoption of Einstein's gravity, or a version of the so-called modified gravity. (For an introduction to modified gravity theories, one may consult Refs. [12, 13].)

To begin with, it is convenient to quote from Ref. [11] the classification of possible future singularities:

(i) Type I ("Big Rip"): For $t \to t_s,\ a \to \infty,\ \rho \to \infty$, and $|p| \to \infty$, or p and ρ are finite at $t = t_s$.

(ii) Type II ("sudden"): For $t \to t_s,\ a \to a_s,\ \rho \to \rho_s$, and $|p| \to \infty$,

(iii) Type III: For $t \to t_s,\ a \to a_s,\ \rho \to \infty$, and $|p| \to \infty$,

(iv) Type IV: For $t \to t_s,\ a \to a_s,\ \rho \to 0,\ |p| \to 0$, or p and ρ are finite. Higher order derivatives of H diverge.

Here the notation is standard, a meaning the scale factor and t_s referring to the instant of the singularity. The above classification was introduced in the context of ideal, i.e., nonviscous, cosmology. We can however make use of the same classification also in the viscous case.

In the following, we will present salient features of 4D, respective 5D, viscous cosmology theory, and thereafter focus on the classification of the various alternatives.

2 Viscous 4D theory: Basics

We include this basic material mainly for reference purposes. We consider the standard FRW metric,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{3}$$

and set the spatial curvature k, as well as the 4D cosmological constant Λ_4, equal to zero. The Hubble parameter is $H = \dot{a}/a$, the scalar expansion is $\theta = U^\mu{}_{;\mu} = 3H$ with U^μ the four-velocity of the fluid, and $\kappa_4^2 = 8\pi G_4$ is the gravitational coupling. Of main interest are the (tt) and (rr) components of the Friedmann equations. They are

$$\theta^2 = 3\kappa_4^2\,\rho, \tag{4}$$

$$\frac{2\ddot{a}}{a} + H^2 = -\kappa_4^2\,\tilde{p}, \tag{5}$$

where $\tilde{p} = p - \zeta\theta$ is the effective pressure. From the differential equation for energy, $T^{0\nu}{}_{;\nu} = 0$, we get

$$\dot{\rho} + (\rho + p)\theta = \zeta\theta^2. \tag{6}$$

As a consequence of positive entropy change in an irreversible process, we must require the value of ζ to be non-negative. From the equations above we obtain the following differential equation for the scalar expansion,

$$\dot{\theta} - \frac{f(\rho)}{2\rho}\theta^2 - \frac{3}{2}\kappa_4^2\,\zeta\theta = 0. \tag{7}$$

In view of the relationship $\dot{\theta} = (\sqrt{3}\,\kappa_4/2)\dot{\rho}/\sqrt{\rho}$ we can alternatively reformulate this equation as an equation for the density,

$$\dot{\rho} - \kappa_4\,\sqrt{3\rho}\,f(\rho) - 3\kappa_4^2\,\zeta\rho = 0. \tag{8}$$

The solution is (cf. Eq. (9) in [14])

$$t = \frac{1}{\sqrt{3}\,\kappa_4}\int_{\rho_*}^{\rho}\frac{d\rho}{\sqrt{\rho}f(\rho)[1+\kappa_4\,\zeta\sqrt{3\rho}/f(\rho)]}. \tag{9}$$

We here let $t=0$ be the initial (present) time, and let the corresponding initial density be ρ_*. The functional form of the bulk viscosity ζ is so far unspecified. The shear viscosity is omitted, due to the assumed spatial isotropy in the cosmic fluid. Note the dimensions: $[\kappa_4^2] = \mathrm{cm}^2$, $[f(\rho)] = [\rho] = \mathrm{cm}^{-4}$, $[\zeta] = \mathrm{cm}^{-3}$. Viscous cosmology are treated at various places, for instance, in Refs. [3, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

3 Specific cases in 4D

We are now in a position to discuss various cases in 4D explicitly. We have to distinguish between several alternatives: (i) use of Einstein or modified gravity; (ii) possible density dependence of the thermodynamic parameter $w = w(\rho)$; and (iii) possible time dependence of the bulk viscosity $\zeta(t)$.

3.1 Einstein gravity, w and ζ being constants

Let $f(\rho) = \alpha\rho$, with α a constant. The equation of state is then

$$p = w\rho = -(1+\alpha)\rho. \tag{10}$$

We can now solve explicitly for the Hubble parameter [14, 26],

$$H(t) = \frac{H_* e^{t/t_c}}{1 - \frac{3\alpha}{2}H_* t_c(e^{t/t_c}-1)}, \tag{11}$$

where H_* is the present-time value of H and t_c is the 'viscosity time',

$$t_c = \left(\frac{3}{2}\kappa_4^2\zeta\right)^{-1}. \tag{12}$$

From Eq. (0.10) it is seen that $H(t)$ becomes singular when the denominator vanishes. Let us first for reference purposes set $\zeta = 0$:

The nonviscous case. If t_{s0} designates the singularity time, we have

$$t_{s0} = \frac{2}{3\alpha H_*}. \tag{13}$$

Then [26],

$$H(t) = \frac{H_* t_{s0}}{t_{s0} - t}, \tag{14}$$

$$a(t) = \frac{a_* t_{s0}^{2/3\alpha}}{(t_{s0} - t)^{2/3\alpha}}, \tag{15}$$

$$\rho(t) = \frac{\rho_* t_{s0}^2}{(t_{s0} - t)^2}. \tag{16}$$

The viscous case. If now $t_{s\zeta}$ denotes the singularity time, we get from Eq. (0.10)

$$t_{s\zeta} = t_c \ln \left[1 + \frac{2}{3\alpha} \frac{1}{H_* t_c} \right]. \tag{17}$$

and

$$H(t) \to \frac{H_* t_{s0}}{t_{s0} - t}, \quad t \to t_{s\zeta}. \tag{18}$$

Close to the singularity we thus obtain the same singular behavior as in the nonviscous case. Moreover, we get the following forms,

$$a(t) \sim (t_{s\zeta} - t)^{-2/3\alpha}, \quad t \to t_{s\zeta}, \tag{19}$$

$$\rho(t) \sim (t_{s\zeta} - t)^{-2}, \quad t \to t_{s\zeta}. \tag{20}$$

The viscosity tends to shorten the singularity time,

$$t_{s\zeta} < t_{s0}, \tag{21}$$

but it does *not* modify the exponents in the singularity. The singularity is of Type I if $\alpha > 0$, and of Type II if $\alpha < 0$.

3.2 Einstein gravity, $f(\rho) = A\rho^\beta$, and ζ being constant

We shall assume that $\beta \geq 1$. From Eq. (0.8) it is apparent that the last term in the denominator dominates for large ρ. Near the singularity we obtain the form

$$\rho(t) \sim (t_{s\zeta} - t)^{\frac{-2}{2\beta-1}}, \quad t \to t_{s\zeta}, \tag{22}$$

which generalizes Eq. (0.19) and reduces to it when $\beta = 1$. Thus $\rho \to \infty$ implying, according to Eq. (1.2), that also $|p| \to \infty$. The Hubble parameter becomes

$$H(t) \sim (t_{s\zeta} - t)^{\frac{-1}{2\beta-1}}. \tag{23}$$

If $\beta > 1$, $a \to a_s$ (a finite value) when $t \to t_{s\zeta}$. The singularity is of Type III. If $\beta = 1$, the singularity is of Type I.

The material of this subsection was discussed also in Ref. [26], whereas the equation of state corresponding to (0.21) and(0.22) was discussed by Nojiri and Odintsov [27].

3.3 Modified gravity, w being constant, and $\zeta = \tau\theta^{2\alpha-1}$

Consider now the following gravity model,

$$S = \frac{1}{2\kappa_4^2} \int d^4x\sqrt{-g}(f_0 R^\alpha + L_m), \tag{24}$$

where f_0 and α are constants, L_m being the matter Lagrangian. This model has been considered before; cf., for instance, Refs. [28, 29, 30, 23]. The case $f_0 = 1$ and $\alpha = 1$ yields Einstein's gravity. The equations of motion following from the action above are

$$-\frac{1}{2} f_0 g_{\mu\nu} R^\alpha + \alpha f_0 R_{\mu\nu} R^{\alpha-1} - \alpha f_0 \nabla_\mu \nabla_\nu R^{\alpha-1} + \alpha f_0 g_{\mu\nu} \nabla^2 R^{\alpha-1} = \kappa_4^2 T_{\mu\nu}, \tag{25}$$

where $T_{\mu\nu}$ corresponds to the term L_m in the Lagrangian. For the cosmic fluid we have

$$T_{\mu\nu} = \rho U_\mu U_\nu + \tilde{p} h_{\mu\nu}, \tag{26}$$

where $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ is the projection tensor and $\tilde{p} = p - \zeta\theta$ the effective pressure. In comoving coordinates, $U^0 = 1$, $U^i = 0$. We assume now the simple equation of state given in Eq. (1.1).

Of main interest is the (00)-component of Eq. (0.24). Using that $R = 6(\dot{H} + 2H^2)$, $T_{00} = \rho$, as well as the energy conservation equation (0.5) which in turn follows from $\nabla^\nu T_{\mu\nu} = 0$, we obtain

$$\frac{3}{2}\gamma f_0 R^\alpha + 3\alpha f_0 [2\dot{H} - 3\gamma(\dot{H} + H^2)] R^{\alpha-1} + 3\alpha(\alpha - 1) f_0 [(3\gamma - 1) H\dot{R} + \ddot{R}] R^{\alpha-2}$$

$$+3\alpha(\alpha - 1)(\alpha - 2) f_0 \dot{R}^2 R^{\alpha-3} = 9\kappa_4^2 \zeta H, \tag{27}$$

with $\gamma = w + 1$. The important point now is that this complicated equation for $H(t)$ is satisfied with the following form

$$H = H_*/X, \quad \text{where} \quad X = 1 - BH_*t, \tag{28}$$

B being a nondimensional parameter. For Big Rip to occur, B has to be positive.

Taking the bulk viscosity to have the form

$$\zeta = \tau\theta^{2\alpha-1} = \tau(3H)^{2\alpha-1} \tag{29}$$

with τ a positive constant, the time-dependent factors in Eq. (0.26) drop out. There remains an algebraic equation, determining B.

Of main interest is the time-dependent forms

$$\zeta = \tau(3H_*/X)^{2\alpha-1}, \quad \rho = \rho_*/X^{2\alpha}. \tag{30}$$

As an example, the case $\alpha = 2$ turns out to yield a cubic equation for B. There is one positive root (assuming f_0 positive), leading to a viscosity-generated Big Rip. If $\alpha < 0$, typically $\alpha = -1$, there may still be positive solutions for B implying that $H = H_*/X$ is diverging. By contrast, $\zeta \propto X^{-(2\alpha-1)}$ and $\rho \propto X^{-2\alpha}$ go to zero.

4 Relationship to 5D viscous theory

Let us investigate the possible link between the 4D theory above and the analogous viscous theory in 5D space. To this end we consider a spatially flat ($k = 0$) brane located at the fifth dimension $y = 0$, surrounded by an anti-de-Sitter (AdS) space. If the 5D cosmological constant, called Λ, is negative, the configuration is that of the Randall-Sundrum II model (RSII) [31]. The 5D coordinates are denoted $x^A = (t, \mathbf{x}, y)$, and the 5D gravitational coupling is $\kappa_5^2 = 8\pi G_5$. The Einstein equations are

$$R_{AB} - \frac{1}{2}g_{AB}R + g_{AB}\Lambda = \kappa_5^2 T_{AB}, \tag{31}$$

and the metric is

$$ds^2 = -n^2dt^2 + a^2\delta_{ij}dx^idx^j + dy^2, \tag{32}$$

where $n(t,y)$ and $a(t,y)$ are to be determined from the Einstein equations.

Of main interest are the (tt) and (yy) components of the field equations. They are

$$3\left\{\left(\frac{\dot{a}}{a}\right)^2 - n^2\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\right\} - \Lambda n^2 = \kappa_5^2 T_{tt}, \tag{33}$$

$$3\left\{\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{1}{n^2}\left[\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n}\right) + \frac{\ddot{a}}{a}\right]\right\} + \Lambda = \kappa_5^2 T_{yy} \tag{34}$$

(cf. for instance, Refs. [32, 33, 34, 19, 26]). Overdots and primes mean derivatives with respect to t and y respectively. On the brane $y = 0$ we assume there is a constant tension σ, and an isotropic fluid with time-dependent energy density $\rho = \rho(t)$. The energy-momentum tensor is now

$$T_{AB} = \delta(y)(-\sigma\delta_{\mu\nu} + \rho U_\mu U_\nu + \tilde{p}h_{\mu\nu})\delta_A^\mu \delta_B^\nu. \tag{35}$$

Applying the junction conditions across the brane we obtain, for arbitrary y, after integration with respect to y [35],

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{1}{6}\Lambda + \left(\frac{a'}{a}\right)^2 + \frac{C}{a^4}, \tag{36}$$

$$H_0^2 = \frac{1}{6}\Lambda + \frac{\kappa_5^4}{36}(\sigma + \rho)^2. \tag{37}$$

We here let subscript zero refer to the brane. On the brane, $n_0(t) = 1$. Recall that Λ and σ are constants, and that Eq. (0.2) is a 5D, not a 4D, equation. Its essential new feature is that it contains a ρ^2 term. The equation functions as a bridge between 4D and 5D cosmologies.

We observe the solution for $a_0(t)$ if $\rho = 0$:

$$a_0(t) = e^{\sqrt{\lambda}\,t}, \quad \lambda = \frac{1}{6}\Lambda + \frac{1}{36}\kappa_5^4\sigma^2, \tag{38}$$

normalized such that $a_0(0) = 1$.

Inserting $\rho = \rho_*/X^{2\alpha}$ into Eq. (0.2) we get

$$H_0^2 = \frac{1}{6}\Lambda + \frac{\kappa_5^4}{36}\left[\sigma + \frac{\rho_*}{(1 - BH_*t)^{2\alpha}}\right]^2. \tag{39}$$

Near the Big Rip, $t_s = 1/(BH_*)$, the quantities Λ and σ become unimportant, and we get

$$a_0(t) \sim \exp\left[\frac{(\kappa_5^2/6)\rho_*}{(2\alpha-1)(BH_*)^{2\alpha}(t_s-t)^{2\alpha-1}}\right], \tag{40}$$

showing that if $\alpha > 1/2$, $a_0(t)$ has an essential singularity. Einstein's gravity corresponds to $\alpha = 1$. The singularity becomes stronger, the higher is the value of α. If $\alpha < 1/2$, $a_0(t)$ does not diverge at t_s. From Eq. (0.36),

$$a^2(t,y) = \frac{1}{2}a_0^2(t)\left[\left(1+\frac{\kappa_5^4\,\sigma^2}{6\Lambda}\right)+\left(1-\frac{\kappa_5^4\,\sigma^2}{6\Lambda}\right)\cosh(2\mu\,y)-\frac{\kappa_5^2\,\sigma}{3\mu}\sinh(2\mu|y|)\right], \tag{41}$$

with $\mu = \sqrt{-\Lambda/6}$. The important point here is that the *Big Rip divergence on the brane becomes transferred to the bulk.* The bulk scale factor $a(t,y)$ diverges for arbitrary y at $t = t_s$ if $a_0(t)$ diverges at t_s. There is no fundamental difference between an Einstein fluid and a modified gravity fluid in this respect; their behavior is essentially the same.

In summary, we have discussed viscous dark energy as a particular representative of inhomogeneous equation-of-state fluids and the appearance of finite-time future singularities for such energies. It is of interest to note that due to the relationship between modified gravity and inhomogeneous equation-of-state ideal fluids [36], our findings may be useful in the study of future singularities in modified gravity [37].

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Dark energy, exotic matter and properties of horizons in black hole physics and cosmology

Dedicated to Sergei D. Odintsovon the occasion of his 50th birthday

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Abstract

We summarize recent results on properties of near-horizon metrics in different spherically symmetric space-times, including Kantowski-Sachs cosmological models whose evolution begins with a horizon (the so-called Null Big Bang) and static metrics related to black holes. We describe the types of matter compatible with cosmological and black hole horizons. It turns out, in particular, that a black hole horizon can be in equilibrium with a fluid of disordered cosmic strings ("black holes can have curly hair"). We also discuss different kinds of horizons from the viewpoint of the behavior of tidal forces acting on an extended body and recently classified as "usual", "naked" and "truly naked" ones; in the latter case, tidal forces are infinite in a freely falling reference frame. It is shown that all truly naked horizons, as well as many of those previously characterized as naked and even usual ones, do not admit an extension and therefore must be considered as singularities. The whole analysis is performed locally (in a neighborhood of a candidate horizon) in a model-independent manner. Finally, the possible importance of some of these models in generating dynamic, perturbatively small vacuum fluctuation contributions to the cosmological constant (within a cosmological Casimir-effect approach to this problem) is discussed too.

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1 Introduction

The remarkable discovery that our Universe is accelerating [1] and its explanation, in the framework of general relativity, in terms of the so-called dark energy, have posed a number of questions. The distinctive feature of dark energy, irrespective of its specific nature, consists in the violation of the standard energy conditions, including the Null Energy Condition (NEC). Unusual properties of this hypothetic source make us return to the issues which had been seemingly clarified a long time ago but for sources that satisfy the standard energy conditions. The cosmological challenge has an impact on other areas of gravitational physics. It concerns the existence and properties of wormholes for which NEC violation is necessary. In black hole physics, the necessary conditions of regularity include the (marginal) validity of the NEC at the horizon [2], but the entire relationship between the properties of matter and the near-horizon geometry remains unclear. In addition, NEC violation can play a significant role in the possible emergence of the so-called truly naked black holes (TNBHs) [3, 4], a class of objects in which infinite tidal accelerations in a freely falling reference frame is compatible with finiteness of the algebraic curvature invariants like the Kretschmann scalar. In cosmology, near-horizon phenomena are especially relevant in the context of the so-called Null Big Bang scenarios [5, 6] where the cosmological evolution itself begins with a horizon. Also, the possible importance of some of these models in generating dynamic, perturbatively small vacuum fluctuation contributions to the cosmological constant (within the dynamical Casimir effect [7] approach to this problem) will be considered, too. In this paper, we briefly review some recent results in this area. We will consider three different but related issues: Null Big Bang scenarios, possible black hole hair of matter characterized by macroscopic equations of state, and a relationship between two different classifications of near-horizon geometries according to their analyticity properties (hence extensibility beyond the horizon) and the properties of tidal forces acting on a freely falling body.

In this paper, for simplicity, we restrict ourselves to spherically symmetric space-times, though extension of the results to more general geometries would surely be of interest. More details can be found in our works [8, 9, 10].

2 Null Big Bang and its matter sources

We begin our considerations with spherically symmetric cosmological models characterized by the general Kantowski-Sachs (KS) metric

$$ds^2 = b^2 dt^2 - a^2 dx^2 - r^2(t) d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{1}$$

and supported by a source with the stress-energy tensor

$$T_{\mu\,(\mathrm{tot})}^{\nu} = T_{\mu\,(\mathrm{vac})}^{\nu} + T_{\mu\,(\mathrm{matt})}^{\nu}, \tag{2}$$

where

$$T_{\mu\,(\mathrm{vac})}^{\nu} = \mathrm{diag}(\rho_v,\ \rho_v,\ -p_{v\perp},\ -p_{v\perp}) \tag{3}$$

describes a "vacuum fluid" (defined by the condition $T_{0\,(\mathrm{vac})}^{0} = T_{1\,(\mathrm{vac})}^{1}$ which guarantees invariance of $T_{\mu\,(\mathrm{vac})}^{\nu}$ under any Lorentz boosts in the distinguished x-direction [11]) and

$$T_{\mu\,(\mathrm{matt})}^{\nu} = \mathrm{diag}(\rho_m,\ -p_{mx},\ -p_{m\perp},\ -p_{m\perp}) \tag{4}$$

is the contribution of matter (anisotropic fluid) taken in the most general form compatible with the symmetry of the metric (1). We shall see that the properties of the system strongly depend on whether or not there is a "vacuum" admixture to such matter.

In what follows, it is helpful to use the so-called quasiglobal time coordinate, such that $b = a^{-1}$. The coordinate defined in this way, as well as its counterpart in static spherically-symmetric metrics, has two important advantages [12, 13]: (i) it always takes finite values $t = t_h$ at Killing horizons that separate static or cosmological regions of space-time from one another; (ii) near a horizon, the increment $t - t_h$ is a multiple (with a nonzero constant factor) of the corresponding increments of manifestly well-behaved Kruskal-type null coordinates, used for analytic continuation of the metric across the horizon. This condition implies the analyticity requirement for both metric functions $a^2(t)$ and $r^2(t)$ at $t = t_h$. Though, for our consideration, it is quite sufficient to require that these functions belong to class C^2 of smoothness.

With this coordinate gauge, two independent combinations of Einstein's equations, chosen as $\binom{0}{0} - \binom{1}{1}$ and $\binom{0}{0}$, read (the dot denotes d/dt)

$$\frac{2\ddot{r}}{r}a^2 = -8\pi(\rho_m + p_{mx}). \tag{5}$$

$$\frac{1}{r^2}(1 + \dot{r}^2 a^2 + 2a\dot{a}r\dot{r}) = 8\pi(\rho_m + \rho_v). \tag{6}$$

Assuming the absence of interaction between matter and vacuum, the conservation law $\nabla_\nu T^\nu_\mu = 0$ should hold for each of them separately. Taking the component with $\mu = 0$, we obtain

$$\dot{\rho}_m + \frac{\dot{a}}{a}(\rho_m + p_{mx}) + \frac{2\dot{r}}{r}(\rho_m + p_{m\perp}) = 0 \tag{7}$$

for matter and a similar equality for vacuum.

Let us assume $\rho_m \geq 0$ and consider different kinds of matter: "normal" matter that respects the NEC,

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0, \qquad \xi_\mu\xi^\mu = 0, \tag{8}$$

and "phantom" matter that violates it. Taking in (8) the null vector $\xi^\mu = (a,\ a^{-1},\ 0,\ 0)$ we obtain the necessary conditions for NEC validity

$$\rho_m + p_{mx} \geq 0. \tag{9}$$

For normal matter, by definition, Eq. (9) holds, and consequently, according to Eq. (5), $\ddot{r} \leq 0$. So we can repeat the argument of [6]: let the system be expanding ($\dot{r} > 0$) at some t_1. Then, either $r \to 0$ at some earlier instant $t_s < t_1$ (which means a curvature singularity) or the singularity is not reached, which can only happen due to a Killing horizon at some instant $t_h > t_s$. We have the following general result:

(i) *With any normal matter, regular cosmological evolution can only begin with a Killing horizon.*

Now, let us assume that there is a horizon at some $t = t_h$, so that, as $t \to t_h$, r remains finite while

$$a^2(t) \approx a_0(t - t_h)^n, \qquad n \in \mathbb{N}, \tag{10}$$

where n is the order of the horizon. Then it immediately follows from (5) and the horizon regularity requirement (which implies analyticity of $r(t)$ and, in particular, finiteness of $\ddot{r}$) that

$$\rho_m + p_{mx} \to 0 \quad \text{as} \quad t \to t_h. \tag{11}$$

Furthermore, we can generically assume that near the horizon the pressure of our matter behaves as $p_{mx} \approx w\rho_m$, $w = \text{const}$. Then (assuming $|p_\perp|/\rho < \infty$) Eq. (7) implies the approximate equality

$$\rho_m \approx \text{const} \cdot a^{-(w+1)}, \tag{12}$$

near the horizon. This leads to the following two inferences:

(ii) *Non-interacting normal matter cannot exist in a KS cosmology with a horizon; thus it can only appear there due to interaction with the vacuum fluid.*

(iii) *Normal matter could only appear after a null big bang due to interaction with a sort of vacuum.*

This generalizes the conclusions made in [6] for KS cosmologies with dustlike matter.

As far as phantom matter is concerned, the presence of a Killing horizon is not necessary for obtaining a nonsingular cosmology. If such a horizon does exist, phantom matter can be present but with the restriction $w \leq -3$. Then, an analysis of Eqs. (5)–(7) near the horizon shows that we can have a simple or multiple horizon with $\rho_v(r_h) \neq 0$ or only a simple one with $\rho_v(r_h) = 0$. In such cases, a universe appearing in a Null Bang is initially contracting in the two spherical directions, $\dot{r} < 0$.

There is also a variant in which the Universe began its evolution infinitely long ago from an almost static state, which kind of evolution has been called "emergent universes" [14]. It should be pointed out here that, in the KS framework, unlike isotropic cosmologies, there exists a variant of nonsingular evolution in which one of the scale factors in the metric (1), namely, $a(t)$, vanishes as $\tau \to -\infty$ (where τ is the cosmological proper time, related to the quasiglobal time t by $d\tau = \int dt/a(t)$) while the other, $r(t)$, remains finite in the same limit, and both timelike and null geodesics starting from $\tau = -\infty$ are complete. This is what can be called a *remote horizon* in the past, by analogy with remote horizons in static space-times mentioned in [13, 9].

We will illustrate this opportunity with two examples of such a generic behaviour as $\tau \to -\infty$:

$$\mathbf{A}: \qquad a \approx a_0\, \mathrm{e}^{Ht} \sim 1/|\tau|, \qquad r \approx r_0 + r_1\, \mathrm{e}^{Ht} \tag{13}$$

$$\mathbf{B}: \qquad a \approx a_0[t_0/(-t)]^q \sim |\tau|^{-q/(q+1)}, \qquad r \approx r_0 + r_1[t_0/(-t)]^{-s}, \tag{14}$$

where $a_0,\ r_0,\ r_1,\ H,\ t_0,\ s,\ q = \text{const} > 0$. Then, an analysis shows that in case A $w = -4$ and in case B $w = -3 - (s+2)/q < -3$. Also, in both cases, the conservation equation leads to the asymptotic structure of the vacuum stress tensor

$$T^\nu_{\mu\,(\text{vac})} = \text{diag}(\rho_v,\ \rho_v,\ 0,\ 0). \tag{15}$$

Thus a combination of the Null Big Bang and emergent universe scenarios is possible but only under some special conditions: matter with $w \leq -3$ and a particular structure of the vacuum stress-energy tensor in the remote past.

In our reasoning, relying on the asymptotic behaviour of the density and pressure near the horizon, we did not assume any particular equation of state and even did not restrict the behaviour of the transverse pressure except for its regularity requirement. In this sense, our conclusions are model-independent. The fact that the very assumption of the existence of a cosmological horizon entails a number of rather general conclusions resembles, to some extent, the situation in black hole physics where the presence of the horizon greatly simplifies the description of the system and reduces the number of possibilities.

3 A black hole surrounded by matter

In the previous section, we dealt with cosmological evolution. Now, let us discuss the relationship between the properties of matter and the near-horizon geometry in static, spherically symmetric space-times. Such a problem arises in black hole physics. In real astrophysical conditions, black holes do not exist in empty space but are rather surrounded by some kind of matter which is either in equilibrium with the black hole or is falling on it. Meanwhile, the famous no-hair theorems (see, e.g., [2, 15] and references therein) are not directly applicable to such situations of evident astrophysical interest.

As before, we will rely on the horizon regularity condition, the Einstein equations and the conservation law for matter. The manner of reasoning is close to that of previous section. Instead of (1), we now have the metric

$$ds^2 = A(u)dt^2 - \frac{du^2}{A(u)} - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2), \tag{16}$$

which is written using the quasiglobal radial coordinate u, similar to the quasiglobal time t of Section 2 and specified by the "gauge" condition $g_{00}g_{11} = -1$. We suppose that the vacuum fluid and matter have the stress-energy tensor (SET) given by (3) and (4), where p_{mx} (pressure of matter in the "longitudinal" direction in KS cosmology) is replaced by the radial pressure p_{mr}. Note that, in static, spherically symmetric space-times, examples of vacuum fluids are the cosmological constant ($p_{v\perp} = -\rho_v = \Lambda$), linear or nonlinear electric or magnetic fields in the radial direction ($p_{v\perp} = \rho_v$) and other forms which may be specified, e.g., by ρ_v as a function of r [11, 16, 17, 18].

Two independent combinations of Einstein's equations, similar to (5) and (6), read (the prime means d/du)

$$G_0^0 - G_1^1 \equiv 2A\frac{r''}{r} = -8\pi(\rho_m + p_{mr}), \tag{17}$$

$$G_1^1 \equiv \frac{1}{r^2}[-1 + A'rr' + Ar'^2] = -8\pi(\rho_v - p_{mr}). \tag{18}$$

We again suppose that matter and the vacuum fluid do not interact with each other. Then the conservation law for matter reads

$$p_r' + \frac{2r'}{r}(p_{mr} - p_{m\perp}) + \frac{A'}{2A}(\rho_m + p_{mr}) = 0. \tag{19}$$

Now, assuming that there is a horizon at some $u = u_h$, a necessary condition of its regularity is that in its neighbourhood

$$A(u) \approx a_0(u - u_h)^n, \qquad n \in \mathbb{N}, \tag{20}$$

where n is the order of the horizon. Another regularity condition is a smooth (at least C^2) behaviour of the other metric coefficient, $r^2(u)$.

One more assumption is that near the horizon the radial pressure of matter behaves as $p_{mr} \approx w\rho_m$, $w = \text{const}$. Then, on the basis of Eqs. (17)–(19) and the horizon regularity conditions, we can prove the following.

Theorem 1. *A spherically symmetric black hole can be in equilibrium with a static matter distribution with the SET (4) only if near the event horizon ($u \to u_h$, where u is the quasiglobal radial coordinate) either (i) $w \to -1$ (matter in this case has the form of a vacuum fluid) or (ii) $w \to -1/(1+2k)$, where $w \equiv p_r/\rho$ and k is a positive integer. In case (i),* *the horizon can be of any order n, and $\rho(u_h)$ is nonzero. In case (ii), the horizon is simple, and $\rho \sim (u - u_h)^k$.*

The generic case of such a non-vacuum hairy black hole is k=1, implying $w = -1/3$. In the case of an isotropic fluid, $p_r = p_\perp$, it corresponds to a distribution of disordered cosmic strings [19]. Since such strings are, in general, arbitrarily curved and may be closed, one can express the meaning of the theorem by the words "non-vacuum black holes can have curly hair". Recall, however, that in general our w characterizes the radial pressure, while the transverse one is only restricted by the condition $|p_\perp|/\rho < \infty$.

Other values of k ($k = 2,\ 3$ etc.) represent special cases obtainable by fine-tuning the parameter w.

In the presence of vacuum matter with the SET (3), the following theorem holds:

Theorem 2. *A spherically-symmetric black hole can be in equilibrium with a non-interacting mixture of static non-vacuum matter with the SET (4) and vacuum matter with the SET (3) only if, near the event horizon ($u \to u_h$), $w \equiv p_r/\rho \to -n/(n+2k)$, where $n \in \mathbb{N}$ is the order of the horizon, $n \le k \in \mathbb{N}$ and $\rho \sim (u - u_h)^k$.*

Thus a horizon of a static black hole can in general be surrounded by vacuum matter and matter with $w = -1/3$, which is true for any order of the horizon (i.e., including extremal and superextremal black holes) if $n = k$. There can also be configurations with $k > n$ and fine-tuned equations of state where $w = -n/(n+2k) > -1/3$. An arbitrarily small amount of other kinds of matter, normal or phantom, added to such a configuration, should break its static character by simply falling onto the horizon or maybe even by destroying the black hole. In other words, black holes may be hairy, or "dirty", but the possible kinds of hair are rather special in the near-horizon region: normal (with $p_r \ge 0$) or phantom hair are completely excluded. In an equilibrium configuration, all "dirt" is washed away from the near-horizon region, leaving there only vacuumlike or modestly exotic, probably "curly" hair.

In particular, a static black hole cannot live inside a star of normal matter with nonnegative pressure unless there is an accretion region around the horizon or a layer of string and/or vacuum matter.

We did not discuss the behaviour of $p_{m\perp}$ and $p_{v\perp}$. In fact, these quantities are inessential for our reasoning. The latter is entirely local, restricted to the neighborhood of the horizon, and the results, which involve the single parameter $w = p_r/\rho\,\big|_{\rm horizon}$, are in other respects model-independent. Meanwhile, a full analysis of specific systems would require the knowledge of the equation of state (including the properties of $p_{m\perp}$ and $p_{v\perp}$) and conditions on the metric in the whole space (e.g., the asymptotic flatness condition). Such an analysis depends on the model in an essential way and is beyond the scope of this paper. One can add that the equations of state well-behaved near the horizon are often incompatible with reasonable conditions at infinity (see, e.g., the example of an exact solution with string fluid in [9]); it simply means that such matter does not extend to infinity and can only occupy a finite region around the horizon.

Our inferences are quite general and hold for all kinds of hair: for instance, in all known examples of black holes with scalar fields (see, e.g., [20] and references therein), the SETs near the horizon must satisfy the above conditions, which may be directly verified.

Also, our approach is relevant to semiclassical black holes in equilibrium with their Hawking radiation (the Hartle-Hawking state), whose SET essentially differs from that of a perfect fluid. Since the density of quantum fields is, in general, nonzero at the horizon (see Sec. 11 of the textbook [2] for details), the regularity condition (3.3) (with t replaced by r) tells us that such quantum radiation should behave near the horizon like a vacuum fluid. Our results show that a black hole can be in equilibrium with a mixture of Hawking radiation and some kinds of classical matter with $-1 < w < 0$ (including the important case of a Pascal perfect fluid with $p_r = p_\perp$). Possible effects of this circumstance for semiclassical black holes need a further study. Moreover, large enough black holes, for which the Hawking radiation may be neglected, can be in equilibrium with classical matter alone, also including the case of a perfect fluid.

It would be of interest to generalize our results to nonspherical and rotating distributions of matter.

4 Truly naked horizons and their sources

In the previous two sections, the restriction on possible matter sources supporting geometries with Killing horizons essentially relied on the horizon regularity condition, which essentially meant analyticity. Meanwhile, the notion of regularity is by itself not as obvious as one could think. In particular, it turns out that there exist such horizons that all scalars composed algebraically from the components of the curvature tensor are finite there but some separate curvature components (responsible for the transverse tidal forces) enormously grow when approaching the horizon [13, 21, 10, 22, 23]. From a mathematical viewpoint, such cases represents interesting examples of so-called nonscalar singularities [24]. This makes especially important a careful analysis of the metric near the surfaces which can be called candidate horizons and a comparison between the properties of tidal forces on such surfaces and the conditions under which the metric can be extended beyond them.

Let the metric in the Schwarzschild-like coordinates be written as

$$ds^2 = \mathrm{e}^{2\gamma} dt^2 - \mathrm{e}^{2\alpha} dr^2 - r^2 d\Omega^2. \tag{21}$$

Let us assume that near a candidate horizon $\mathbb{H}: \ r = r_h$ (where, by definition, $\mathrm{e}^\gamma \to 0$) the metric coefficients behave as follows:

$$\mathrm{e}^{2\gamma} \sim (r - r_h)^q, \qquad \mathrm{e}^{2\alpha} \sim (r - r_h)^p, \tag{22}$$

with $p > 0$ and $q > 0$. As follows from the geodesic deviation equations, the tidal forces experienced by bodies in the gravitational field are conveniently characterized by the combination of components of the curvature tensor $Z := R^{12}{}_{12} - R^{02}{}_{02}$ in the static reference frame and by $\bar{Z} = Z\,\mathrm{e}^{-2\gamma}$ in a freely falling reference frame near $\mathbb{H}$. These quantities have been used in [4] to distinguish usual ($Z = 0 = \bar{Z}$,), naked ($Z = 0$, $\bar{Z} \neq 0$ is finite) and truly naked ($Z = 0$, $\bar{Z} = \infty$) horizons. In all cases we consider surfaces $\mathbb{H}$ at which all algebraic curvature invariants are finite, and this is so under the condition

$$p \geq 2 \quad \text{or} \quad 2 > p \geq 1, \quad p + q = 2. \tag{23}$$

The comparison is carried out by rewriting the metric (21) in terms of the quasiglobal coordinate u [Eq. (16)] and imposing the requirement that the metric coefficients $A(u)$ and $r^2(u)$ should be analytic at $\mathbb{H}$, where $A(u) = \mathrm{e}^{2\gamma(r)} = 0$. In particular, we obtain the condition

$$q(n-2) = n(p-2). \tag{24}$$

which selects a sequence of lines in the (p, q) plane, intersecting at the point $(-2, 0)$.

The results of such a comparison are presented in Table 1 [10].

We see from the table that all truly naked horizons are, in fact, singularities since the metric cannot be extended beyond them; moreover, some naked and even usual horizons turn out to be singular.

Table 1: Horizon types according to the properties of tidal forces and regularity (extensibility) of the metric; $n \in \mathbb{N}$ is the order of the horizon.

No.	$p,\ q$	type by tidal forces	regularity
1	$p = q = 1$	usual or naked	regular, $n = 1$
2	$1 < p < 3/2,\ q = 2 - p$	truly naked	singular
3	$p = 3/2,\ q = 1/2$	naked	regular, $n = 1$
4	$3/2 < p < 2,\ q = 2 - p$	usual	regular, $n = 1$
5	$p \geq 2,\ q > p$	truly naked	singular
6	$p \geq 2,\ q = p$	usual or naked	regular if $p = q = n$, otherwise singular
7	$p \geq 2,\ p - 1 < q < p$	truly naked	singular
8	$p \geq 2,\ q = p - 1$	naked	regular if $p = 1 + n/2$, otherwise singular
9	$p \geq 2,\ p - 2 < q < p - 1$	usual	regular if (24) holds, $n \in \mathbb{N}$, otherwise singular
10	$p \geq 2,\ q \leq p - 2$	usual	remote horizon

As to possible source of gravity leading to different types of horizons, the situation turns out to be the following. If we consider arbitrary one-component matter with $p_r/\rho = w =$ const $\neq -1$ (at least near the surface $\mathbb{H}$), the only possible solutions correspond to a simple horizon, such that $A(u) \sim u - u_h$, and, provided $w = -1/(1 + 2k)$ where k is a positive integer, we obtain regular solutions in full agreement with Section 3. Solutions with truly naked horizons are not obtained.

If, however, we consider a mixture of the two kinds of matter described by (4) and (3), there appear solutions containing matter with $0 > w > -1$ and $\rho \sim A^{-(w+1)/(2w)}$. Furthermore, if we turn to the curvature coordinates, we obtain, for $p \neq q$, the relation $w = -q/(q + 2p - 2)$, and it appears that $\rho > 0$ for $q > p$ and $\rho < 0$ for $q < p$. Thus, any p and q satisfying the condition (23) are admissible, except for those with $p = q$. In the latter case, solutions can also exist, with w satisfying the requirement $w > -p/(3p-2)$. All kinds of solutions mentioned in Table I are possible, and the values of w cover the whole range from 0 to -1. This is related to the underdetermined nature of the system since the function $\rho_{(\rm vac)}(u)$ remains arbitrary.

If we put $\rho_v = \Lambda/(8\pi) =$ const, thus specifying the vacuum as a cosmological constant, the Einstein equations relate the exponents p and q characterizing the metric to the matter parameter w. Namely, we have either (i) $p = q = 1$ (a simple regular horizon) and $w = -1/(1 + 2k)$ (as described in Section 3 and [9]) or (ii) $p = 2$, $q = -2w/(w + 1)$, $w \neq -1/2$. The parameter w can take any value in the range $(-1,\ 0)$ except $-1/2$.

5 Conclusion

In a model-independent way, we have etablished the correspondence between the equation of state (in terms of the parameter $w = p_r/\rho$) and the type of horizon both for cosmological scenarios and for black holes. We found the interval of w for which regular or (truly) naked horizons occur. Certain discrete values of w characterize possible “hair” around a regular black hole horizon. Thus we have used a unified approach to so seemingly different physical objects and phenomena as Null Big Bang, the hair properties of black holes and (truly) naked black holes.

This consideration, along with [6, 8], suggests an interesting type of cosmological scenarios, with such stages as (i) a static or stationary core, (ii) a de Sitter-like horizon, (iii) particle creation and isotropization, (iv) a hot stage and further on according to the Standard Model.

In the context of the early Universe, in addition to particle creation, it would be of interest to take into account one more quantum phenomenon, the ordinary [25] and the dynamical [26] Casimir effects related to the nontrivial topology of KS models, e.g., in the manner of Refs. [27, 28], and its possible influence on the structure of singularities like those discussed in this paper. It has been argued that Casimir considerations can play no role in trying to solve the problem of the cosmological constant, in its hard form. This seems actually to be true, but provided a drastic suppression of the main contributions to the same for some particular topology could be proven to happen in some model, then additional, sort of perturbative contributions coming from some adjustments in the topology or the evolution of our universe could provide a clue to solve the issue of its value being so small. It is in this context that Casimir-like calculations as mentioned could be of importance. At the very least, proving that these additional contributions are of the same order of magnitude as the observed value of the universe acceleration is already a first step, that has been undertaken in some specific cases [29].

It would also be of interest to relate the origin of singularities in KS cosmology subject to quantum effects with their effective 2D description, in the manner of [30], where quantum-corrected KS cosmologies were investigated. The effective 2D description makes the presentation qualitatively easier and may reveal a fundamental structure behind singularities, related to quantum effects.

As to singular horizons in KS cosmology as discussed in this paper (at least simple ones), they can be considered as examples of the so-called finite-time singularities. Four types of such singularities are known and classified for isotropic FRW models in [31]. Among them, the Big Rip (or type I singularity) is the most well-known and is widely discussed in connection with different models of dark energy. It is clear that in KS models, where we have two scale factors, such singularities may occur and their properties should be more diverse, and the corresponding classification should be naturally extended as compared with the one-scale-factor FRW cosmology. Such an extended classification, which should also apply to static, spherically symmetric analogues of KS cosmologies as well as to other, more complicated anisotropic cosmologies, is of significant interest. We hope to present a detailed description of such singularities in our future publications.

It is a pleasure for us to dedicate this paper to Sergei Odintsov on the occasion of his 50th birthday.

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BRST charge for nonlinear algebras

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Abstract

We study the construction of the classical nilpotent canonical BRST charge for the nonlinear gauge algebras where a commutator (in terms of Poisson brackets) of the constraints is a finite order polynomial of the constraints.

1 Introduction

The BRST charge, corresponding to the Noether current of BRST symmetry [1], is one of the most efficient tools for studing the classical and quantum aspects of constrained systems. The properties of the BRST charge, especially its nilpotency, are the base of modern quantization methods of gauge theores in both Lagrangian [2] and Hamiltonian [3] formalism.

In this paper we discuss the form of the canonical BRST charge for a general enough class of gauge theories. Classical formulation of the gauge theory in phase space is characterized by first class constraints $T_\alpha = T_\alpha(p,q)$ with p_i and q^i being canonically conjugate phase variables. Constraints T_α satisfy the involution relations in terms of the Poisson bracket

$$\{T_\alpha, T_\beta\} = f^\gamma_{\alpha\beta} T_\gamma \tag{1}$$

with structure functions $f^\gamma_{\alpha\beta}$. In Yang-Mills type theories the structure functions are constants and the nilpotent BRST charge $\mathcal{Q}$ ($\{\mathcal{Q},\mathcal{Q}\} = 0$) can be written in a closed form. For general gauge theories the structure functions depend on phase variables $f^\gamma_{\alpha\beta} = f^\gamma_{\alpha\beta}(p,q)$ and the existence theorem for the nilpotent BRST charge has been proved [4]. It allows to present $\mathcal{Q}$ by series expansion (in general, infinite) in ghost variables

$$\mathcal{Q} = c^\alpha T_\alpha - \frac{1}{2} c^\alpha c^\beta f^\gamma_{\alpha\beta} \mathcal{P}_\gamma + \cdots = \mathcal{Q}_1 + \mathcal{Q}_2 + \cdots. \tag{2}$$

Here c^α and $\mathcal{P}_\alpha$ are canonically conjugate ghost variables and the dots mean the terms of higher orders in ghost variables conditioned by p,q dependence of structure functions. The problem, which we discuss here, consists in construction for a given constrained theory the higher order contributions to $\mathcal{Q}$ in terms of its structure functions. In general, solution to this problem is unknown.

We consider a class of gauge theories which can be described in terms of constraints T_α satisfying the relation (1) with nonconstant structure functions which form a finite order polynomial in the constraints T_α

$$f^\gamma_{\alpha\beta} = F_{\alpha\beta}^{\ \ \gamma} + V^{(1)\gamma\beta_1}_{\alpha\beta} T_{\beta_1} + \cdots + V^{(n-1)\gamma\beta_1\ldots\beta_{n-1}}_{\alpha\beta} T_{\beta_1} \cdots T_{\beta_{n-1}} \tag{3}$$

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where $F_{\alpha\beta}^{\ \ \gamma}, V_{\alpha\beta}^{(1)\gamma\beta_1}, ..., V_{\alpha\beta}^{(n-1)\gamma\beta_1\dots\beta_{n-1}}$ are constants. Algebras of such kind appeared to be in conformal field theories (the so-called $\mathcal{W}_N$ algebras) [5], in theories with quantum groups [6], in higher spin theories on AdS space [7].

Construction of the nilpotent BRST charge for quadratically nonlinear algebras ($V^{(2)} = ... = V^{(n-1)} = 0$) subjected to an additional special assumption concerning structure constants $V_{\alpha\beta}^{(1)\gamma\delta} = V_{\alpha\beta}^{\gamma\delta}$ was performed in [8] with the result

$$\mathcal{Q} = c^\alpha T_\alpha - \frac{1}{2}c^\alpha c^\beta F_{\alpha\beta}^{\gamma}\mathcal{P}_\gamma - \frac{1}{2}c^\alpha c^\beta V_{\alpha\beta}^{\gamma}\mathcal{P}_\gamma - \frac{1}{24}c^\alpha c^\beta c^\gamma c^\delta V_{\alpha\beta}^{\mu\nu}V_{\gamma\delta}^{\rho\sigma}F_{\mu\rho}^{\lambda}\mathcal{P}_\nu\mathcal{P}_\sigma\mathcal{P}_\lambda. \tag{4}$$

Notice that BRST analysis for quadratic algebras of different special kinds including the case considered in [8], was also given in [9]. As to general nonlinear algebras of the form (3), to our knowledge, the problem of construction of a nilpotent the BRST charge is open in this case.

In this paper we find some special restrictions on structure constants when the nilpotent BRST charge (2) can be presented in the simplest form including terms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ only. The details are given in [10].

2 BRST charge for generic nonlinear algebras

Let us consider a theory with nonlinear algebras as described above (3). The structure constants $F_{\alpha\beta}^{\ \ \gamma}$ and $V_{\alpha\beta}^{(k-1)\alpha_1\dots\alpha_k}$ $(k = 2,3,...,n)$ are antisymmetric in lower indices and $V_{\alpha\beta}^{(k-1)\alpha_1\dots\alpha_k}$ $(k = 2,3,...,n)$ are totally symmetric in upper indices.

The Jacobi identities for these algebras have the form

$$F_{[\alpha\beta}^{\ \ \gamma}F_{\lambda]\gamma}^{\ \ \delta} = 0\ , \quad F_{[\alpha\beta}^{\ \ \rho}V_{\lambda]\rho}^{(1)\beta_1\beta_2} + V_{[\alpha\beta}^{(1)\rho(\beta_1}F_{\lambda]\rho}^{\ \ \beta_2)} = 0\ , \tag{5}$$

$$\begin{aligned} &F_{[\alpha\beta}^{\ \ \rho}V_{\lambda]\rho}^{(m)\beta_1\dots\beta_m\beta_{m+1}} + V_{[\alpha\beta}^{(m)\rho(\beta_1\dots\beta_m}F_{\lambda]\rho}^{\ \ \beta_{m+1})} + \\ &+\sum_{k=1}^{m-1} C_{mk}V_{[\alpha\beta}^{(k)\rho(\beta_1\dots\beta_k}V_{\lambda]\rho}^{(m-k)\beta_{k+1}\dots\beta_{m+1})} = 0\ (m = 2,3,...,n-1)\ , \end{aligned} \tag{6}$$

$$\sum_{k=m-n+1}^{n-1} C_{mk}V_{[\alpha\beta}^{(k)\rho(\beta_1\dots\beta_k}V_{\lambda]\rho}^{(m-k)\beta_{k+1}\dots\beta_{m+1})} = 0\ (m = n,...,2n-2)\ , \tag{7}$$

where

$$C_{mk} = \frac{(k+1)!(m-k+1)!}{(m+1)!}. \tag{8}$$

In Eqs. (6), (7) symmetrization includes two sets of symmetric indices. We assume that in the symmetrization only one representative among equivalent ones obtained by permutation of indices into these sets is presented.

Contribution of the first order in ghost fields c^α, $\mathcal{Q}_1 = c^\alpha T_\alpha$, defines the nilpotency equation in the second order which has the solution

$$\mathcal{Q}_2 = -\frac{1}{2}c^\alpha c^\beta \left(F_{\alpha\beta}^{\ \ \gamma} + \bar{V}_{\alpha\beta}^{\gamma\beta}T_\beta \right)\mathcal{P}_\gamma. \tag{9}$$

Here the notation

$$\bar{V}_{\alpha\beta}^{\gamma\beta} = \sum_{k=1}^{n-1} V_{\mu\nu}^{(k)\alpha\beta\sigma_1\dots\sigma_{k-1}}T_{\sigma_1}\cdots T_{\sigma_{k-1}} \tag{10}$$

is used. Analyzing the nilpotency equation in the third order in ghost fields c^α we can find that if the following restrictions on structure constants of the algebra (3)

$$\begin{aligned} &V_{[\alpha_1\alpha_2}^{(k)\sigma\beta_1\sigma_1\dots\sigma_{k-1}}V_{\alpha_3]\sigma}^{(m-k)\sigma_k\dots\sigma_m} = 0, \\ &k = 1,2,...,n-1, \quad m > k, \quad m = 2,3,...,2n-2. \end{aligned} \tag{11}$$

are fulfilled, then the contribution to the BRST charge in the third order is equal to zero, $\mathcal{Q}_3 = 0$. Further analysis of the nilpotency equation in the forth order leads to the following contribution to the BRST charge

$$\mathcal{Q}_4 = -\frac{1}{24} c^{\alpha_1} c^{\alpha_2} c^{\alpha_3} c^{\alpha_4} \;\; \bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2} \bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4} F^{\beta_3}_{\sigma\rho} + 2\bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4} F^{\beta_3}_{\sigma\rho} + \tilde{V}^{\sigma\beta_1}_{\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4} F^{\beta_3}_{\sigma\rho} \;\; \mathcal{P}_{\beta_1} \mathcal{P}_{\beta_2} \mathcal{P}_{\beta_3}, \quad (12)$$

where the notation

$$\tilde{V}^{\alpha\beta}_{\mu\nu} = \sum_{k=1}^{n-1} (k-1) V^{(k)\alpha\beta\sigma_1\ldots\sigma_{k-1}}_{\mu\nu} T_{\sigma_1} \cdots T_{\sigma_{k-1}}. \quad (13)$$

was used. In the case of quadratically nonlinear algebras $\bar{V}^{\gamma\beta}_{\alpha\beta} = V^{(1)\gamma\beta}_{\alpha\beta}, \tilde{V}^{\alpha\beta}_{\mu\nu} = 0$ and from (12) it follows the result (4) in the forth order. The relations

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} F^{\beta_3)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} + \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_3]\rho}_{\alpha_3\alpha_4]} F^{\beta_2}_{\sigma\rho} = 0, \quad (14)$$

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} + cycle(\beta_2, \beta_3, \lambda) = 0, \quad (15)$$

$$\tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} F^{\beta_3)}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda)}_{\sigma\rho} = 0 \quad (16)$$

derived from the Jacobi identities (5) – (7) and the restrictions (11) were used to obtain the contribution (12). We point out that the restrictions (11) lead to equalities

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0. \quad (17)$$

If now we additionally assume the following restrictions on the structure constants

$$\begin{gathered} V^{(k)\beta\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2} V^{(m-k)\beta_2\gamma\sigma_k\ldots\sigma_{m-2}}_{\alpha_3\alpha_4]} F^{\beta_3}_{\beta\gamma} = 0, \\ k = 1, ..., n-1, \quad m > k, \quad m = 2, ..., 2n-2, \end{gathered} \quad (18)$$

then we have

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} = 0 \quad (19)$$

and as the result $\mathcal{Q}_4 = 0$. Therefore there exists a unique form of the nilpotent BRST charge $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ if conditions (11), (18) are fulfilled. Although these conditions look like very restrictive, there exist the interesting algebras where they are fulfilled. For example, the conditions (11), (18) take place for Zamolodchikov's W_3 algebra with central extension and for the higher spin algebras in AdS space [7]. Of course, there exist non-linear algebras for which the conditions (11) and (18) are not fulfilled, e.g. these relations are not valid for $so(N)$-extended superconformal algebras with central extension [5] (see also [8]).

3 Summary

In this paper we have studied a construction of the nilpotent classical BRST charge for nonlinear algebras of the form (3) which are characterized by the structure constants $F^{\gamma}_{\alpha\beta}, V^{(1)\alpha_1\alpha_2}_{\alpha\beta}, ...,$ $V^{(n-1)\alpha_1\ldots\alpha_n}_{\alpha\beta}$. We have proved that if the conditions (11) and (18) are satisfied and a set of constraints T_α is linearly independent, the BRST charge is given in the universal form $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. Also we have proved that suitable quantities in terms of which one can efficiently analyze general nonlinear algebras (3) are $\bar{V}^{\mu\nu}_{\alpha\beta}, \tilde{V}^{\mu\nu}_{\alpha\beta}$.

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Twistor String Structure of the Kerr-Schild Geometry and Consistency of the Dirac-Kerr System

Paper dedicated to the Jubilee of Professor S. Odintsov

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Abstract

Kerr-Schild (KS) geometry of the rotating black-holes and spinning particles is based on the associated twistor structure which is defined by the Kerr theorem and determined by analytic generating function $F(Z)$ of the projective twistor coordinates $Z \in CP^3$.

On the other hand, there is a Newman initiated complex representation which describes the source of Kerr-Newman solution as a "particle" propagating along a complex world-line $X(t + i\sigma)$ which also determines generating function of the Kerr theorem $F(Z) = F_X(Z)$. Since a complex world line $X(t + i\sigma)$ is really a 2D world-sheet, the source of Kerr-Newman solution represents a complex Euclidean string extended in the imaginary time direction σ. Due to relation $F(Z) = F_X(Z)$, the twistor KS structure turns out to be adjoined to the Kerr's complex string source, forming a natural twistor-string construction similar to the Nair-Witten twistor-string. We show that twistor polarization of the Kerr-Newman solution may be matched with the *massless* solutions of the Dirac equation, providing consistency of the Dirac-Kerr model of spinning particle and natural area for the Nair-Witten concept of the scattering gauge amplitudes in twistor space.

1 Introduction

Kerr-Schild (KS) geometry is a background of the rotating black-hole (BH) solutions. On the other hand, the charged Kerr-Newman solution has $g = 2$ as that of the Dirac electron, and the KS geometry acquires central role in the structure of spinning particles in gravity, in particular, forming a space-time background for the Dirac-Kerr model of electron [1]. In this paper we show that consistency of the Dirac-Kerr system for massive particles may be provided by a twistor-string source formed by the *massless* solutions of the Dirac equation aligned with the twistor structure of the KS background. In many respects this twistor-string is similar to the Nair-Witten twistor-string model, allowing us to join to the Nair-Witten concept on the scattering amplitudes in twistor space

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[2, 3, 4]. The KS solutions in 4D are algebraically special solutions of type D which has two (doubled) Principal Null Congruences (PNC) corresponding to geodesic lines of outgoing or ingoing photons. Tangent vectors to these congruences, $k_\mu(x)^\pm$, are null and determine two different coordinate system of the black-hole solutions related to the 'in' or 'out' congruence, [5]. KS metrics have very simple KS form

$$g_{\mu\nu} = \eta_{\mu\nu} - 2Hk_\mu k_\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is metric of auxiliary Minkowski space-time, and the in (or out) null vector field $k^\mu(x)$, $x \in M^4$ determines symmetry of space-time, in particular, direction of gravitational 'dragging'. For rotating BH geometry the congruence is twisting which causes the difficulties for its derivation and analysis.

The obtained exact solutions of the Einstein-Maxwell system of equations [6] indicate that electromagnetic (em) field is not free with respect to the choice of the in or out representation. The direction of congruence related with em field must be the same as it is for metric, i.e. vector potential of em field A_μ has to satisfy the alignment condition, $A_\mu k^\mu = 0$. Since the classical em-fields are determined by the retarded potentials, the in-out symmetry of gravity turns out to be broken and the solutions of the Einstein-Maxwell system have to be based on the out-congruences, which has important physical consequences. The structure of Kerr congruence for the stationary Kerr-Newman solution at rest is shown in Fig.1.

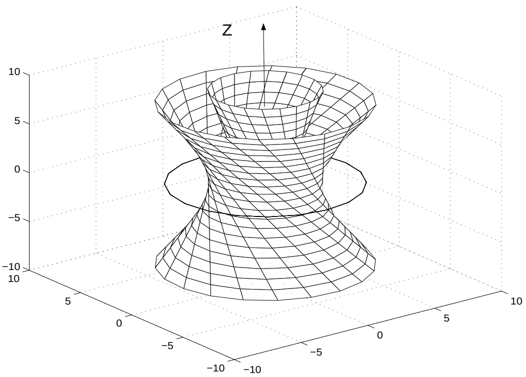

Figure 1: The Kerr singular ring and the Kerr congruence.

It displays very specific twosheetedness of the KS space-time. The Kerr singular ring represents a branch line of space-time, showing that congruence is propagating from negative sheet of metric onto positive one, and the outgoing congruence is analytic extension of the ingoing one.

The KS metrics have a rigid connection with metric of auxiliary Minkowski space-time $\eta_{\mu\nu}$ and are practically unfastened from the position of horizon. It allows one to obtain solutions which are form-invariant with respect to position of the horizons and to consider the black-holes and spinning particles without horizon on the common basis [7], as well as perform the analysis of deformations of the black-hole horizon [8, 9].

2 The Kerr theorem

Kerr congruence forms a fiber bundle of the KS space-time which is determined by the Kerr theorem [10, 11, 12, 13]. The lightlike fibers of the Kerr congruence are *real* twistors (intersections of the complex conjugated null planes [11]). The Kerr theorem gives a rule to generate geodesic and shear-free (GSF) null congruences in Minkowski space-time M^4. Due to the specifical form of the KS metrics, these congruences turn out to be also geodesic and shear-free in the curved KS space-times which justifies application of the KS formalism [6] to solutions of the Einstein-Maxwell field equations.

The Kerr theorem states that any GSF congruence in M^4 is determined by some holomorphic generating function $F(Z)$ of the projective twistor coordinates

$$Z = (Y, \quad \zeta - Yv, \quad u + Y\bar{\zeta}\ ), \tag{2}$$

being an analytic solution of the equation $F(Z) = 0$ in projective twistor space, $Z \in CP^3$. [2] The variable Y plays especial role, being projective spinor coordinate $Y = \pi^2/\pi^1$ and simultaneously the projective angular coordinate

$$Y = e^{i\phi} \tan \frac{\theta}{2}. \tag{3}$$

The dependence $Y(x)$ is the output of the Kerr theorem which determines the Kerr congruence as a field of null directions

$$k_\mu dx^\mu = P^{-1}(du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv). \tag{4}$$

The null field k^μ may also be expressed in spinor form

$$k^\mu = \bar{\pi}\sigma^\mu \pi \tag{5}$$

via the Pauli matrices σ^μ, or in the terms of spinor components [12, 3, 2]

$$k_{a\dot{a}} = \sigma^\mu_{a\dot{a}} k_\mu = \pi_a \bar{\pi}_{\dot{a}}. \tag{6}$$

Each real null ray represents a twistor which is fixed by projective twistor coordinates (2), or by homogenous coordinates

$$Z^\alpha = \{\pi^a, \mu_{\dot{a}}.\} \tag{7}$$

The spinor π^a determines the null direction k^μ, and spinor $\mu_{\dot{a}} = x_\nu \sigma^\nu_{\dot{a}a} \pi^a$ fixes the position (equation) of a real null ray (or of the corresponding complex null plane).

Quadratic in Y functions $F(Y)$ determine two roots $Y^\pm(x)$ of the equation $F = 0$ which corresponds to twosheetedness of the Kerr space-time.

It was shown in seminal work [6] that any quadratic generating function $F(Y)$ of the Kerr theorem determines a class of the exact solutions of the Einstein-Maxwell field equations. The case of quadratic in Y functions $F(Y) = A(x)Y^2 + B(x)Y + C(x) \equiv A(x)(Y - Y^+)(Y - Y^-)$ was studied in details [11, 14]. In fact, it describes the Kerr-Newman solution in a general position with arbitrary spin-orientation and Lorentz boost. This information is encoded in the coefficients A, B, C, and the KS formalism allows one to write down the corresponding form of congruence and the exact solutions for metric and em field.

3 Complex string as source of Kerr geometry

Structure of the Kerr-Newman solution admits a complex interpretation (suggested by Newman [15]) as being generated by a source propagating along a complex world-line $X^\mu(\tau) \in CM^4$, and parameters of the quadratic in Y function $F(Y)$ may be easily determined from parameters of the world-line [11, 14]. Since a complex world-line $X^\mu(\tau)$ in CM^4 is parametrized by complex time $\tau = t + i\sigma$, it is really a complex world-sheet with target space CM^4, [16]. Thereby, the complex Kerr-Newman source is equivalent to some complex string. The KS twistor structure may be described by a complex retarded-time construction, in which twistors of the Kerr congruence are the real sections of the complex light cones emanated from the complex world-line[3]. In this way the Kerr-Newman solution may be generalized to solutions of a broken $N = 2$ supergravity [17]. Projection of the *real* KS twistors onto complex world line selects on the world-sheet a strip $Im\ \tau \in [-a, a]$, forming an open Euclidean string extended along the complex time direction $\sigma = Im\tau$. The parameter σ is linked

[2] Twistor coordinates are defined via the null Cartesian coordinates $2^{\frac{1}{2}}\zeta = x + iy, \quad 2^{\frac{1}{2}}\bar{\zeta} = x - iy, \quad 2^{\frac{1}{2}}u = z - t, \quad 2^{\frac{1}{2}}v = z + t.$

[3] Newman suggested this construction for CM^4, [15], while the application to curved space-times demands the KS representation for metric [11, 4].

with angular directions of twistor lines, $\sigma = a\cos\theta$.[4] Condition of the existence of real slice for the complex twistors determines the end points of the string $\tau = t \pm ia$ [16]. In the same time, consistency of boundary conditions demands orientifolding this string, by doubling of the world-sheet and turning it into a closed but folded string [16, 4]. The joined to the end points of this string twistors form two especial null rays emanated in the North and South directions, $k_N = \bar{\pi}_N\sigma^\mu\pi_N$ and $k_S = \bar{\pi}_S\sigma^\mu\pi_S$. [1]. They play peculiar role in the KS geometry, controlling parameters of the function F, and therefore, the twistorial structure in whole. The corresponding end points of complex string may be marked by quark indices $\Psi^\alpha_{N(S)}$. It was obtained in [1, 4] that the null directions k_N and k_S may be set in the one-to-one correspondence with solutions of the Dirac equation $\Psi = (\phi_\alpha, \chi^{\dot{\alpha}})$. Putting $\pi_N = \phi$ and $\pi_S = \bar{\chi}$, one sees that the Dirac wave function Ψ manages the position, orientation and boost of the Kerr source. It gives a combined Dirac-Kerr model of spinning particle, in which twistor structure is controlled by the solutions of the Dirac equation [1]. Plane waves of the Dirac wave function propagate along the null directions k_N and k_S.

The essential drawback of this model is that the Dirac equation and its wave functions are considered in M^4, instead of the consistent treatment on the KS background. However, the exact solutions of the massive Dirac equation on the Kerr background are unknown, and moreover, there are evidences that they cannot be consistent with KS structure in principle. Note also that the plane waves are inconsistent with the Kerr background and there is a related problem with consistency of the conventional Fourier transform on the twosheeted Kerr background.

4 Consistency of the Dirac-Kerr system

Consistent solutions of the Dirac equation may easily be obtained for the massless fields which are aligned with the Kerr congruence. For the aligned to PNC solutions, the Dirac spinor $\Psi^\dagger_D = (\phi, \chi) = (A,\ B,\ C,\ D)$ has to satisfy the relation $\Psi_D k_\mu \gamma^\mu = 0$, which yields, [17, 18], $A = D = 0$, and the functions B and C take the form ([17], App. B)

$$B = f_B(\bar{Y}, \bar{\tau})/\bar{\tilde{r}}, \qquad C = f_C(Y, \tau)/\tilde{r}, \quad \tilde{r} = r + ia\cos\theta, \tag{8}$$

where f_B and f_C are arbitrary analytic functions of the complex angular variable Y and the retarded-time $\tau = t - \tilde{r}$, obeying the relations $\tau,_2 = \tau,_4 = 0$, and $Y,_2 = Y,_4 = 0$. Due to this analyticity, the wave solutions have poles at some values of Y, or a series of such poles $f_B = \sum_i a_i/(Y - Y_i)$, leading to singular beams along some of twistor lines [5]. It turns out that the treatment of the massless solutions resolves four problems:

1) existence of the exact self-consistent solutions,

2) the Dirac plane waves on the KS background turn into the lightlike beams concentrating near singular twistor lines with directions k_N and k_S.

3) there appears natural relation to the massless core of superstring theory [19] and to the massless Nair-Witten twistor models for scattering, [3, 2].

4) the Dirac massless solutions form the lightlike momentum $p(Y)$ distributed over sphere, $Y \in S^2$, and the total nonzero mass appears after averaging over sphere.

The real source of Kerr-Newman solution is a disk (D2-brane). The total mass has a few contributions [7], including the source itself and the mass-energy of the fields distributed over null rays. It depends on $Y \in S^2$ which may be factorized into angular part $e^{i\phi}$, and parameter $\sigma = a\cos\theta = a\frac{1-Y\bar{Y}}{1+Y\bar{Y}}$ along the complex Kerr string. Orientifolding is expected to be equivalent to normal ordering

$$p^\mu = <: \bar{\Psi}\gamma^\mu\Psi :> = \int_{S^2} \mu(Y) dY d\bar{Y} : \bar{\Psi}\gamma^\mu\Psi : . \tag{9}$$

[4] In fact, each point of the complex string has adjoined complex twistor parameter Y with the additional freedom of the rotations $e^{i\phi}$ around z-axis. Up to this $U(1)$ symmetry, each point of the complex Kerr string has an adjoined real twistor line of the Kerr congruence, forming the Kerr twistor-string structure [4].

[5] It is valid for em-field too, [8]

In the rest frame

$$m =< p^0(Y) >= \int_{S^2} \mu(Y) dY d\bar{Y} p^0(Y). \tag{10}$$

Therefore, mechanism of the origin of mass from the massless solutions is similar to that of the dual (super)string models and corresponds to an initial Wheeler's idea of a geon [20] ("mass without mass" obtained from the averaged em field + gravity), as well as to the initial Ramond idea on averaging of the local string structure in the dual string models [21]. The corresponding wave solutions on the Kerr background have singular beams which take asymptotically the form of the exact singular pp-wave solutions by A. Peres [22].

5 Singular Beams, Multi-particle KS solutions and Quantum Gravity

Note that singular null-strings (lightlike beams) along the Kerr congruence appear instead of the conventional smooth harmonic functions in several problems related with the exact solutions of the massless equation on the KS background [17, 25, 8]. In particular, the beam-like solutions break the topological structure of the black-hole horizons leading to instability of black holes with respect to electromagnetic excitations [25, 8].

Similar beams appear also in the multi-particle KS solutions which are described by functions $F(Y)$ of higher degrees in Y, [23]. The corresponding KS congruences form multisheeted Riemann surfaces, and the resulting space-times turn out to be multisheeted too, which generalizes the known twosheetedness of the Kerr geometry.

If we have a system of k particles with known parameters q_i, one can form the function F as a product of the k given blocks $F_i(Y) = F(Y|q_i)$ with a known dependence $F(Y|q_i)$,

$$F(Y) \equiv \prod_{i=1}^{k} F_i(Y). \tag{11}$$

The solution of the equation $F = 0$ acquires $2k$ roots $Y_i^{\pm}$, leading to 2k-sheeted twistor space.

The twistorial structure on the i-th $(+)$ or $(-)$ sheet is determined by the equation $F_i = 0$ and does not depend on the other functions $F_j, \quad j \neq i$. Therefore, the particle i does not feel the twistorial structures of other particles. Similar, singular sources of the k Kerr's spinning particles are determined by equations $F = 0, \; d_Y F = 0$ which acquires the form

$$\prod_{l=1}^{k} F_l = 0, \qquad \sum_{i=1}^{k} \prod_{l \neq i}^{k} F_l d_Y F_i = 0 \tag{12}$$

and splits into k independent relations

$$F_i = 0, \quad \prod_{l \neq i}^{k} F_l d_Y F_i = 0, \tag{13}$$

showing that i-th particle does not feel also the singular sources of other particles. The space-time splits on the independent twistorial sheets, and therefore, the twistor structure related to the i-th particle plays the role of its "internal space".

It looks wonderful. However, it is a natural generalization of the well known twosheetedness of the Kerr space-time which remains one of the mysteries of the Kerr solution for the very long time.

In spite of independence of the twistor structures positioned on the different sheets, there is an interaction between them via the singular lightlike beams (pp-strings) which appear on the common twistor lines connecting the different sheets of the particles [23] and play the role of propagators in the Nair-Witten concept on scattering amplitudes in twistor space [3]. Multisheetedness of the KS spacetimes cannot be interpreted in the frame of classical gravity, however it has far-reaching

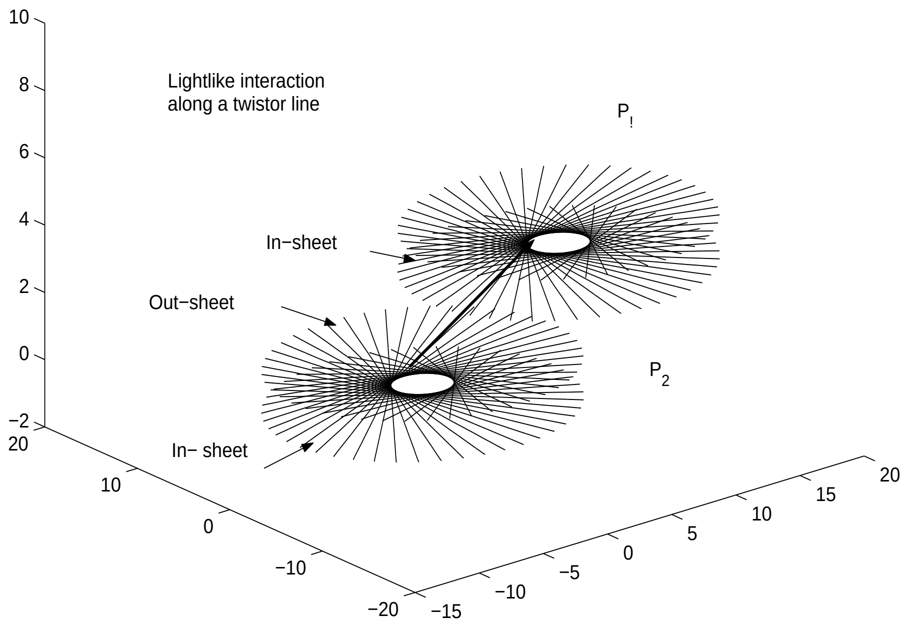

Figure 2: The lightlike interaction in Multi-sheeted twistor space via a common twistor line connecting the out-sheet of one particle to the in-sheet of another.

consequences for quantum gravity [24], in which metric is considered as operator $\hat{g}_{\mu\nu}$ leading to an "effective geometry",

$$< out|\hat{g}_{\mu\nu}|in >= g_{\mu\nu} < out|in > . \tag{14}$$

6 KS twistor structure and the Nair twistor WZW model

Nair considers S^2 as a momentum space of the massless gauge bosons and constructs a WZW model over sphere S^2 which is parametrized by the spinor coordinates π^a, [2]. This construction is close to our field of null directions $k^\mu(\pi^a)$ in M^4, which is parametrized by $Y \in S^2,\ Y = \pi^2/\pi^1$. The suggested by Nair and Witten treatment of the scattering amplitudes in twistor space CP^3 means that the time evolution occurs in M^4 along twistor null lines $Z = const.$, while S^2 assumes topological duty. Twistors form a trivial bundle over S^2, $\pi : M^4 \to S^2$. Introducing two-dimensional spinor fields $\Psi(\pi)$ on the sphere, Nair considers correlation function

$$< \Psi_r(\pi)\Psi_s(\pi') >= \frac{\delta_{rs}}{\pi\pi'} \sim \frac{\delta_{rs}}{Y - Y'} \tag{15}$$

and the local currents $J^a = \bar{\Psi}t^a\Psi$, showing that OPE takes in homogenous spinor coordinates the usual form of the corresponding Kac-Moody algebra of the WZW model[26] with central extension $k = 1$. In this case the complex-analytic structure of WZW model is lifted from S^2 to twistor space CP^3 over M^4. Twistor string structure of the KS geometry has much in common with the Nair WZW model and Witten's twistor string B-model, as well as many specifical features. However, in all the cases twistor approach displays the principal advantage, providing a natural extension of CFT from

S^2 to M^4. Since the massless KS twistor space allows one to describe massive spinning particles, the Nair-Witten treatment of the scattering amplitudes of the gauge fields may be extended in the KS geometry to describe scattering of the gauge fields on the massive black-holes and spinning particles.

The simplest illustration is the scattering of a plane wave on the Kerr's twistor structure. In the frame of the generalized KS formalism [23] it may be considered as a two-particle solution , twistor structure of which is described by the Kerr generating function $F(Y)$ of third degree in Y,

$$F_3(Y) = A(x)(Y - Y_f)(Y - Y^+)(Y - Y^-), \tag{16}$$

formed as a product of the linear in Y function

$$F_f = (Y - Y_f) \tag{17}$$

corresponding to a plane wave propagating in angular direction defined by $Y_f = const.$, and the usual quadratic in Y function of the Kerr geometry

$$F_K(Y) = A(x)(Y - Y^+(x))(Y - Y^-(x)), \tag{18}$$

forming the usual twosheeted Kerr space. Twistor structure described by F_f represents the third f-sheet covered by the system of parallel twistor lines oriented along Y_f The twistor structures of the f-sheet and the two Kerr's $K^\pm$-sheets are independent of each other, as well as the corresponding exact solutions. However, the KS-formalism [23] allows us to consider twistor structure defined by function F_3 as a single three-sheeted spacetime and obtain a single three-sheeted solution. Performing the KS computations, we find out a wonderful thing, the resulting single solution acquires on the f-sheet a pole $\sim 1/(Y - Y_f)$ along the unique twistor line which is common for the plane wave and the Kerr solution and determined by equation $Y^-(x) = Y_f = const.$ The plane wave solution on the f-sheet is turned into a pp-wave solution with a singular source at this common twistor line. Similar, there appears the pole $\sim 1/(Y - Y_f)$ on the K^--sheet of the Kerr solution along the same singular line. One sees that it is similar to the discussed by Witten transformation of plane waves in the momentum space into the waves with singular support in twistor space obtained by the suggested by Witten twistor Fourier transform [3](ch. 2.5).

7 Conclusion

We have showed that consistency of the Dirac-Kerr model for a *massive* spinning particle in gravity (in particular electron) has to be based on the *massless* solutions of the Dirac equation aligned with the KS twistor background. It provides a natural twistor string structure of the KS geometry and generalization of the Nair-Witten approach to scattering of the gauge amplitudes in twistor space to massive cases. The above considered aligned solutions of the massless Dirac equation were extracted from fermionic part of the super-Kerr-Newman solutions to broken $N = 2$ supergravity [17]. It suggests that the massless solutions of this type have natural twistor-string structure to be of interest to joining quantum theory and gravity.

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Stochastic Background of Gravitational Waves as a Benchmark for Extended Theories of Gravity

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Abstract

The cosmological background of gravitational waves can be tuned by Extended Theories of Gravity. In particular, it can be shown that assuming a generic function $f(R)$ of the Ricci scalar R gives a parametric approach to control the evolution and the production mechanism of gravitational waves in the early Universe.

In the last thirty years several shortcomings came out in Einstein General Relativity (GR) and people began to investigate whether it is the only theory capable of explaining gravitational interactions. Such issues sprang up in Cosmology and Quantum Field Theory. In the first case, the Big Bang singularity, the flatness and horizon problems led to the result that the Standard Cosmological Model is inadequate to describe the Universe in extreme regimes. Besides, GR is a *classical* theory which does not work as a fundamental theory. Due to these facts and to the lack of a self-consistent Quantum Gravity theory, alternative theories have been pursued. A fruitful approach is that of Extended Theories of Gravity (ETGs) which have become a sort of paradigm based on corrections and enlargements of GR.

These theories have acquired interest in Cosmology owing to the fact that they "naturally" exhibit inflationary behaviors [1]. Recently, ETGs are playing an interesting role in describing the today observed Universe. In fact, the amount of good quality data of last decade has made it possible to shed new light on the effective picture of the Universe. In particular, the *Concordance Model* predicts that baryons contribute only for $\sim 4\%$ of the total matter - energy budget, while the exotic *cold dark matter* represents the bulk of the matter content ($\sim 25\%$) and the cosmological constant Λ plays the role of the so called "dark energy" ($\sim 70\%$). Although being the best fit to a wide range of data, the ΛCDM model is severely affected by strong theoretical shortcomings that have motivated the search for alternative models. Dark energy models mainly rely on the implicit assumption that GR is the correct theory of gravity indeed. Nevertheless, its validity on the larger astrophysical and cosmological scales has never been tested, and it is therefore conceivable that both cosmic speed up and dark matter represent signals of a breakdown in GR [2, 3, 4, 5, 6, 7, 8].

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From an astrophysical viewpoint, ETGs do not require to find out candidates for dark energy and dark matter at fundamental level; the approach starts from taking into account only the "observed" ingredients (i.e. gravity, radiation and baryonic matter); this is in agreement with the early spirit of GR which could not act in the same way at all scales. In fact, GR has been definitively probed in the weak field limit and up to Solar System scales. However, a comprehensive effective theory of gravity, acting consistently at any scale, is far, up to now, to be found, and this demands an improvement of observational datasets and the search for experimentally testable theories. A pragmatic point of view could be to "reconstruct" the suitable theory of gravity starting from data. The main issues of this "inverse " approach is matching consistently observations at different scales and taking into account wide classes of gravitational theories where "ad hoc" hypotheses are avoided. In principle, the most popular dark energy models can be achieved by considering $f(R)$ theories of gravity and the same track can be followed to match galactic dynamics [16]. This philosophy can be taken into account also for the cosmological stochastic background of gravitational waves (GW) which, together with CMBR, would carry, if detected, a huge amount of information on the early stages of the Universe evolution.

In this paper, we face the problem to match a generic $f(R)$ theory with the cosmological background of GWs. GWs are perturbations $h_{\mu\nu}$ of the metric $g_{\mu\nu}$ which transform as 3-tensors. The GW-equations in the transverse-traceless gauge are

$$h_i^j = 0\,. \tag{1}$$

Latin indexes run from 1 to 3. Our task is now to derive the analog of Eqs. (1) from a generic $f(R)$ given by the action

$$\mathcal{A} = \frac{1}{2k}\int d^4x\sqrt{-g}f(R)\,. \tag{2}$$

From conformal transformation, the extra degrees of freedom related to higher order gravity can be recast into a scalar field

$$\widetilde{g}_{\mu\nu} = e^{2\Phi}g_{\mu\nu} \qquad \text{with} \qquad e^{2\Phi} = f'(R)\,. \tag{3}$$

Prime indicates the derivative with respect to R. The conformally equivalent Hilbert-Einstein action is

$$\widetilde{\mathcal{A}} = \frac{1}{2k}\int\sqrt{-\widetilde{g}}d^4x\left[\widetilde{R} + \mathcal{L}\quad \Phi,\Phi_{;\mu}\quad\right] \tag{4}$$

where $\mathcal{L}\quad \Phi,\Phi_{;\mu}$ is the scalar field Lagrangian derived from

$$\widetilde{R} = e^{-2\Phi}\quad R - 6\quad \Phi - 6\Phi_{;\delta}\Phi^{;\delta}\quad . \tag{5}$$

The GW-equation is now

$$\widetilde{\ }\widetilde{h}_i^j = 0 \tag{6}$$

where

$$\widetilde{\ } = e^{-2\Phi}\qquad + 2\Phi^{;\lambda}\partial_{;\lambda}\quad . \tag{7}$$

Since no scalar perturbation couples to the tensor part of gravitational waves, we have

$$\widetilde{h}_i^j = \widetilde{g}^{lj}\delta\widetilde{g}_{il} = e^{-2\Phi}g^{lj}e^{2\Phi}\delta g_{il} = h_i^j \tag{8}$$

which means that h_i^j is a conformal invariant.

As a consequence, the plane-wave amplitudes $h_i^j(t) = h(t)e_i^j\exp(ik_mx^m)$, where e_i^j is the polarization tensor, are the same in both metrics. This fact will assume a key role in the following discussion.

In a FRW background, Eq.(6) becomes

$$\ddot{h} + \quad 3H + 2\dot{\Phi}\quad \dot{h} + k^2a^{-2}h = 0 \tag{9}$$

being $a(t)$ the scale factor, k the wave number and h the GW amplitude. Solutions are combinations of Bessel's functions. Several mechanisms can be considered for the production of cosmological GWs.

In principle, we could seek for contributions due to every high-energy process in the early phases of the Universe.

In the case of inflation, GW-stochastic background is strictly related to dynamics of cosmological model. This is the case we are considering here. In particular, one can assume that the main contribution to the stochastic background comes from the amplification of vacuum fluctuations at the transition between the inflationary phase and the radiation era. However, we can assume that the GWs generated as zero-point fluctuations during the inflation undergo adiabatically damped oscillations ($\sim 1/a$) until they reach the Hubble radius H^{-1}. This is the particle horizon for the growth of perturbations. Besides, any previous fluctuation is smoothed away by the inflationary expansion. The GWs freeze out for $a/k \gg H^{-1}$ and reenter the H^{-1} radius after the reheating. The reenter in the Friedmann era depends on the scale of the GW. After the reenter, GWs can be detected by their Sachs-Wolfe effect on the temperature anisotropy $\triangle T/T$ at the decoupling. If Φ acts as the inflaton, we have $\dot{\Phi} \ll H$ during the inflation. Adopting the conformal time $d\eta = dt/a$, Eq. (9) reads

$$h'' + 2\frac{\chi'}{\chi}h' + k^2 h = 0 \tag{10}$$

where $\chi = ae^{\Phi}$. The derivation is now with respect to η. Inside the radius H^{-1}, we have $k\eta \gg 1$. Considering the absence of gravitons in the initial vacuum state, we have only negative-frequency modes and then the solution of (10) is

$$h = k^{1/2}\sqrt{2/\pi}\,\frac{1}{aH}C\exp(-ik\eta)\,. \tag{11}$$

C is the amplitude parameter. At the first horizon crossing ($aH = k$) the averaged amplitude $A_h = (k/2\pi)^{3/2}\,|h|$ of the perturbation is

$$A_h = \frac{1}{2\pi^2}C\,. \tag{12}$$

When the scale a/k becomes larger than the Hubble radius H^{-1}, the growing mode of evolution is constant, i.e. it is frozen. It can be shown that $\triangle T/T \lesssim A_h$, as an upper limit to A_h, since other effects can contribute to the background anisotropy. From this consideration, it is clear that the only relevant quantity is the initial amplitude C in Eq. (11), which is conserved until the reenter. Such an amplitude depends on the fundamental mechanism generating perturbations. Inflation gives rise to processes capable of producing perturbations as zero-point energy fluctuations. Such a mechanism depends on the gravitational interaction and then ($\triangle T/T$) could constitute a further constraint to select a suitable theory of gravity. Considering a single graviton in the form of a monochromatic wave, its zero-point amplitude is derived through the commutation relations:

$$[h(t,x),\, \pi_h(t,y)] = i\delta^3(x-y) \tag{13}$$

calculated at a fixed time t, where the amplitude h is the field and π_h is the conjugate momentum operator. Writing the Lagrangian for h

$$\widetilde{\mathcal{L}} = \frac{1}{2}\sqrt{-\widetilde{g}}\widetilde{g}^{\mu\nu}h_{;\mu}h_{;\nu} \tag{14}$$

in the conformal FRW metric $\widetilde{g}_{\mu\nu}$, where the amplitude h is conformally invariant, we obtain

$$\pi_h = \frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{h}} = e^{2\Phi}a^3\dot{h} \tag{15}$$

Eq. (13) becomes

$$\left[h(t,x),\, \dot{h}(y,y)\right] = i\frac{\delta^3(x-y)}{a^3e^{2\Phi}} \tag{16}$$

and the fields h and $\dot{h}$ can be expanded in terms of creation and annihilation operators. The commutation relations in conformal time are

$$hh'^{*} - h^{*}h' \ = \frac{i(2\pi)^3}{a^3e^{2\Phi}}\,. \tag{17}$$

From (11) and (12), we obtain $C = \sqrt{2}\pi^2 H e^{-\Phi}$, where H and Φ are calculated at the first horizon-crossing and, being $e^{2\Phi} = f'(R)$, the relation

$$A_h = \frac{H}{\sqrt{2f'(R)}}\,, \tag{18}$$

holds for a generic $f(R)$ theory. This is the central result of this paper and deserves some discussion. Clearly the amplitude of GWs produced during inflation depends on the theory of gravity which, if different from GR, gives extra degrees of freedom. On the other hand, the Sachs-Wolfe effect could constitute a test for gravity at early epochs. This probe could give further constraints on the GW-stochastic background, if ETGs are independently probed at other scales.

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Cosmological dynamics of fourth order gravity

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Abstract

We discuss the dynamical system approach applied to Higher Order Theories of Gravity. We show that once the theory of gravity has been specified, the cosmological equations can be written as a first-order autonomous system and we give several examples which illustrate the utility of our method. We also discuss a number of results which have appeared recently in the literature.

1 Introduction

Although there are many good reasons to consider General Relativity (GR) as the best theory for the gravitational interaction, in the last few decades the advent of precision cosmology tests appears more and more to suggest that this theory may be incomplete. In fact, besides the well known problems of GR in explaining the astrophysical phenomenology (i.e., the galactic rotation curves and small scale structure formation), cosmological data indicates an underlying cosmic acceleration of the Universe which cannot be recast in the framework of GR without resorting to additional exotic matter components. Several models have been proposed [1] in order to address this problem and currently the one which best fits all available observations (Supernovae Ia [2], Cosmic Microwave Background anisotropies [3], Large Scale Structure formation [4], baryon oscillations [5], weak lensing [6]), turns out to be the *Concordance Model* in which a tiny cosmological constant is present [7] and ordinary matter is dominated by a Cold Dark component. However, given that the Λ-CDM model is affected by significant fine-tuning problems related to the vacuum energy scale, it seems desirable to investigate other viable theoretical schemes.

It is for these reasons that in recent years many attempts have been made to generalize standard Einstein gravity. Among these models the so-called Extended Theory of Gravitation (ETG) and, in particular, *non-linear gravity theories* or *higher-order theories of gravity* (HTG) have provided interesting results on both cosmological [9, 8, 1, 11, 12, 13] and astrophysical [11, 14] scales. These models are based on gravitational actions which are non-linear in the Ricci curvature R and/ or contain terms involving combinations of derivatives of R [15, 16, 17]. The peculiarity of these models is related to the fact that the gravitational field equations can be recast in such a way that the higher order corrections provide an energy - momentum tensor of geometrical origin describing an "effective" source term on the right hand side of the standard Einstein field equations [8, 1]. In this scenario, the cosmic acceleration can be shown to result from such a new geometrical contribution to the cosmic energy density budget, due to higher order corrections to the Hilbert-Einstein Lagrangian.

Because the field equations resulting from HTG are extremely complicated, the theory of dynamical systems provides a powerful scheme for investigating the physical behaviour of such theories (see for example [18, 19]). In fact, studying cosmologies using the dynamical systems approach has the advantage of providing a relatively simple method for obtaining exact solutions (even if these only represent the asymptotic behavior) and obtain a (qualitative) description of the global dynamics of these models. Consequently, such an analysis allows for an efficient preliminary investigation of these theories, suggesting what kind of models deserve further investigation. Of particular importance are those theories that admit solutions that have an expansion history similar to the standard ΛCDM model and are therefore worth considering as background models for a description of the growth of structure in HTG [20].

In this paper, using the Dynamical Systems Approach (DSA) approach suggested by Collins and then by Ellis and Wainwright (see [21] for a wide class of cosmological models in the GR context), we develop a completely general scheme, which in principle allows one to analyze every fourth order gravity Lagrangian. Our study generalizes [18], which considered a generic power law function of the Ricci scalar $f(R) = R^n$ and extends the general approach given in a recent paper [22]. Here a general analysis was obtained using a one-parameter description of any $f(R)$ model, which unfortunately turns out to be somewhat misleading.

The aim of this paper is to illustrate the general procedure for obtaining a phase space analysis for any analytical $f(R)$ Lagrangian, which is regular enough to be well defined up to the third derivative in R. After a short preliminary discussion about fourth order gravity, we will discuss this general procedure, giving particular attention to clarifying the differences between our approach and the one worked out in [22]. In order to illustrate these differences and the problems that exist in [22], we will apply our method to two different families of Lagrangian $R^p \exp qR$ and $R + \chi R^n$. The last part of the paper is devoted to discussion and conclusions. Unless otherwise specified, we will use natural units ($\hbar = c = k_B = 8\pi G = 1$) and the $(+,-,-,-)$ signature.

2 Fourth Order Gravity Models

If one relaxes the assumption of linearity of the gravitational action the most general fourth order Lagrangian in an homogeneous and isotropic spacetime can be written as:

$$L = \sqrt{-g}\,[f(R) + \mathcal{L}_M] \ . \tag{1}$$

By varying equation (1), we obtain the fourth order field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = f'(R)^{;\alpha\beta}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}\right) + \tilde{T}^M_{\mu\nu}\,, \tag{2}$$

where $\tilde{T}^M_{\mu\nu} = \dfrac{2}{\sqrt{-g}}\dfrac{\delta(\sqrt{-g}L_M)}{\delta g_{\mu\nu}}$ and the prime denotes the derivative with respect to R. Standard Einstein equations are immediately recovered if $f(R) = R$. When $f'(R) \neq 0$ the equation (2) can be recast in the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{TOT}_{\mu\nu} = T^R_{\mu\nu} + T^M_{\mu\nu}\,, \tag{3}$$

where

$$T^R_{\mu\nu} = \frac{1}{f'(R)}\quad \frac{1}{2}g_{\mu\nu}\quad f(R) - Rf'(R)\quad + f'(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) \quad , \tag{4}$$

represent the stress energy tensor of an effective fluid sometimes referred to as the "curvature fluid" and

$$T^{M}_{\mu\nu} = \frac{1}{f'(R)}\tilde{T}^{M}_{\mu\nu} \ , \tag{5}$$

represents an effective stress-energy tensor associated with standard matter.

The conservation properties of these effective fluids are given in [20, 23] but it is important to stress that even if the *effective tensor* associated with the matter is not conserved, standard matter still follows the usual conservation equations $\tilde{T}^{M;\nu}_{\mu\nu} = 0$.

Let us now consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = dt^2 - a^2(t) \quad \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{6}$$

For this metric the action the field equations (4) reduce to

$$H^2 + \tfrac{k}{a^2} = \tfrac{1}{3f'}\left\{\tfrac{1}{2}\left[f'R - f\right] - 3H\dot{f}' + \mu_m\right\} \ , \tag{7}$$

$$2\dot{H} + H^2 + \tfrac{k}{a^2} = -\tfrac{1}{f'}\left\{\tfrac{1}{2}\left[f'R - f\right] + \ddot{f}' - 3H\dot{f}' + p_m\right\} \ , \tag{8}$$

and

$$R = -6 \quad 2H^2 + \dot{H} + \frac{k}{a^2} \quad , \tag{9}$$

where $H \equiv \dot{a}/a$, $f' \equiv \frac{df(R)}{dR}$ and the "dot" is the derivative with respect to t. The system (7) is closed by the Bianchi identity for $\tilde{T}^{M}_{\mu\nu}$:

$$\dot{\mu}_m + 3H(\mu_m + p_m) = 0 \ , \tag{10}$$

which corresponds to the energy conservation equation for standard matter.

3 The dynamical system approach in fourth order gravity theories

Following early attempts (see for example [24]), the first extensive analysis of cosmologies based on fourth order gravity theory using the DSA as defined in [21] was given in [18]. Here the phase space of the power law model $f(R) = \chi R^n$ was investigated in great detail, exact solutions were found and their stability determined. Following this, several authors have applied a similar approach to other types of Lagrangians [26], and very recently this scheme was generalized in [22].

In this paper we give a self consistent general technique that allows us to perform a dynamical system analysis of any analytic fourth order theory of gravity in the case of the FLRW spacetime.

The first step in the implementation of the DSA is the definition of the variables. Following [18], we introduce the general dimensionless variables :

$$x = \frac{\dot{f}'}{f'H}, \qquad y = \frac{R}{6H^2}, \qquad z = \frac{f}{6f'H^2}, \qquad \Omega = \frac{\mu_m}{3f'H^2}, \qquad K = \frac{k}{a^2H^2} \ , \tag{11}$$

where μ_m represents the energy density of a perfect fluid that might be present in the model.

The cosmological equations (7) are equivalent to the autonomous system:

$$\frac{dx}{dN} = \varepsilon\left(2K + 2z - x^2 + (K + y + 1)x\right) + \Omega\varepsilon\left(-3w - 1\right) + 2, \tag{12}$$

$$\frac{dy}{dN} = y\varepsilon\left(2y + 2K + x\Upsilon + 4\right), \tag{13}$$

$$\frac{dz}{dN} = z\varepsilon\left(2K - x + 2y + 4\right) + \varepsilon\, xy\Upsilon, \tag{14}$$

$$\frac{d\Omega}{dN} = \Omega\varepsilon\left(2K - x + 2y - 3w + 1\right), \tag{15}$$

$$\frac{dK}{dN} = 2K\varepsilon\left(K + y + 1\right), \tag{16}$$

where $N = |\ln a|$ is the logarithmic time and $\varepsilon = |H|/H$. In addition, we have the constraint equation

$$1 = -K - x - y + z + \Omega\,, \tag{17}$$

which can be used to reduce the dimension of the system. If one chooses to eliminate K, the variable associated with the spatial curvature, we obtain

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon\,(4z - 2x^2 + (z-2)x - 2y) + \Omega\varepsilon\,(x - 3w + 1),\\
\frac{dy}{dN} &= y\varepsilon\,[2\Omega + 2(z+1) + x(\Upsilon - 2)],\\
\frac{dz}{dN} &= z\varepsilon\,(2z + 2\Omega - 3x + 2)z + x\varepsilon\, y\Upsilon,\\
\frac{d\Omega}{dN} &= \Omega\,\varepsilon\,(2\Omega - 3x + 2z - 3w - 1),\\
K &= z + \Omega - x - y - 1\,.
\end{aligned} \tag{18}$$

The quantity Υ is defined, in analogy with [22], as

$$\Upsilon \equiv \frac{f'}{Rf''}\,. \tag{19}$$

The expression of Υ in terms of the dynamical variables is the key to closing the system (39) and allows one to perform the analysis of the phase space. The crucial aspect to note here is that Υ is a function of R only, so the problem of obtaining $\Upsilon = \Upsilon(x, y, z, \Omega)$ is reduced to the problem of writing $R = R(x, y, z, \Omega)$. This can be achieved by noting that the quantity

$$r \equiv -\frac{Rf'}{f}\,, \tag{20}$$

is a function of R only and can be written as

$$r = -\frac{y}{z}\,. \tag{21}$$

Solving the above equation for R allows one to write R in terms of y and z and close the system (18).

In this way, once a Lagrangian has been chosen, we can in principle write the dynamical system associated with it using (18), substituting into it the appropriate form of $\Upsilon = \Upsilon(y, z)$. This procedure does however require particular attention. For example, there are forms of the function f for which the inversion of (21) is highly non trivial (e.g., $f(R) = \cosh(R)$). In addition, the function Υ could have a non-trivial domain, admit divergences or may not be in the class C^1, which makes the analysis of the phase space a very delicate problem. Finally, the number m of equations of (18) is always $m \geq 3$ and this implies that fourth order gravity models can admit chaotic behaviour. While this is not surprising, it makes the deduction of the non–local properties of the phase space a very difficult task.

The solutions associated with the fixed points can be found by substituting the coordinates of the fixed points into the system

$$\dot{H} = \alpha H^2\,, \qquad \alpha = -1 - \Omega_i + x_i - z_i\,, \tag{22}$$

$$\dot{\mu}_m = -\frac{3(1+w)}{\alpha\, t}\mu_m\,, \tag{23}$$

where the subscript "i" stands for the value of a generic quantity in a fixed point. This means that for $\alpha \neq 0$ the general solutions can be written as

$$a = a_0(t - t_0)^{1/\alpha}\,, \tag{24}$$

$$\mu_m = a_0(t - t_0)^{-\frac{3(1+w)}{\alpha}}\,. \tag{25}$$

The expression above gives the solution for the scale factor and the evolution of the energy density for every fixed point in which $\alpha \neq 0$. When $\alpha = 0$ the (22) reduces to $\dot{H} = 0$ which correspond to either a static or a de Sitter solution.

The solutions obtained in this way have to be considered particular solutions of the cosmological equations which are found by using a specific ansatz (i.e. the fixed point condition [25]). For this reason it is important to stress that only direct substitution of the results derived from this approach in the cosmological equations can ensure that the solution is physical (i.e. it satisfies the cosmological equations (7)). This check is also useful for understanding the nature of the solutions themselves e.g., to calculate the value of the integration constant(s).

Also, the fact that different fixed points correspond to the same solutions is due to the fact that at the fixed points the different terms in the equation combine in such a way to obtain the same evolution of the scale factor. This means that although two solutions are the same in terms of time dependence, the physical mechanism that realizes them can be different

One difference betwen our approach and the one in [22] is that we consider a non-zero spatial curvature k. The choice of including a non-zero spatial curvature k has been made with the aim of obtaining a completely general analysis of a fourth order cosmology from the dynamical systems point of view. In addition, since most of the observational values for the cosmological parameters are heavily model dependent, we chose to limit as much as possible the introduction of priors in the analysis. However, as we write in the footnote in section 3, the limit of flat spacelike sections ($K \to 0$) can be obtained in a straightforward way for our examples. In fact, each fixed point is associated with a specific value of the variable K (i.e. a value for k) and the stability of these points is independent of the value of K. As matter of fact in order to consider fixed points living on the hypersurface $K = 0$, one has just to exclude the fixed points associated with $K \neq 0$. In addition to that, looking at the dynamical equations one realizes that $K = 0$ is an invariant submanifold, i.e., an orbit with initial condition $K = 0$ will not escape the subspace $K = 0$ and orbits with initial condition $K \neq 0$ can approach the hyperplane $K = 0$ only asymptotically. As a consequence, one does not need to have any other information on the rest of the phase space to characterize the evolution of the orbits in the submanifold $K = 0$. The authors of [22] proposed that the function $m(r) = \Upsilon(r)^{-1}$ could be used as a parameter associated with the choice of $f(R)$, thus obtaining a "one parameter approach" to the dynamical systems analysis of $f(R)$ gravity. Unfortunately their method has several problems that lead to incorrect results. These problems can be avoided only if one considers the framework presented above.

Let us look at this issue in more detail [1]. In [22] the system equivalent to (39) is associated with the relation

$$\frac{dr}{dN} = r(1 + m(r) + r)\frac{\dot{R}}{HR}\,, \tag{26}$$

which is clearly a combination of the equations for z and y. In order to ensure that the variable r and consequently the parameter m is constant they require the RHS of the above equation to be zero. Their solution to this problem is the condition $1 + m(r) + r = 0$, which is an equation for r when the function $m(r)$ has been substituted for and is also the bases of their method of analysis.

The problem here is that this equation has not been fully expressed in terms of the dynamical system variables. In fact, one can rewrite (26) in the form:

$$\frac{dr}{dN} = \frac{r(1 + m(r) + r)}{m(r)}x\,, \tag{27}$$

which means that the condition $\dfrac{dr}{dN} = 0$ in fact corresponds to

$$\frac{r(1 + m(r) + r)}{m(r)}x = 0/,, \tag{28}$$

[1]It is important to note that in [22] the signature is not the same of the one used here (e.g -,+,+,+ instead of +,-,-,-) and the definition of the variables are slightly different. The transformation from one variable to another is as follows:

$$x \to -x_1, \quad y \to -x_3, \quad z \to x_2, \quad K \to 0, \quad w \to 0.$$

However, as expected, this does not affect our conclusions.

rather than $1+m(r)+r=0$. Equation (28) has a solution if

$$x = 0, \tag{29}$$
$$r = 0, \tag{30}$$
$$\frac{(1+m(r)+r)}{m(r)} = 0\,, \tag{31}$$

and this leads to solutions for r which are in general different from the values of r obtained from $1+m(r)+r=0$. This inconsistency has major consequences for the rest of the analysis in [22], leading to changes in the number of fixed points as well as their stability (see below for details).

In fact, a more careful analysis reveals that for some of the fixed points (e.g. $P_1, ...P_4$) the values of r obtained from the relation $r=-y/z$ either cannot be determined unambiguously or do not solve the condition $1+m(r)+r=0$, which is claimed to come from (26) in [22].

This is a clear indication that the approach used in [22] is both incomplete and leads to wrong conclusions. It is also interesting to stress that if one substitutes the expression for m in terms of the dynamical system variables in (26-29) of [22], the results match the one obtained in our formalism. This implies that the reason the method described in [22] fails has its roots in the attempt to describe the phase space of a whole class of fourth order theories of gravity with only one parameter.

In the following we will present a number of examples of $f(R)$ theories that can be analyzed with this method and we compare the results obtained with those given in [22].

4 Examples of $f(R)$ - Lagrangians

In this section we will show, with the help of some examples, how the DSA developed above can be applied. In particular we will consider the cases $f(R) = R^p\exp(qR)$ and $f(R) = R+\chi R^n$. Since the aim of the paper is to provide only the general setting with which to develop the dynamical system approach in the framework of fourth order gravity, we will not give a detailed analysis of these models. Istead, we will limit ourselves to the finite fixed points, their stability and the solutions associated with them. A comparison with the results of [22] will also be presented.

4.1 The $f(R) = R^p\exp(qR)$ case

Let us consider the Lagrangian $f(R) = R^p\exp(qR)$. As explained in the previous section, the dynamical system equations for this Lagrangian can be obtained by calculating the form of the parameter Υ. We have

$$\Upsilon(y,z) = \frac{y\,z}{y^2-p\,z^2}\,. \tag{32}$$

Substituting this function into (39) we obtain

$$\frac{dx}{dN} = \varepsilon\,[4z-2x^2+(z-2)x-2y]+\Omega\varepsilon\,(x-3w+1), \tag{33}$$
$$\frac{dy}{dN} = y\varepsilon\quad 2\Omega+2z+2+\frac{x\,z}{y^2-p\,z^2}-2x\quad, \tag{34}$$
$$\frac{dz}{dN} = z\varepsilon\quad 2z+2\Omega-3x+2+\frac{x\,y}{y^2-p\,z^2}\quad, \tag{35}$$
$$\frac{d\Omega}{dN} = \Omega\,\varepsilon\,(2\Omega-3x+2z-3w-1), \tag{36}$$
$$K = z+\Omega-x-y-1\,. \tag{37}$$

The most striking feature of this system is the fact that two of the equations have a singularity in the hypersurface $y^2 = p\,z^2$. This, together with the existence of the invariant submanifolds $y=0$ and $z=0$ heavily constrains the dynamics of the system. In particular, it implies that no global attractor is present, thus no general conclusion can be made on the behavior of the orbits without first providing information about the initial conditions. The finite fixed points can be obtained by setting the LHS of (33) to zero and solving for (x, y, z, Ω), the results are shown in Table 1.

The solutions corresponding to these fixed points can be obtained by substituting the coordinates into the system (22) and are shown in Table 2 [2]. The stability of the finite fixed points can be found using the Hartman-Grobman theorem [27]. The results are shown in Table 3. Note that some of the eigenvalues diverge for $p = 0, 1$. This happens because in the operations involved in the derivation of the stability terms $p - 1$ and/or p appear in the denominators. However this is not a real pathology of the method but rather a consequence of the fact that for these two values of the parameter the cosmological equations assume a special form. In fact it is easy to prove that if one starts the calculations using these critical values of p one ends up with eigenvalues that present no divergence [23].

Let us now compare our results with the ones in [22]. The number of fixed points obtained for this Lagrangian, when $K = 0$, matches the ones obtained in [22]. This result can be explained by the fact that the solutions of the constraint equation for m (26) coincide with the ones coming from the correct constraint equation (27) (the matching between the two systems can be obtained setting $w = 0$ in Table 1). However, when one calculates the stability of these points our results are strikingly different to those presented in [22]. For example, in our general formalism it turns out that the fixed point $\mathcal{N}$ (corresponding to P_5 of [22]) is a saddle for any value of the parameter p and, as consequence, it can represent only a transient phase in the evolution of this class of models. Instead, in [22] the authors find that this point can be stable (not necessarily always a spiral) and argue that this fact prevents the existence of cosmic histories in which a decelerated expansion is followed by an accelerated one. From this they also conclude that an entire subclass of these models ($m = m(p) > 0$) can be ruled out. Our results show clearly that this is not the case. Another example is the point $\mathcal{M}$ corresponding to P_6 of [22] . In [22] the authors find that this point can be stable or a saddle as we do, but the intervals of values of the parameters for which this happens are different (see Table 3). As explained above, the reason behind these differences is the fact that the method used in [22] leads to incorrect results when, like in this case, there is no unambiguous way of determining the parameter $r = -y/z$ from the coordinates of the fixed points. Consequently the conclusions in [22] relating to the properties of these points are incorrect and have no physical meaning.

Table 1: Fixed points of $R^p \exp(qR)$.The superscript "*" represents a point corresponding to a double solution.

Point	Coordinates (x, y, z, Ω)	K
$\mathcal{A}$	$(0, 0, 0, 0)$	-1
$\mathcal{B}$	$(-1, 0, 0, 0)$	0
$\mathcal{C}$	$(-1 - 3w, 0, 0, -1 - 3w)$	-1
$\mathcal{D}$	$(1 - 3w, 0, 0, 2 - 3w)$	0
$\mathcal{E}$	$(2, 0, 2, 0)$	-1
$\mathcal{F}^*$	$(1, -2, 0, 0)$	0
$\mathcal{G}$	$(0, -2, -1, 0)$	0
$\mathcal{H}$	$(4, 0, 5, 0)$	0
$\mathcal{I}^*$	$(-3(1 + w), -2, 0, -4 - 3w)$	0
$\mathcal{L}$	$(2 - 2p, 2p(1 - p), 2 - 2p, 0)$	$2p(p - 1) - 1$
$\mathcal{M}$	$\left(\frac{4-2p}{1-2p}, \frac{(5-4p)p}{2p^2-3p+1}, \frac{5-4p}{(p-1)(2p-1)}, 0\right)$	0
$\mathcal{N}$	$\left(\frac{-3(1+w)(p-1)}{p}, \frac{3(1+w)-4p}{2p}, \frac{-4p+3w+3}{2p^2}, \frac{p(9w-2p(3w+4)+13)-3(w+1)}{2p^2}\right)$	0

[2]Note that even if the parameter q is not present in the dynamical equations it appears in the solutions because we have calculated the integration constants via direct substitution in the cosmological equations.

Table 2: Solutions associated with the fixed points of $R^p\exp(qR)$. The solutions are physical only in the intervals of p mentioned in the last column.

Point	Scale Factor	Energy Density	Physical
$\mathcal{A}$	$a(t)=(t-t_0)$	0	$p\geq 1$
$\mathcal{B}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$p\geq 2$
$\mathcal{C}$	$a(t)=(t-t_0)$	0	$p\geq 1$
$\mathcal{D}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$p\geq 2$
$\mathcal{E}$	$a(t)=(t-t_0)$	0	$p\geq 1$
$\mathcal{F}^*$	$\left\{\begin{array}{l} a(t)=a_0,\\ a(t)=a_0\,f(t)\end{array}\right.$	0	$p\geq 0$ $\left\{\begin{array}{l} p<\frac{2}{3}\\ q>0\end{array}\right.\vee\left\{\begin{array}{l} p>\frac{2}{3}\\ q<0\end{array}\right.$
$\mathcal{G}$	$\left\{\begin{array}{l} a(t)=a_0,\\ a(t)=a_0\,f(t)\end{array}\right.$	0	$p\geq 0$ $\left\{\begin{array}{l} p<\frac{2}{3}\\ q>0\end{array}\right.\vee\left\{\begin{array}{l} p>\frac{2}{3}\\ q<0\end{array}\right.$
$\mathcal{H}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$p\geq 2$
$\mathcal{I}^*$	$\left\{\begin{array}{l} a(t)=a_0,\\ a(t)=a_0\,f(t)\end{array}\right.$	0	$p\geq 0$ $\left\{\begin{array}{l} p<\frac{2}{3}\\ q>0\end{array}\right.\vee\left\{\begin{array}{l} p>\frac{2}{3}\\ q<0\end{array}\right.$
$\mathcal{L}$	$a(t)=$ $(t-t_0)\sqrt{1-2p(p-1)}$	0	$1\leq p\leq\frac{1}{2}+\frac{\sqrt{3}}{2}$
$\mathcal{M}$	$a(t)=a_0\,(t-t_0)^{\frac{2p^2-3p+1}{2-p}}$	$\mu_m=\mu_{m\,0}t^{\frac{3\left(2p^2-3p+1\right)(w+1)}{p-2}}$	$p=\frac{1}{2},1,\frac{5}{4}$
$\mathcal{N}$	$a(t)=a_0\,(t-t_0)^{\frac{2p}{3(w+1)}}$	$\mu_m=\mu_{m\,0}(t-t_0)^{-2p}$	$p=\frac{3(w+1)}{4}\quad(\mu_{m\,0}=0)$

where $f(t)=\exp(\pm\sqrt{2-3p}(t-t_0)/6\sqrt{q})$

Table 3: The stability associated with the fixed points in the model $R^p\exp(qR)$. With the index $^+$ we have indicated the attractive nature of the spiral points.

Point	Stability
$\mathcal{A}$	saddle
$\mathcal{B}$	$\left\{\begin{array}{ll} \text{repellor} & 0<w<2/3\\ \text{saddle} & \text{otherwise}\end{array}\right.$
$\mathcal{C}$	saddle
$\mathcal{D}$	$\left\{\begin{array}{ll} \text{repellor} & 2/3<w<1\\ \text{saddle} & \text{otherwise}\end{array}\right.$
$\mathcal{E}$	saddle
$\mathcal{F}$	saddle
$\mathcal{G}$	$\left\{\begin{array}{ll} \text{attractor} & 0<w<1\cup 2<p\leq\frac{68}{25}\\ \text{spiral}^+ & 0\leq w\leq 1\cup\frac{68}{25}<p<4\\ \text{saddle} & \text{otherwise}\end{array}\right.$
$\mathcal{H}$	saddle
$\mathcal{I}$	non hyperbolic
$\mathcal{L}$	$\left\{\begin{array}{ll} \text{attractor} & \frac{1}{2}-\frac{\sqrt{3}}{2}<p\leq 0\vee\frac{4}{3}\leq p<\frac{1}{2}+\frac{\sqrt{3}}{2}\\ \text{spiral}^+ & 0<p<\frac{4}{3}\\ \text{saddle} & \text{otherwise}\end{array}\right.$
$\mathcal{M}$	$\left\{\begin{array}{ll} \text{attractor} & p<\frac{1}{2}(1-\sqrt{3})\vee\frac{1}{2}(1+\sqrt{3})<p<2\\ \text{saddle} & \text{otherwise}\end{array}\right.$
$\mathcal{N}$	saddle

4.2 The case $f(R) = R + \chi R^n$

Let us discuss now the case of a Lagrangian corresponding to a power law correction of the Hilbert - Einstein gravity Lagrangian $f(R) = R + \chi R^n$. In this case, the characteristic function $\Upsilon(y,z)$ reads:

$$\Upsilon(y,z) = \frac{y}{n(z-y)}\,, \tag{38}$$

and substituting this relation into the system of equations (39) one obtains

$$\begin{aligned}
\frac{dx}{dN} &= -2x^2 + (z-2)x - 2y + 4z + \Omega(x - 3w + 1), && (39)\\
\frac{dy}{dN} &= y\varepsilon\,[2\Omega + 2(z+1) + \frac{x\,y}{n(z-y)} - 2x], && (40)\\
\frac{dz}{dN} &= 7z\varepsilon\,(2z + 2\Omega - 3x + 2) + \varepsilon\,\frac{x\,y^2}{n(z-y)}, && (41)\\
\frac{d\Omega}{dN} &= \Omega\,\varepsilon\,(2\Omega - 3x + 2z - 3w - 1), && (42)\\
K &= z + \Omega - x - y - 1\,. && (43)
\end{aligned}$$

As in the case of $f(R) = R^p \exp(qR)$, the system is divergent on a hypersurface (this time $y = z$) but it admits only one invariant submanifold, namely $y = 0$. This, again, implies that no global attractor is present and no general conclusion can be made on the behavior of the orbits without giving information about the initial conditions. The finite fixed points, their stability and the solutions corresponding to them are summarized in Tables 4, 5 and 6.

As before our results are different from those given in [22]. First of all, our set of fixed points do not coincide with the ones presented in [22]. In particular, in our analysis there is no fixed point corresponding to P_{5a}. Again, the reason for this difference is to be found in the constraint equation (26), which in this case gives the incorrect set of solutions and therefore affects the set of fixed points. In fact, if one substitutes the expression for $m(r)$ of [22] in terms of the coordinates in equations (34)-(39), it is easy to verify that two of these equations diverge at this point.

The differences between the results in our approach and the one presented in [22] are even more evident when the stability analysis is considered. For example, the point $\mathcal{E}$, corresponding to P_1, is always a saddle, except into the region $0 < n < 2$ when it is attractive. This behavior is recovered in [22] only for $-2 < n < -41/25$. Also, points $\mathcal{G}$ (corresponding to P_4 of [22]) and $\mathcal{D}$ (corresponding to P_3 of [22]), which in our approach are always saddles in the dust case, are always repellers in [22]. Finally, also the stability of $\mathcal{I}$ corresponding to P_6 appears to be different from the one presented in [22].

Table 4: Coordinate of the finite fixed points for $R + \chi R^n$ gravity.

Point	Coordinates (x, y, z, Ω)	K
$\mathcal{A}$	$(0,0,0,0)$	-1
$\mathcal{B}$	$(-1,0,0,0)$	0
$\mathcal{C}$	$(-1-3w,0,0,-1-3w)$	-1
$\mathcal{D}$	$(1-3w,0,0,2-3w)$	0
$\mathcal{E}$	$(0,-2,-1,0)$	0
$\mathcal{F}$	$(2,0,2,0)$	-1
$\mathcal{G}$	$(4,0,5,0)$	0
$\mathcal{H}$	$(2(1-n),2n(n-1),2(1-n),0)$	$2n(n-1)-1$
$\mathcal{I}$	$\left(\frac{2(n-2)}{2n-1},\frac{(5-4n)n}{2n^2-3n+1},\frac{5-4n}{2n^2-3n+1},0\right)$	0
$\mathcal{L}$	$\left(-\frac{3(n-1)(w+1)}{n},\frac{-4n+3w+3}{2n},\frac{-4n+3w+3}{2n^2},\frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	0

Table 5: The stability of the fixed points in the model $R+\chi R^n$. The quantities B_i related to the fixed point $\mathcal{L}$, represent some non fractional numerical values ($B_1 \approx 1.220$, $B_1 \approx 1.224$, $B_3 \approx 1.470$).

Point	Stability
$\mathcal{A}$	saddle
$\mathcal{B}$	$\begin{cases} \text{repellor} & 0<w<2/3 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{C}$	saddle
$\mathcal{D}$	$\begin{cases} \text{repellor} & 2/3<w<1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{E}$	$\begin{cases} \text{attractor} & \frac{32}{25}\leq n<2 \\ \text{spiral}^+ & 0<n<\frac{32}{25} \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{F}$	saddle
$\mathcal{G}$	saddle
$\mathcal{H}$	$\begin{cases} \text{attractor} & \frac{1}{2}(1-\sqrt{3})<n\leq 0 \\ \text{spiral}^+ & 0<n<1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{I}$	$\begin{cases} \text{attractor} & n<\frac{1}{2}(1-\sqrt{3})\cup n>2, \\ \text{repeller} & \begin{cases} 1<n<\frac{5}{4}, (w=0,1/3), \\ 1<n<\frac{1}{14}(11+\sqrt{37}), (w=1) \end{cases} \\ \text{saddle} & \text{otherwise,} \end{cases}$
$\mathcal{L}$	$\begin{cases} w=0,1/3 & \text{saddle,} \\ w=1 & \begin{cases} \text{repellor} & B_1<n\leq B_2\cup B_3<n<\frac{3}{2}, \\ \text{saddle} & \text{otherwise} \end{cases} \end{cases}$

Table 6: Solutions associated to the fixed points of $R+\chi R^n$. The solutions are physical only in the intervals of p mentioned in the last column.

Point	Scale Factor	Energy Density	Physical
$\mathcal{A}$	$a(t)=(t-t_0)$	0	$n\geq 1$
$\mathcal{B}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$n\geq 1$
$\mathcal{C}$	$a(t)=(t-t_0)$	0	$n\geq 1$
$\mathcal{D}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$n\geq 1$
$\mathcal{E}^*$	$\begin{cases} a(t)=a_0, \\ a(t)=a_0\,f(t) \end{cases}$ $\gamma=\frac{1}{2(1-n)}$	0	$n\geq 0$ $n<\frac{2}{3},\chi>0\ \vee$ $n>\frac{2}{3},\chi<0$
$\mathcal{F}$	$a(t)=(t-t_0)$	0	$n\geq 1$
$\mathcal{G}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$n\geq 1$
$\mathcal{H}$	$a(t)=\sqrt{1-2n(n-1)}\,(t-t_0)$	0	$1\leq n\geq\frac{1}{2}+\frac{\sqrt{3}}{2}$
$\mathcal{I}^*$	$a(t)=a_0\,(t-t_0)^{\frac{2n^2-3n+1}{2-n}}$	$\mu_m=\mu_{m\,0}t^{-\frac{3\left(2n^2-3n+1\right)(w+1)}{n-2}}$	$n=\frac{1}{2},\mu_{m\,,0}=0$
$\mathcal{L}$	$a(t)=a_0\,(t-t_0)^{\frac{2n}{3(w+1)}}$	$\mu_m=\mu_{m\,,0}(t-t_0)^{2p}$	non physical

where $f(t)=\exp\left[\pm 2\sqrt{3}\chi^{\gamma}(2-3n)^{\gamma}(t-t_0)\right]$

5 Conclusions

In this paper we have presented a general formalism that allows one to apply DSA to a generic fourth order Lagrangian. The crucial point of this method is to express the two characteristic functions [22]:

$$\Upsilon = \frac{f'}{Rf''}\,, \quad r = -\frac{Rf'}{f}\,, \tag{44}$$

in terms of the dynamical variables, which, in principle, allows one to obtain a closed autonomous system for any Lagrangian density $f(R)$.

The resulting general system admits many interesting features, but is very difficult to analyze without specifying the function Υ (i.e. the form of $f(R)$). Consequently, a "one parameter" approach can lead to a number of misleading results.

Even after substituting for Υ, the dynamical system analysis is still very delicate; in fact, Υ could be discontinuous, admit singularities or generate additional invariant submanifolds that influence deeply the stability of the fixed points as well as the global evolution of the orbits.

After describing the method, we applied it to two classes of fourth order gravity models: $R+\chi R^n$ and $R^p \exp(qR)$, finding some very interesting preliminary results for the finite phase space. Both these models have fixed points with corresponding solutions that admit accelerated expansion and, consequently can model either inflation or dark energy eras (or both). In addition, there are other fixed points which are linked to phases of decelerated expansion which can in principle allow for structure formation. These latter solutions are not physical for every value of their parameters, but this is not necessarily a problem. In fact, in order to obtain a Friedmann cosmology evolving towards a dark energy era, these points are required to be unstable i.e., cosmic histories coast past them for a period which depends on the initial conditions. This means that the general integral of the cosmological equations corresponding to such an orbit will only approximate the fixed point solution and this approximate behavior might still allow structures to form.

It is also important to mention the fact that even if one has the desired fixed points and desired stability, this does not necessarily imply that there is an orbit connecting them. This is due to the presence of singular and invariant submanifolds that effectively divide the phase space into independent sectors. Of course one can implement further constraints on the parameters in order to have all the interesting points in a single connected sector, but this is still not sufficient to guarantee that an orbit would connect them. The situation is made worse by the fact that, since the phase space is of dimension higher than three, chaotic behavior can also occur. It is clear then, that any statement on the global behavior of the orbits is only reliable if an accurate numerical analysis is performed. However, these issues (and others) will be investigated in more detail in a series of forthcoming papers.

A final comment is needed regarding the differences between our results and the ones given in [22]. Even if the introduction of Υ and r, was suggested for the first time in that paper, the results above (and in particular the existence of a viable matter era) are in disagreement with the ones given in that paper. The reason is that the authors of [22] used "a one parameter description" in order to deal with (39) in general. We were able to prove that, unfortunately, not only are the equations given in [22] incomplete, but also that the method also gives both incorrect and misleading conclusions.

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Generalized modified gravity models: the stability issue

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Abstract

A brief introduction on the issue of stability in generalized modified gravity is presented and the dynamical system methods are used in the investigation of the stability of spatially flat homogeneous cosmologies within a large class of generalized modified gravity models in the presence of a relativistic matter-radiation fluid.

1 Introduction

To start with, we recall that recent cosmological data support the fact that there is a good evidence for a late accelerated expansion of the observable universe, apparently due to the presence of an effective positive and small cosmological constant of unknown origin. This is known as dark energy issue (see for example [39]).

Among several existing explanation, the so called modified gravity models are possible realizations of dark energy (for a recent review and alternative approaches see [2, 3, 4]), which may offer a quite natural geometrical approach again in the spirit of the original Einstein theory of gravitation. In fact, the main idea underlying these approaches to dark energy puzzle is quite simple and consists in adding to the gravitational Einstein-Hilbert action other gravitational terms which may dominate the cosmological evolution during the very early or the very late universe epochs, but in such a way that General Relativity remains valid at intermediate epochs and also at non cosmological scales.

The Λ-CDM model is the simplest possibility but, it is worth investigating more general modifications, possible motivations run from quantum corrections to string models. We shall first consider the simpler modification of the kind $F(R) = R + f(R)$, and then discuss the related generalizations. Models based on $F(R)$ are not new and they have been used in the past by many authors, for example as models for inflation, $f(R) = aR^2$ [5]. Recently their interest in cosmology was triggered by the model $f(R) = -\mu^4/R$, proposed in order to describe the current acceleration of the observable universe [6, 7]. For incomplete list of references, see [8].

It is important to stress that these $F(R)$ models are conformally equivalent to Einstein's gravity, coupled with a self-interacting scalar field, Einstein frame formulation. We will consider only the

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Jordan frame, in which the dynamics of gravity is described by $F(R)$ with minimally coupled matter. Observations are typically interpreted in this Jordan frame.

We would like also to mention the so called viable $F(R)$ models, which have recently been proposed [9, 10, 11, 12, 13, 14], with the aim to describe the current acceleration with a suitable choice of $F(R) = R + f(R)$, but also to be compatible with local stringent gravitational tests of Einstein gravity $F(R) = R$. The main idea is the so called disappearing of cosmological constant for low curvature, and mimicking the Λ-CDM model for high curvature. Thus, the requirements are:

a. $f(R) \to 0\,, \quad R \to 0\,$, compatibility with local tests;

b. $f(R) \to -2\Lambda_0\,, \quad R \to +\infty$, description of current acceleration;

c. Local stability of the matter.

As a illustration, we recall a recent example of viable model [15]

$$f(R) = -\alpha \quad \tanh \quad \frac{b\,(R - R_0)}{2} \quad + \tanh \quad \frac{bR_0}{2}$$

where R_0, and Λ_0 are suitable constants. Its advantages are a better formulation in the Einstein frame and a generalization that may also include the inflation era.

2 The de Sitter stability issue for $F(R)$ models

The stability of the de Sitter solution, relevant for Dark energy, may be investigated in the $F(R)$ models in several ways. We limit ourselves to the following three approaches:

i. perturbation of the equations of motion in the Jordan frame;

ii. one-loop gravity calculation around de Sitter background;

iii. dynamical system approach in FRW space-time.

We shall briefly discuss first two approaches, and then concentrate to the third one, since it is the only that can be easily extended to more general modified gravitational models, which is the main issue of this short review.

i. Stability of $F(R)$ model in the Jordan frame

The starting point is the trace of the equations of motion, which is trivial in Einstein gravity $R = -\kappa^2 T$, but, for a general $F(R)$ model, reads

$$3\nabla^2 f'(R) - 2f(R) + Rf'(R) - R = \kappa^2 T\,.$$

The new non trivial extra degree of freedom is the scalaron: $1 + f'(R) = e^{-\chi}$. Requiring $R = R_0 = CST$, one has de Sitter existence condition in vacuum

$$R_0 + 2f(R_0) - R_0 f'(R_0) = 0\,.$$

Perturbing around dS: $R = R_0 + \delta R$, with $\delta R = -\frac{1+f'(R_0)}{f''(R_0)}\delta\chi$, one arrives at the scalaron perturbation equation

$$\nabla^2\delta\chi - M^2\delta\chi = -\frac{\kappa^2}{6(1+f'(R_0))}T\,.$$

One may read off the scalaron effective mass

$$M^2 \equiv \frac{1}{3} \quad \frac{1+f'(R_0)}{f''(R_0)} - R_0 \quad .$$

Thus, if $M^2 > 0$, one has stability of the dS solution and the related condition reads

$$\frac{1+f'(R_0)}{R_0 f''(R_0)} > 1\,.$$

If $M^2 < 0$, there is a tachyon and instability. Furthermore, one may show that M^2 has to be very large in order to pass both the local and the astronomical tests and $1 + f'(R) > 0$, in order to have a positive effective Newton constant. The same result has been obtained within a different more general perturbation approach in [16].

ii. One-loop $F(R)$ quantum gravity partition function

Here we present the generalization to the modified gravitational case of the study of Fradkin and Tseytlin [17], concerning Einstein gravity on dS space. One works in the Euclidean path integral formulation, with dS existence condition $2F_0 = R_0F_0$, assumed to be satisfied. The small fluctuations around this dS instanton may be written as

$$g_{ij} = g_{0,ij} + h_{ij}\,, \qquad g^{ij} = g_0^{ij} - h^{ij} + h^{ik}h_k^j + \mathcal{O}(h^3)\,, \qquad h = g_0^{ij}h_{ij}\,.$$

Making use of the standard expansion of the tensor field h_{ij} in irreducible components, and making an expansion up to second order in all the fields, one arrives at a very complicated Lagrangian density L_2, not reported here, describing Gaussian fluctuations around dS space. As usual, in order to quantise the model described by L_2, one has to add gauge fixing and ghost contributions. Then, the computation of Euclidean one-loop partition function reduces to the computations of functional determinants. These functional determinants are divergent and may be regularized by the well known zeta-function regularization. The evaluation requires a complicated calculation and, neglecting the so called multiplicative anomaly, potentially present in zeta-function regularized determinants (see [18]), one arrives at the one-loop effective action [19] (here written in the Landau gauge)

$$\begin{aligned}\Gamma_{on-shell} \;=\; & \frac{24\pi F_0}{GR_0^2} + \frac{1}{2}\log\det\ \ell^2 \ -\Delta_2 + \frac{R_0}{6} \\ & -\frac{1}{2}\log\det\ \ell^2 \ -\Delta_1 - \frac{R_0}{4} \\ & +\frac{1}{2}\log\det\ \ell^2 \ -\Delta_0 - \frac{R_0}{3} + \frac{2F_0}{3R_0F_0''} \quad .\end{aligned}$$

The last term is absent in the Einstein theory. As a result, in the scalar sector one has an effective mass $M^2 = \frac{1}{3}\ \frac{2F_0}{R_0F_0''} - R_0$. Stability requires $M^2 > 0$, in agreement with the previous scalaron analysis, and with the inhomogeneous perturbation analysis [16].

The dynamical system approach

The main idea of such a method is to convert the generalized Einstein-Friedman equations in an autonomous system of first order differential equations and makes use of the theory of dynamical systems (see [20, 21, 22, 23, 24, 25] and references therein). We remind that the stability or instability issue is really relevant in cosmology. For example, in the ΛCDM model it ensures that no future singularities will be present in the solution. Within cosmological models, the stability or instability around a solution is of interest at early and also at late times.

The stability of de Sitter solutions has been discussed in several places, an incomplete list being [16, 26, 27, 28, 28, 29, 30, 31, 32, 33]. More complicated is the analysis of other critical points, associated with the presence of non vanishing matter and radiation. For example, in the paper [14], a non local model of modified gravity $F(R)$ has been investigated by means of this approach.

Here we shall extend to general modified gravity in the presence of matter the method given in Ref. [22], which permits to determine all critical points of a $F(R)$ model. Ordinary matter is important in reconstructing the expansion history of the Universe and probing the phenomenological relevance of the models (see for example the recent papers [22, 23, 24, 25], where the $F(R)$ case has been discussed in detail). Our generalisation consists in the extension of that method in order to include all possible geometrical invariants. This means that F could be a generic scalar function of curvature, Ricci and Riemann tensors.

There are some theoretical (quantum effects and string-inspired) motivations in order to investigate gravitational models depending on higher-order invariants. The "string-inspired" scalar-Gauss-Bonnet gravity case $F(R, G)$ has been suggested in Ref. [35] as a model for gravitational dark energy, while some time ago it has been proposed as a possible solution of the initial singularity problem [36]. The investigation of different regimes of cosmic acceleration in such gravity models has been carried out in Refs. [35, 37, 38, 21, 40, 8, 42, 43, 44, 45, 46]. In particular, in [43] a first attempt to the study of the stability of such kind of models has been carried out using an approach based on quantum field theory.

The method we shall use in the present paper is based on a classical Lagrangian formalism, see, for example, [47, 48, 49], inspired by the paper [5], where quantum gravitational effects were considered for the first time. With regard to this, it is well known that one-loop and two-loops quantum effects induce higher derivative gravitational terms in the effective gravitational Lagrangian. Instability due to quadratic terms have been investigated in [50]. A particular case has been recently studied in [51] and general models depending on quadratic invariants have been investigated in [52, 53].

A stability analysis of nontrivial vacua in a general class of higher-derivative theories of gravitation has already been presented in [54]. Our approach is different from the one presented there since we are dealing with scalar quantities and moreover it is more general, since it is not restricted to the vacuum invariant submanifold.

Finally, it should be stressed that the stability studied here is the one with respect to homogeneous perturbations. For the $F(R)$ case, the stability criterion for homogeneous perturbations is equivalent to the inhomogeneous one [16]. In the following we will summarize the results obtained in [55].

2.1 The dynamical system approach: The general case

To start with, let us consider a Lagrangian density which is an arbitrary function of all algebraic invariants built up with the Riemann tensor of the FRW space-time we are dealing with, that is

$$\mathcal{L} = -\frac{1}{2\chi}\, F(R, P, Q, ...) + \mathcal{L}_m \,, \tag{2.1}$$

where R is the scalar curvature, $P = R^{\mu\nu}R_{\mu\nu}$ and $Q = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ are the two quadratic invariants and the dots means other independent algebraic invariants of higher order, and $\mathcal{L}_{\Updownarrow}$ is the matter Lagrangian which depends on ρ and $p = p(rho)$, the density and pressure of the matter.

For the sake of convenience we write the metric in the form

$$ds^2 = -e^{2n(t)}dt^2 + e^{2\alpha(t)}d\vec{x}^{\,2} \,, \qquad N(t) = e^{n(t)} \,, \qquad a(t) = e^{\alpha(t)} \,. \tag{2.2}$$

In this way $\dot\alpha(t) = H(t)$ is the Hubble parameter and a generic invariant geometrical quantity U has the form

$$U = e^{-2pn(t)}\, u(\dot n, \dot\alpha, \ddot\alpha) = e^{-2pn(t)}\, u(\dot n, H, \dot H) = H^{2p}\, e^{-2pn(t)}\, u(X) \,, \tag{2.3}$$

where $2p$ is the dimension (in mass) of the invariant under consideration and $X = (\dot H/H^2 - \dot n/H)$ (see [55]). In particular one has

$$\begin{aligned} R &= 6H^2e^{-2n}(2+X) \,, \\ P &= 12H^4e^{-4n}(3+3X+X^2) \,, \\ Q &= 12H^4e^{-4n}(2+2X+X^2) \,, \end{aligned} \tag{2.4}$$

Using this notation, the action reads

$$S = -\int d^3x \int dt\, L(n, \dot n, \alpha, \dot\alpha, \ddot\alpha) = \frac{1}{2\chi}\int d^3x \int dt\, e^{n+3\alpha}\, F(n, \dot n, \dot\alpha, \ddot\alpha) + S_m \,, \tag{2.5}$$

and the Lagrange equations corresponding to the two Lagrangian variables $n(t)$ and $\alpha(t)$ are given by

$$E_n = \frac{\partial L}{\partial n} - \frac{d}{dt}\frac{\partial L}{\partial \dot n} = 2\sqrt{-g}\, T_{00}g^{00} = 2\rho\,\sqrt{-g} \,, \tag{2.6}$$

$$E_\alpha = \frac{\partial L}{\partial \alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot\alpha} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot\alpha} = \sqrt{-g}\, T_{ab}g^{ab} = -6p\,\sqrt{-g} \,. \tag{2.7}$$

It has to be noted that since $n(t)$ is a "gauge function", which corresponds to the choice of the evolution parameter, Eqs. (2.6) and (2.7) are not independent and in fact they satisfy the differential equation

$$\frac{dE_n}{dt} = \dot{n}E_n + \dot{\alpha}E_n\,, \tag{2.8}$$

which is equivalent to the conservation of energy-momentum tensor. Furthermore, we may use the gauge freedom and fix the cosmological time by means of the condition $N(t) = 1$, that is $n(t) = 0$. From now on it is understood that all quantities will be evaluated in such a gauge and so the parameter t corresponds to the standard cosmological time. In this way Eqs. (2.6) and (2.7) read

$$H\dot{F}_{\dot{H}} - HF_H + F - \dot{H}F_{\dot{H}} + 3H^2F_{\dot{H}} = 2\rho\,, \tag{2.9}$$

$$\ddot{F}_{\dot{H}} - \dot{F}_H + 6H\dot{F}_{\dot{H}} - 3HF_H + 3F + 3\dot{H}F_{\dot{H}} + 9H^2F_{\dot{H}} = -6p\,. \tag{2.10}$$

The latter equations are the generalisation to arbitrary action of the well known Friedmann equations.

Now we shall replace Eqs. (2.9)-(2.10) with an autonomous system of first order differential equations. To this aim we first observe that in pure Einstein gravity, that is for $F = R$, (2.9)-(2.10) read

$$H^2F_{\dot{H}} = F_X = 2\rho \qquad \Longrightarrow \qquad \Omega_\rho = 1\,, \tag{2.11}$$

$$(3H^2 + 2\dot{H})F_{\dot{H}} = (3 + 2X)F_X = -6p \qquad \Longrightarrow \qquad \Omega_p = -1 - \frac{2}{3}X\,, \tag{2.12}$$

where we have introduced the dimensionless variables

$$\Omega_\rho = \frac{2\rho}{H^2F_{\dot{H}}} = \frac{2\rho}{F_X}\,, \qquad \Omega_p = \frac{2p}{H^2F_{\dot{H}}} = \frac{2p}{F_X}\,, \tag{2.13}$$

which in this special case are given by the usual values $\Omega_\rho = \rho/3H^2$ and $\Omega_p = p/3H^2$. From Eqs. (2.11) and (2.12) it follows

$$w \equiv \frac{p}{\rho} = \frac{\Omega_\rho}{\Omega_p} = -1 - \frac{2}{3}X\,. \tag{2.14}$$

In the general case, Eqs. (2.9) and (2.10) have more terms with respect to (2.11) and (2.12) and it is quite natural to interpret them as corrections due to the presence of higher-order terms in the action. Then we define

$$\Omega_\rho^{eff} = \Omega_\rho + \Omega_\rho^{curv} = 1\,, \qquad \Omega_p^{eff} = \Omega_p + \Omega_p^{curv} = -1 - \frac{2}{3}X\,, \tag{2.15}$$

$$w_{eff} \equiv \frac{\Omega_p^{eff}}{\Omega_\rho^{eff}} = -1 - \frac{2}{3}X\,, \tag{2.16}$$

where Ω_ρ^{curv} and Ω_p^{curv} are complicated expressions, which only depend on the function F. They can be derived from (2.9) and (2.10), but they explicit form is not necessary for our aims. The effective quantity w_{eff} is equal to the ratio between the effective density and the effective pressure and it could be negative even if one considers only ordinary matter. It is known that the current-measured value of w_{eff} is near -1.

In order to get all critical points of the system now we follow the method, as described, for example, in [22]. First of all, we introduce the dimensionless variables

$$\Omega_\rho = \frac{2\rho}{H^2F_{\dot{H}}} = \frac{2\rho}{F_X}\,, \qquad \Omega_p = \frac{2p}{H^2F_{\dot{H}}} = \frac{2p}{F_X}\,, \tag{2.17}$$

$$X = \frac{\dot{H}}{H^2}\,, \qquad Y = \frac{F - HF_H}{H^2 F_{\dot{H}}} = \frac{F}{F_X} - X\,, \qquad Z = \frac{\dot{F}_{\dot{H}} - F_H}{HF_{\dot{H}}} = \frac{F'_X}{F_X} - 2X - \xi\,, \tag{2.18}$$

where the prime means derivative with respect to α and the quantity

$$\xi = \xi(X,Y) = \frac{F_H}{HF_{\dot{H}}} = \frac{HF_H}{F_X}\,, \tag{2.19}$$

has to be considered as a function of the variables X and Y. In general it is a function of X and H, but this latter quantity can be expressed in terms of X and Y as a direct consequence of the definition of Y itself. Then we derive an autonomous system by taking the derivatives of such variables. From Eq. (2.9) we have the constraint

$$\Omega_\rho = Y + Z + 3 \qquad \Longrightarrow \qquad \Omega_{curv} = -(Y + Z + 2)\,, \tag{2.20}$$

which reduces to the standard one when F is linear in R (general relativity with cosmological constant).

Deriving the variables above by taking into account of (2.10) and (2.20) we get the system of first order differential equations

$$\begin{cases} X' = -2X^2 - \gamma X + \beta(Z + \xi) \\ Y' = -(2X + Z + \xi)Y - XZ \\ Z' = -3(1+w)(Z + Y + 3) - (Z + \xi)(Z + 3) - X(Z + 6) \end{cases} \tag{2.21}$$

where $X' \equiv \frac{dX}{d\alpha} = \frac{1}{H}\frac{dX}{dt}$ (and so on) and for simplicity we have set $p = w\rho$, with constant w. For ordinary matter $0 \le w \le 1/3$ ($w = 0$ corresponds to dust, while $w = 1/3$ to pure radiation), but in principle one could also consider "exotic" matter with $w < 0$ or cosmological constant which corresponds to $w = -1$. We have also set

$$\beta = \beta(X,Y) = \frac{F_{\dot{H}}}{H^2 F_{\dot{H}\dot{H}}} = \frac{F_X}{F_{XX}}\,, \qquad \gamma = \gamma(X,Y) = \frac{F_{H\dot{H}}}{HF_{\dot{H}\dot{H}}} = \frac{HF_{HX}}{F_{XX}} = \beta\xi_X + \xi\,. \tag{2.22}$$

It is understood that $F_{\dot{H}\dot{H}} \neq 0$ has been assumed. The quantity Ω_ρ at the critical points will be determined by means of Eq. (2.20).

The critical points are obtained by putting $X' = 0, Y' = 0, Z' = 0$ in the system (2.21). The number and the position of such points depends on the Lagrangian throughout the functions β, γ and ξ. In principle, given F one can derive all critical points, but in practice for a generic F the algebraic system could be very complicated and the solutions quite involved. We shall consider in detail some particular cases at the end of the Section.

As already said above, the critical points of (2.21) depends on ξ, β, γ, which in general are complicated functions of X and Y, then it is not possible to determine general solutions without to choose the model, nevertheless it is convenient to distinguish two distinct classes of solutions characterised by the values of $w \neq -1$ and $w = -1$. For the sake of completeness we consider $w \le 1/3$ and so we write the solutions also for "exotic" matter, that is quintessence ($-1 < w < 0$) and phantom ($w < -1$). Of course, such solutions have to be dropped if one is only interested in ordinary matter/radiation. We have

- $w \neq -1$ — The critical points are the solutions of the system of three equations

$$\begin{cases} 2X^2 + \gamma X - \beta(Z + \xi) = 0 \\ (2X + Z + \xi)Y + XZ = 0 \\ 3(1+w)(Z + Y + 3) + (Z + \xi)(Z + 3) + X(Z + 6) = 0 \end{cases} \qquad w \neq -1\,, \tag{2.23}$$

  where ξ, β, γ are functions of X, Y determined by Eqs. (2.19) and (2.22). The stability matrix has three eigenvalues and the point is stable if the real parts of all of them are negative.

  The latter system has always the de Sitter solution $P_0 \equiv (X = 0, Y = 1, Z = -4)$, where $\Omega_\rho = 0$ and $w_{eff} = -1$. Note however that such a solution could exist also in the presence of matter, since the existence of P_0 critical point only implies that the critical value for Ω_ρ vanishes.

- $\Omega_\rho \neq 0$, $w=-1$ — The critical points are given by

$$\begin{cases} 2X^2+\gamma X-\beta(Z+\xi)=0 \\ (2X+Z+\xi)Y+XZ=0 \\ (Z+\xi)(Z+3)+X(Z+6)=0 \end{cases} \qquad w=-1\,. \tag{2.24}$$

For this class of solutions, the non-singular stability matrix has three eigenvalues and the point is stable if the real parts of all of them are negative.

We see that there is at least one singular case (critical line) when $X=0$ and $Z=-\xi=-4$. In fact in such a case Y or Ω_ρ are undetermined since

$$\Omega_\rho = Y+3-\xi(0,Y) = Y-1 \qquad \Longrightarrow \qquad Y=1+\Omega_\rho\,, \qquad \Omega_\rho \text{ arbitrary}\,. \tag{2.25}$$

Such a solution can be seen as a generalisation of the de Sitter solution for a model with cosmological constant. The de Sitter critical point for the model $\tilde F = F-2\Lambda$ reads $(X=0,\tilde Y=1,\tilde Z=-4)$. Such a solution follows from Eq. (2.25) if we choose $\rho_0\equiv\Lambda$. In fact, on the critical point $(X=0,Y=1+\Omega_\rho,Z=-4)$ (Eq. (2.25)) and from definitions (2.18) we get

$$\Omega_\rho = \frac{2\rho}{H^2F_{\dot H}} = \frac{F}{H^2F_{\dot H}}-1 \qquad \Longrightarrow \qquad \tilde Y \equiv \frac{\tilde F}{H^2\tilde F_{\dot H}} = 1\,, \tag{2.26}$$

which corresponds to de Sitter critical point for $\tilde F$. Eq. (2.25) is more general than the case with pure cosmological constant since ρ is not necessary a constant, and for this special class of solutions $w_{eff}=-1$. Note also that the stability matrix has always a vanishing eigenvalue and the stability of the system is determined by the other two eigenvalues.

For some models, but just for technical reasons, it could be convenient to treat the cosmological constant as matter, using the previous identification we have done.

Explicit examples

In order to see how the method works, now we give explicit solutions for some models and, when possible we also study the stability of the critical points. We restrict our analysis to the values $0\le w\le 1/3$ and to the special value $w=-1$, which corresponds to the pure cosmological constant, but in principle any negative value of w could be considered, even if this will be in contrast with the aim of modified gravity. In fact, modified gravity can generate an effective negative value of w without the use of phantom or quintessence.

It as to be stressed that in general, due to technical difficulties, one has to study the models by a numerical analysis. Only for some special cases one is able to find analytical results. Here we report the results for some models of the latter class in which the analytical analysis can be completely carried out. We also study more complicated models and for those we limit our analysis to the de Sitter solutions.

In the following we shall use the compact notation

$$P\equiv(X,Y,Z,\Omega_\rho,w_{eff})\,, \qquad P_0\equiv(0,1,-4,0,-1)\,, \qquad P_\Lambda=(0,1+\Omega_\Lambda,-4,\Omega_\Lambda,-1)\,. \tag{2.27}$$

The latter is an additional critical point that we have for the choice $w=-1$ and can be seen as the de Sitter solution in the presence of cosmological constant.

$F=R-\mu^4/R$ — This is the well known model introduced in [6, 7] and discussed in [22]. For this model the system (2.21) with arbitrary w has six different solutions, but only two of them effectively correspond to physical critical points, if $0\le w\le 1/3$. In principle there are other critical points for negative values of w (phantom or quintessence) and moreover there is also a particular solution for $w=-1$ which corresponds to the model with a cosmological constant Λ.

Solving the autonomous system one finds

- $P = P_0$: unstable de Sitter critical point. The critical value for the scalar curvature reads $R_0 = \sqrt{3}\,\mu^2$.
- $P = (-1/2, -1, -2, 0, -2/3)$: stable critical point. At the critical value, $H_0 = 0$.
- $P = (3(1+w)/2, -(5+3w), -2(5+3w), -3(4+3w), -(2+w))$: unstable critical point where $H_0 = 0$.
- $P = P_\Lambda$: unstable critical point. At the critical value one has $H_0^2 = (\Lambda/6)(1+\sqrt{1+3\mu^4/4\Lambda^2}$.

$F = R + aR^2 + bP + cQ$ — (Starobinsky-like model). Here we have to assume $3a+b+c \neq 0$ otherwise the quadratic term becomes proportional to the Gauss-Bonnet invariant. For $0 \leq w \leq 1/3$, this model has only one critical point. In order to have a de Sitter solution, we have to introduce a cosmological constant Λ. We have in fact

- $P = P_0$: Minkowskian solution with $R_0 = 0$, which is stable if $3a + b + c > 0$.
- $P = P_\Lambda$: de Sitter critical point with $R_0 = 6\Lambda$, which is stable if $3a + b + c > 0$, in agreement with [53].

$F = R - d^2 Q_3$ — This is the simplest toy model with the cubic invariant $Q_3 = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta}$. For this model we have the following critical points:

- $P = P_0$: unstable de Sitter solution with $R_0 = 6/d$.
- $P = P_0$: stable Minkowskian solution with $R_0 = 0$.
- $P = (0.05, 0.60, -3.60, 0, -1.03)$: stable solution with $H_0 = 0$.
- $P = P_\Lambda$: this point exists, but not for any value of d and Λ. Also the value of H_0 and the stability depend on the parameters.

$F = R + aR^2 + bP + cQ - d^2 Q_3$ — This is a generalisation of the previous two models. It may be motivated by the two-loop corrections in quantum gravity [56, 57]. The de Sitter critical points have been studied in Ref. [58]. The algebraic equations (2.21) are too complicated to be solved analytically, but it is easy to verify that there are at least the following solutions:

- $P = P_0$: de Sitter solution with $R_0 = 6/d$. This is stable if $3a + b + c + 3d > 0$.
- $P = P_0$: Minkowskian solution with $R_0 = 0$, which is stable if $3a + b + c > 0$.
- $P = P_\Lambda$: also in this case this point exists and is stable depending on the parameters (see [58]).

3 Conclusion

In this contribution, after a short review on the $F(R)$ case, we have presented the dynamical system approach which has permitted to arrive at a first order autonomous system of differential equations classically equivalent to the equations of motion for models of generalized modified gravity based on an arbitrary function $F(R, P, Q, Q_3 ...)$, namely built up with all possible geometric invariant quantities of the FRW space-time. We have shown that, in the special case of $F(R)$ theories, the method gives rise to the well known results [22, 23, 24], but in principle it can be applied to the study of much more general cases.

As illustrations, we have discussed some simple models, for which a complete analytical analysis concerning the critical points has been carried out. However, in general, due to technical difficulties, a numerical analysis is required. Among the models investigated, we would like to remind that we were able to deal with one which involves a cubic invariant in the curvature tensor and, to our knowledge, this has never been considered before, and this shows the power of the present approach.

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Modified gravity and Space-Time-Matter theory

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Abstract

The correspondence between $f(R)$ theories of gravity and model theories explaining induced dark energy in a $5D$ Ricci-flat universe, known as the Space-Time-Matter theory (STM), is studied. It is shown that such correspondence may be used to interpret the four dimensional expressions, induced from geometry in $5D$ STM theories, in terms of the extra terms appearing in $f(R)$ theories of gravity. The method is demonstrated by providing an explicit example in which a given $f(R)$ is used to predict the properties of the corresponding $5D$ Ricci-flat universe. The accelerated expansion and the induced dark energy in a 5D Ricci-flat universe characterized by a big bounce is studied and it is shown that an arbitrary function $\mu(t)$ in the 5D solutions can be rewritten, in terms of the redshift z, as a new arbitrary function $F(z)$ which corresponds to the 4D curvature quintessence models.

1 Introduction

The recent distance measurements from the light-curves of several hundred type Ia supernovae [10, 4] and independently from observations of the cosmic microwave background (CMB) by the WMAP satellite [11] and other CMB experiments [4, 5] suggest strongly that our universe is currently undergoing a period of acceleration. This accelerating expansion is generally believed to be driven by an energy source called dark energy. The question of dark energy and the accelerating universe has been therefore the focus of a large amount of activities in recent years. Dark energy and the accelerating universe have been discussed extensively from various point of views over the past few years [6, 7, 8]. In principle, a natural candidate for dark energy could be a small positive cosmological constant. One approach in this direction is to employ what is known as modified gravity where an arbitrary function of the Ricci scalar is added to the Einstein-Hilbert action. It has been shown that such a modification may account for the late time acceleration and the initial inflationary period in

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the evolution of the universe [9, 10]. Alternative approaches have also been pursued, a few example of which can be found in [11, 12, 13]. These schemes aim to improve the quintessence approach overcoming the problem of scalar field potential, generating a dynamical source for dark energy as an intrinsic feature. The goal would be to obtain a comprehensive model capable of linking the picture of the early universe to the one observed today, that is, a model derived from some effective theory of quantum gravity which, through an inflationary period would result in today accelerated Friedmann expansion driven by some Ω_Λ-term. However, the mechanism responsible for this acceleration is not well understood and many authors introduce a mysterious cosmic fluid, the so called dark energy, to explain this effect [14]. As was mentioned above, it has been shown that such an accelerated expansion could be the result of a modification to the Einstein-Hilbert action [8]. A scenario where the issue of cosmic acceleration in the framework of higher order theories of gravity in $4D$ is addressed can be found in [28]. One of the first proposals in this regard was suggested in [9] where a term of the form R^{-1} was added to the usual Einstein-Hilbert action. In $f(R)$ gravity, Einstein equations posses extra terms induced from geometry which, when moved to the right hand side, may be interpreted as a matter source represented by the energy-momentum tensor T^{Curv}, see equation (2.5).

In a similar fashion, the Space-Time-Matter (STM) theory, discussed below, results in Einstein equations in $4D$ with some extra geometrical terms which may be interpret as induced matter. It therefore seems plausible to make a correspondence between the geometrical terms in STM and T^{Curv} resulting in $f(R)$ gravity. We shall explore this idea to show that different choices of the parameter $\mu(t)$ in STM may correspond to different choices of $f(R)$ in curvature quintessence models in modified gravity.

The correspondence discussed above is based on the idea of extra dimensions. The idea that our world may have more than four dimensions is due to Kaluza [17], who unified Einstein's theory of General Relativity with Maxwell's theory of Electromagnetism in a $5D$ manifold. Since then, higher dimensional or Kaluza-Klein theories of gravity have been studied extensively [18] from different angles. Notable amongst them is the STM theory mentioned above, proposed by Wesson and his collaborators, which is designed to explain the origin of matter in terms of the geometry of the bulk space in which our $4D$ world is embedded, for reviews see [20]. More precisely, in STM theory, our world is a hypersurface embedded in a five-dimensional Ricci-flat ($R_{AB} = 0$) manifold where all the matter in our world can be thought of as being manifestations of the geometrical properties of the higher dimensional space. The fact that such an embedding can be done is supported by Campbell's theorem [21] which states that any analytical solution of the Einstein field equations in N dimensions can be locally embedded in a Ricci-flat manifold in $(N + 1)$ dimensions. Since the matter is induced from the extra dimension, this theory is also called the induced matter theory. Applications of the idea of induced matter or induced geometry can also be found in other situations [22]. The STM theory allows for the metric components to be dependent on the extra dimension and does not require the extra dimension to be compact. The sort of cosmologies stemming from STM theory is studied in [23, 24, 26].

In this paper we consider the correspondence between $f(R)$ gravity and STM theory. In section 2 we present a short review of $4D$ dark energy models in the framework of $f(R)$ gravity. In section 3 the field equations are solved in STM theory by fixing a suitable metric and the resulting geometric terms are interpreted as dark energy. The cosmological evolution in STM are considered in section 4. Section 5 deals with an example for a special form of $f(R)$. Conclusions are drawn in the last section.

2 Modified $f(R)$ gravity

General coordinate invariance in the gravitational action, without the assumption of linearity, allows infinitely many additive terms to the Einstein-Hilbert action [25]

$$\mathcal{S} = \int d^4x\sqrt{-g}[c_0R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\lambda\delta}R^{\mu\nu\lambda\delta} + \cdots] + \mathcal{S}_m, \tag{2.1}$$

where R, $R_{\mu\nu}$ and $R_{\mu\nu\lambda\delta}$ are Ricci scalar, Ricci tensor and Reimann tensor, respectively and $\mathcal{S}_m$ is the action for the matter fields. The fourth order term $R_{\mu\nu\lambda\delta}R^{\mu\nu\lambda\delta}$ may be neglected as a consequence of the Gauss-Bonnet theorem. The action (2.1) is not canonical because the Lagrangian function contains derivatives of the canonical variables of order higher than one. This means that, not only do we expect higher order field equations, but also the validity of the Euler-Lagrange equations is compromised. This problem is particularly difficult in the general case, but can be solved for specific metrics. In homogeneous and isotropic spacetimes, the Lagrangian in (2.1) can be further simplified. Specifically, the variation of the term $R_{\mu\nu}R^{\mu\nu}$ can always be rewritten in terms of the variation of R^2. Thus, the "effective" fourth order Lagrangian in cosmology contains only powers of R and we can suppose, without loss of generality, that the general form for a non-linear Lagrangian is given by

$$\mathcal{S} = \int d^4x\sqrt{-g}f(R) + \mathcal{S}_m, \tag{2.2}$$

where $f(R)$ is a generic function of the Ricci scalar[2]. Variation with respect to the metric $g_{\mu\nu}$ leads to the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = f'(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) + \tilde{T}^m_{\mu\nu}\,, \tag{2.3}$$

where

$$\tilde{T}^m_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}} \tag{2.4}$$

and the prime denotes a derivative with respect to R. It is easy to check that standard Einstein equations are immediately recovered if $f(R) = R$. When $f'(R) \neq 0$ the equation (2.3) can be recast in the more expressive form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{\text{Curv}}_{\mu\nu} + T^m_{\mu\nu}\,, \tag{2.5}$$

where an stress-energy tensor has been defined for the curvature contribution

$$T^{\text{Curv}}_{\mu\nu} = \frac{1}{f'(R)}\quad \frac{1}{2}g_{\mu\nu}\quad f(R) - Rf'(R)\quad + f'(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu})\quad , \tag{2.6}$$

and

$$T^m_{\mu\nu} = \frac{1}{f'(R)}\tilde{T}^m_{\mu\nu}\,, \tag{2.7}$$

is an effective stress-energy tensor for standard matter. This step is conceptually very important since a gravity model with a complicated structure converts to a model in which the gravitational field has the standard GR form with a source made up of two fluids: perfect fluid matter and an effective fluid (curvature fluid) that represents the non-Einsteinian part of the gravitational interaction.

We now consider the Robertson-Walker metric for the evolution of the universe

$$ds^2 = dt^2 - a(t)^2\quad \frac{dr^2}{1-kr^2} + r^2d\Omega^2\quad , \tag{2.8}$$

where k is the curvature of the space, namely, $k = 0, 1, -1$ for the flat, closed and open universes respectively. Substituting the above metric with $k = 0$ in equation (2.5) we obtain the $4D$, spatially flat Friedmann equations as follows

$$H^2 = \frac{1}{3}\left(\rho_m + \rho_{\text{Curv}}\right), \tag{2.9}$$

and

$$\dot{H} = -\frac{1}{2}\left[(\rho_m + p_m) + \rho_{\text{Curv}} + p_{\text{Curv}}\right], \tag{2.10}$$

[2] We use units such that $8\pi G_N = c = \hbar = 1$.

where a dot represents derivation with respect to time. Such a universe is dominated by a barotropic perfect fluid with the equation of state (EOS) given by $p_m = w_m \rho_m$ ($w_m = 0$ for pressureless cold dark matter and $w_m = 1/3$ for radiation) and a spatially homogenous curvature quintessence.

The energy density and pressure of the curvature quintessence are

$$p_{\text{Curv}} = \frac{1}{f'(R)} \quad 2 \quad \frac{\dot{a}}{a} \quad \dot{R}f''(R) + \ddot{R}f''(R) + \dot{R}^2 f'''(R) - \frac{1}{2} \quad f(R) - Rf'(R) \quad , \tag{2.11}$$

and

$$\rho_{\text{Curv}} = \frac{1}{f'(R)} \quad \frac{1}{2} \quad f(R) - Rf'(R) \quad - 3 \quad \frac{\dot{a}}{a} \quad \dot{R}f''(R) \quad , \tag{2.12}$$

respectively. The equation of state of the curvature quintessence is

$$w_{\text{Curv}} = \frac{p_{\text{Curv}}}{\rho_{\text{Curv}}}. \tag{2.13}$$

Recently, cosmological observations have indicated that our universe is undergoing an accelerated expanding phase. This could be due to the vacuum energy or dark energy which dominates our universe against other forms of matter such as dark matter and Baryonic matter. We thus concentrate on the vacuum sector *i.e.* $\rho_m = p_m = 0$, from which the evolution equation of curvature quintessence becomes

$$\dot{\rho}_{\text{Curv}} + 3H\left(\rho_{\text{Curv}} + p_{\text{Curv}}\right) = 0, \tag{2.14}$$

which yields

$$\begin{aligned} \rho_{\text{Curv}}(z) &= \rho_{0_{\text{Curv}}} \exp \quad 3 \int_0^z (1 + w_{\text{Curv}}) d\ln(1+z) \\ &\equiv \rho_{0_{\text{Curv}}} E(z), \end{aligned} \tag{2.15}$$

where, $1 + z = \frac{a_0}{a}$ is the redshift and the subscript 0 denotes the current value. In terms of the redshift, the first Friedmann equation can be written as

$$H(z)^2 = H_0^2 \Omega_{0_{\text{Curv}}} E(z), \tag{2.16}$$

where $\Omega_{0_{\text{Curv}}}$ and H_0 are the current values of the dimensionless density parameter and Hubble parameter, respectively. Equation (3.6) is the Friedmann equation in terms of redshift, z, which is suitable for cosmological observations. In fact, equations (3.6) and (3.19), obtained in section 4, are the cosmological connections between $f(R)$ gravity and STM theory.

3 Space-Time-Matter theory

According to the old suggestion of Kaluza and Klein the $5D$ vacuum Kaluza-Klein equations can be reduced under certain conditions to the $4D$ vacuum Einstein equations plus the $4D$ Maxwell equations. Recently, the idea that our four-dimensional universe might have emerged from a higher dimensional space–time is receiving much attention [19]. One current interest is to find out in a more general way how the $5D$ field equations relate to the $4D$ ones. In this regard, a proposal was made recently by Wesson [20] in that the $5D$ Einstein equations without sources $R_{AB} = 0$ (the Ricci flat assumption) may be reduced to the $4D$ ones with sources $G_{ab} = 8\pi G T_{ab}$, provided an appropriate definition is made for the energy-momentum tensor of matter in terms of the extra part of the geometry. Physically, the picture behind this interpretation is that curvature in $(4 + 1)$ space induces effective properties of matter in $(3 + 1)$ space–time. This idea is known as *space time matter* (STM) or modern Kaluza–Klein theory.

In this popular non-compact approach to Kaluza-Klein gravity, the gravitational field is unified with its source through a new type of $5D$ manifold in which space and time are augmented by an extra non-compact dimension which induces $4D$ matter within four dimensional universe. Unlike the usual Kaluza-Klein theory in which a cyclic symmetry associated with the extra dimension is assumed, the new approach removes the cyclic condition and derivatives of the metric with respect to the extra coordinate are retained. This induces non-trivial matter on the hypersurface of $l = constant$. This theory basically is guaranteed by an old theorem of differential geometry due to Campbell [21].

In the context of STM theory, a class of exact $5D$ cosmological solutions has been investigated and discussed in [27]. This solution was further pursued in [23] where it was shown to describe a cosmological model with a big bounce as opposed to the ubiquitous big bang. The $5D$ metric of this solution reads

$$dS^2 = B^2 dt^2 - A^2 \quad \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \quad - dy^2, \tag{3.1}$$

where $d\Omega^2 \equiv \quad d\theta^2 + \sin^2\theta d\phi^2$ and

$$\begin{aligned} A^2 &= \quad \mu^2 + k \quad y^2 + 2\nu y + \frac{\nu^2 + K}{\mu^2 + k}, \\ B &= \quad \frac{1}{\mu}\frac{\partial A}{\partial t} \equiv \frac{\dot{A}}{\mu}. \end{aligned} \tag{3.2}$$

Here $\mu = \mu(t)$ and $\nu = \nu(t)$ are two arbitrary functions of t, k is the $3D$ curvature index ($k = \pm 1, 0$), and K is a constant. This solution satisfies the $5D$ vacuum equation $R_{AB} = 0$. The Kretschmann curvature scalar

$$I_3 = R_{ABCD}R^{ABCD} = \frac{72K^2}{A^8}, \tag{3.3}$$

shows that K determines the curvature of the $5D$ manifold. Such a solution was considered in [27] with a different notation.

Using the $4D$ part of the $5D$ metric (3.7) to calculate the $4D$ Einstein tensor, we obtain

$$\begin{aligned} {}^{(4)}G_0{}^0 &= \frac{3 \quad \mu^2 + k}{A^2}, \\ {}^{(4)}G_1{}^1 &= {}^{(4)}G_2{}^2 = {}^{(4)} G_3{}^3 = \frac{2\mu\dot{\mu}}{A\dot{A}} + \frac{\mu^2 + k}{A^2}. \end{aligned} \tag{3.4}$$

As was mentioned earlier, since the recent observations show that the universe is currently going through an accelerated expanding phase, we assume that the induced matter contains only dark energy with ρ_{DE}, *i.e.* $\rho_m = 0$. We then have

$$\frac{3 \quad \mu^2 + k}{A^2} = \rho_{DE}, \tag{3.5}$$

$$\frac{2\mu\dot{\mu}}{A\dot{A}} + \frac{\mu^2 + k}{A^2} = -p_{DE}. \tag{3.6}$$

From equations (3.11) and (3.12), one obtains the EOS of dark energy

$$w_{DE} = \frac{p_{DE}}{\rho_{DE}} = -\frac{2 \quad \mu\dot{\mu}/\ A\dot{A} + \quad \mu^2 + k \quad A^2}{3\ (\mu^2 + k)/\ A^2}. \tag{3.7}$$

The Hubble and deceleration parameters are given in [23, 26] and can be written as

$$H \equiv \frac{\dot{A}}{AB} = \frac{\mu}{A}, \tag{3.8}$$

and

$$q\,(t, y) \equiv -A\frac{d^2A}{dt^2} \quad \frac{dA}{dt} \quad ^2 = -\frac{A\dot{\mu}}{\mu\dot{A}}, \tag{3.9}$$

from which we see that $\dot{\mu}\,/\mu > 0$ represents an accelerating universe while $\dot{\mu}\,/\mu < 0$ represents a decelerating one. The function $\mu(t)$ therefore plays a crucial role in defining the properties of the universe at late times.

4 Correspondence between modified $f(R)$ gravity and STM theory

In this section we will concentrate on the predictions of the cosmological evolution in the spatially flat case ($k = 0$). To avoid having to specify the form of the function $\nu(t)$, we change the parameter t to z and use $A_0/A = 1 + z$ and define $\mu_0^2 \;\; \mu^2 = F(z)$, noting that $F(0) \equiv 1$. We then find that equations (3.13)-(3.15) reduce to

$$w_{DE}(z) \quad = \quad -\frac{1 + (1+z)\, d\ln F(z)/dz}{3}, \tag{4.1}$$

and

$$q_{DE}(z) \quad = \quad \frac{1 + 3\Omega_{DE} w_{DE}}{2} = -\frac{(1+z)}{2}\frac{d\ln F(z)}{dz}. \tag{4.2}$$

There is an arbitrary function $\mu(t)$ in the present $5D$ model. Different choices of $\mu(t)$ may correspond to different choices of $f(R)$ in curvature quintessence models in modified gravity. Various choices of $\mu(t)$ correspond to the choices of $F(z)$. This enables us to look for the desired properties of the universe via equations (3.16) and (3.18). Using these definitions, the Friedmann equation becomes

$$H^2 = H_0^2(1+z)^2 F(z)^{-1}. \tag{4.3}$$

This would allow us to use the supernovae observational data to constrain the parameters contained in the model or the function $F(z)$. By comparing equation (3.19) with equation (3.6), we find that there exists a correspondence between the functions $f(R)$ and $F(z)$. We thus take $F(z)$ as

$$F(z) = (1+z)^2 \left[\Omega_{0_{\rm Curv}} E(z)\right]^{-1}. \tag{4.4}$$

According to (3.5), it is easy to see that the function $E(z)$ is determined by the particular choice for $f(R)$ which, in turn, determines the function $F(z)$ through equation (3.20). The evolution of the density components and the EOS of dark energy may now be derived. To this end, we must determine the functional form of $f(R)$. Thus, for example, we choose $f(R)$ as a generic power law of the scalar curvature and assume for the scale factor a power law solution in $4D$, investigated in [28]. Therefore

$$f(R) = f_0 R^n, \qquad a(t) = a_0 \quad \frac{t}{t_0} \quad ^{\alpha}. \tag{4.5}$$

The interesting cases are for the values of α satisfying $\alpha > 1$ which would lead to an accelerated expansion of our universe. Let us now concentrate on the case $\rho_m = 0$. Inserting equation (4.5) into the dynamical system (2.9) and (2.10), for a spatially flat space-time we obtain an algebraic system for parameters n and α

$$\begin{cases} \alpha \;\; \alpha(n-2) + 2n^2 - 3n + 1 \;\; = 0, \\ \\ \alpha \;\; n^2 - n + 1 + \alpha(n-2) \;\; = n(n-1)(2n-1), \end{cases} \tag{4.6}$$

from which the allowed solutions are

$$\begin{aligned} &\alpha = 0 \;\; \rightarrow \;\; n = 0,\ 1/2,\ 1, \\ &\alpha = \frac{2n^2 - 3n + 1}{2 - n}, \ \forall n, \quad n \neq 2\,. \end{aligned} \tag{4.7}$$

The solutions with $\alpha = 0$ are not interesting since they provide static cosmologies with a non-evolving scale factor. Note that this result matches the standard General Relativity result $n = 1$ in the absence of matter. On the other hand, the cases with generic α and n furnish an entire family of significant cosmological models. Using equations (2.11) and (3.2) we can also deduce the equation of state for the family of solutions $\alpha = \dfrac{2n^2 - 3n + 1}{2 - n}$ as

$$w_{\mathrm{Curv}}(n) = - \quad \frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \quad , \tag{4.8}$$

where $w_{\mathrm{Curv}} \to -1$ as $n \to \infty$. This shows that an *infinite* n is compatible with recovering an *infinite* cosmological constant. Thus, using equation (4.8), $E(z)$ and $F(z)$ are given by

$$E(z) \quad = \quad (1+z)^{3\left[\frac{-2n+4}{6n^2-9n+3}\right]}, \tag{4.9}$$

$$F(z) \quad = \quad (1+z)^2 \quad \Omega_{0_{\mathrm{Curv}}} (1+z)^{3\left[\frac{-2n+4}{6n^2-9n+3}\right] \quad -1}. \tag{4.10}$$

Now, using the above equations, equations (3.16) and (3.18) can be written as

$$w_{DE}(n) = - \quad \frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \quad , \tag{4.11}$$

and

$$q_{DE}(n) \equiv -\frac{A\dot{\mu}}{\mu \dot{A}} = \frac{-2n^2 + 2n + 1}{2n^2 - 3n + 1}. \tag{4.12}$$

Therefore, within the context of the present investigation, the accelerating, dark energy dominated universe, can be obtained by using the correspondence between $F(z)$ and $f(R)$ in modified gravity theories. We observe that in STM theory, $5D$ dark energy cosmological models correspond to $4D$ curvature quintessence models. This result is consistent with the correspondence between exact solutions in Kaluza-Klein gravity and scalar tensor theory [29]. Note that, as is well known, with a suitable conformal transformation, $f(R)$ gravity reduces to the scalar tensor theory.

From equations (4.9) and (4.10), we can rewrite equation (3.19) as

$$h(z,n) = \Omega_{0_{\mathrm{Curv}}} (1+z)^{3\left[\frac{-2n+4}{6n^2-9n+3}\right]}, \tag{4.13}$$

where $h(z,n) \equiv \frac{H(z)^2}{H_0^2}$ and the contribution of ordinary matter has been neglected. Figure 1 shows the behavior of $h(n)$ as a function of n for $z \sim 1.5$ and $\Omega_{0_{\mathrm{Curv}}} \simeq 0.70$. As can be seen, for $n \longrightarrow \pm\infty$ and $z \longrightarrow 0$ we have $h(z,n) \longrightarrow \Omega_{0_{\mathrm{Curv}}}$, that is, the universe finally approaches the curvature dominant state, thus undergoing an accelerated expanding phase. Figure 2 shows the behavior of $h(z)$ as a function of z for $n = 2, 10, -10$ and $\Omega_{0_{\mathrm{Curv}}} \simeq 0.70$. We see that for small z, $h(z) \longrightarrow 0.70$. Thus, we have obtained late-time accelerating solutions only by using the correspondence between $f(R)$ gravity and STM theory. Here, we have interpreted the properties of $5D$ Ricci-flat cosmologies by dark energy models in modified gravity.

5 Conclusions

In this paper we have studied the correspondence between modified $f(R)$ gravity and Space-Time-Matter theory by investigation of the present accelerated expanding phase of the universe using a general class of $5D$ cosmological models, characterized by a big bounce as opposed to a big bang, which is the standard prediction in $4D$ cosmological models. Such an exact solution contains two arbitrary functions, $\mu(t)$ and $\nu(t)$, which are analogous to different forms of $f(R)$ in curvature quintessence models. Also, once the forms of the arbitrary functions are specified, the

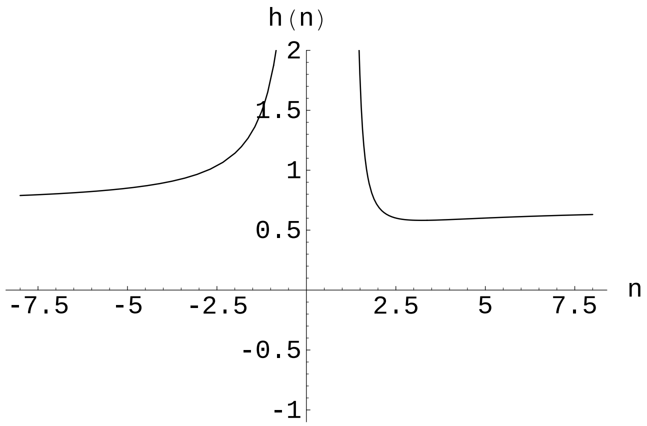

Figure 1: Behavior of $h(n)$ as a function of n for $z \sim 1.5$ and $\Omega_{0_{\rm Curv}} \simeq 0.70$. An accelerating universe occurs for $n \lesssim -2$ and $n \gtrsim 2$.

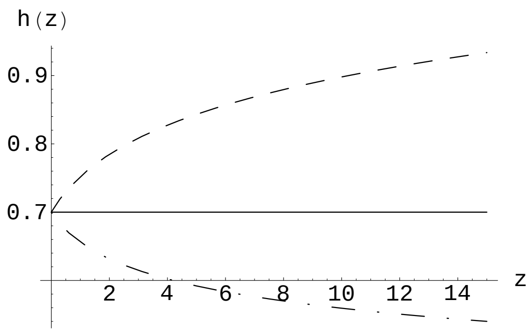

Figure 2: Behavior of $h(z)$ as a function of z for $n = 2$ (solid line), $n = 10$ (dashed line), $n = -10$ (dot-dashed line) and $\Omega_{0_{\rm Curv}} \simeq 0.70$. Note that for $n = 2, 10, -10$, $z \to 0$ and $h(z) \to 0.7$.

characteristic parameters determining the evolution of our universe are specified. We have noted that the correspondence between the functions $F(z)$ and $f(R)$ plays a crucial role and defines the form of the function $F(z)$. Finally, by taking a specific form for $f(R)$ we obtained solutions that describe the late-time acceleration of the universe. Explicitly, the induced dark energy and the resulting accelerated expansion in a 5D Ricci-flat universe is studied and it is shown that an arbitrary function $\mu(t)$ in the 5D solutions can be rewritten as a new arbitrary function $F(z)$ which corresponds to the 4D curvature quintessence models.

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Zeta regularization of infinite series, the linear case: basic concepts and an example

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Dedicated to Sergei D. Odintsov on the occasion of his 50th birthday

Abstract

This is a basic description of some concepts that are necessary for the analytic regularization of infinite, divergent series when using the zeta function procedure. Some fundamental cases corresponding to general first order (Barnes and affine extensions) zeta functions, in arbitrary dimension, will be considered and the physical situation corresponding to a family of harmonic oscillators will be discussed.

1 Introduction

Leonard Euler (1707-1783) was convinced that *"To every series one could assign a number"* [1] (that is, in a reasonable, consistent, and possibly useful way, of course). Euler was unable to prove this statement, but he devised a technique (Euler's summation criterion) in order to 'sum' a large family of divergent series. His statement was however controverted by some other great mathematicians, as Abel, who stated that *"Divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever"*. [2] There is a classical treatise due to G.H. Hardy with the simple and ambitious title *Divergent series* [3] that can be highly recommended to the reader.

Actually, regularization and renormalization procedures are essential in present day Physics. Among the different techniques at hand in order to implement these processes, zeta function regularization is one of the most beautiful. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system, which could, e.g., contribute to the cosmic force leading to the present acceleration of the expansion of our universe. The zeta function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations of QFT reduce basically (from a mathematical point of view) to the computation of determinants of elliptic pseudodifferential operators (ΨDOs) [4]. It is thus no surprise that the preferred definition of determinant for such operators is obtained through the corresponding zeta function (see, e.g., [5]).

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For its application in practice [6, 7], the zeta function regularization method relies on the existence of quite simple formulas that give the analytic continuation of the zeta function, $\zeta(s)$, from the region of the complex plane extending to the right of the abscissa of convergence, Re $s > s_0$, to the rest of the complex plane [5, 8, 9, 10]. These are not only the reflection formula of the corresponding zeta function in each case, but also some other, very fundamental expressions, as the Jacobi theta function identity, Poisson's and Plana's resummation formulas, and the Chowla-Selberg formula. However, some of these powerful expressions are often restricted to specific zeta functions, and their explicit derivation is usually quite involved. For instance, until very recently, the Chowla-Selberg (CS) series formula was only known for the homogeneous, two-dimensional Epstein zeta function. Also, all these formulas make use of the fact that the sum over the index is done over a full lattice in $\mathbf{R}$ or $\mathbf{R}^n$, e.g., extending from $-\infty$ to $+\infty$, and they do not survive in the case of truncated sums (where one gets much more involved, asymptotic expressions only) [8, 9].

A fundamental property shared by all zeta functions is the existence of a reflection formula. For the Riemann zeta function:

$$\Gamma(s/2)\zeta(s) = \pi^{s-1/2}\Gamma(1-s/2)\zeta(1-s). \tag{1.1}$$

For a generic zeta function, $Z(s)$, we may write it as: $Z(\omega - s) = F(\omega, s)Z(s)$. It allows for its analytic continuation in a very easy way —what is, in simple cases, the whole story of the zeta function regularization procedure. But the analytically continued expression thus obtained is just another series, which has again a slow convergence behavior, of power series type [11] (actually the same that the original series had, on its convergence domain). Some years ago, S. Chowla and A. Selberg found a formula, for the series expansion of the Epstein zeta function in the two-dimensional case [12], that yields *exponentially quick convergence everywhere*, not just in the reflected domain. They were very proud of that formula. In Ref. [13], a first attempt was done in order to try to extend this expression to inhomogeneous zeta functions (very important for physical applications, see [14]), but remaining always in *two* dimensions, for this was commonly believed to be an insurmountable restriction of the original formula (see, for instance, Ref. [15]). Later, extensions to an *arbitrary* number of dimensions [16, 17], both for the homogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) cases were constructed. However, some of the new formulas (remarkably the ones corresponding to the zero-mass case, e.g., the original CS framework) were not explicit, since they involved solving a rather non-trivial recurrence (what may also explain why the CS formula had not been extended to higher-dimensional Epstein zeta functions before). In [18] the recurrence was solved and *explicit* formulas where obtained. The linear case is also very important (and difficult too) for its many physical applications (system of harmonic oscillators or a multidimensional oscillator). The most general linear zeta function studied to date is the Barnes' one. Here again many explicit expressions are missing, as for its derivative in the general case. [19]

We will start from some considerations about divergent series and will shortly review the method of zeta function regularization at one-loop order. [16]–[18], [20] Then, we will discuss the case of the Barnes zeta function and extensions thereof, with an aim at the general affine case, a difficult goal. Expressions are obtained for the derivatives of these linear zeta functions. The example of a family of quantum harmonic oscillators will be finally considered.

2 Zeta regularization in a nutshell

Zeta function regularization —which is obtained by analytic continuation in the complex plane of the zeta function of the relevant physical operator in each case— is a most beautiful tool. Assume the system's Hamiltonian operator, H, has a spectral decomposition: $\{\lambda_i, \varphi_i\}_{i\in I}$, being I some set of indices (which can be discrete, continuous, mixed, multiple, ...). Then, the quantum vacuum energy is obtained as: [8]

$$E/\mu = \sum_{i\in I}\langle\varphi_i, (H/\mu)\varphi_i\rangle = \mathrm{Tr}_\zeta H/\mu = \sum_{i\in I}\lambda_i/\mu = \sum_{i\in I}(\lambda_i/\mu)^{-s}\bigg|_{s=-1} = \zeta_{H/\mu}(-1), \tag{2.1}$$

where ζ_A is the zeta function corresponding to the operator A, and the equalities are in the sense of analytic continuation (since, generically, the Hamiltonian operator will not be of the trace class).[2] Note that the formal sum over the eigenvalues is usually ill defined, and that the last step involves analytic continuation, inherent with the definition of the zeta function itself. Also, the unavoidable regularization parameter with dimensions of mass, μ, appears in the process, in order to render the eigenvalues of the resulting operator dimensionless, so that the corresponding zeta function can indeed be defined. We shall not discuss these important details here, which are just at the starting point of the whole renormalization procedure.

This hints towards the use of the zeta function as a summation method. Let us consider two examples: (i) Interpret the series $s_1 = 1+1+1+1+\cdots$ as a particular case of the Riemann zeta function, e.g. for the value $s=0$. This value is on the left hand side of the abscissa of convergence where the series as such diverges but where the analytic continuation of the zeta function provides a perfectly finite value: $s_1 = \zeta(0) = -\frac{1}{2}$. So this is the value to be attributed to the series $1+1+1+1+\cdots$. (ii) The series $s_2 = 1+2+3+4+\cdots$ corresponds to the exponent $s=-1$, so that $s_2 = \zeta(-1) = -\frac{1}{12}$. In practice these (reflection or functional) formulas are not optimal for actual calculations, since they are ordinarily given in terms of power series expansions (as the Riemann zeta itself). Fortunately, sometimes there are more clever expressions, that can be found, which converge exponentially fast, as the Chowla-Selberg [12] formula and some others. [13]-[24] Those give real power to the method of zeta regularization.

3 Barnes' zeta function: Dimension two case

The Barnes zeta function, which corresponds to the linear case, turns out to be even more difficult to extend than the Epstein zeta function (quadratic case) [13, 16], this explains our interest in it here. This fact is easy to understand by looking at the expression Mellin inverse transformed, ordinarily used for the analytic continuation, which has in general a much nicer, Gaussian behavior for the quadratic case (over extensive regions of the complex plane). Generalizations of the Barnes zeta function turn out to be important in different physical applications. [26]

Consider the Barnes zeta function in two dimensions [27]

$$\zeta_B(s;a|\vec{r}) = \sum_{n_1,n_2=0}^{\infty} (a+r_1n_1+r_2n_2)^{-s}, \qquad \mathrm{Re}\, s>2, \quad r_1,r_2>0, \tag{3.1}$$

and the related zeta function

$$\zeta(s;a|\vec{r}) = \sum_{n_1,n_2\in Z}{}' (a+r_1n_1+r_2n_2)^{-s}, \qquad \mathrm{Re}\, s>2, \quad r_1\neq r_2, \tag{3.2}$$

where the prime means that the term with $n_1=n_2=0$ is absent from the sum (actually, we could have defined the Barnes zeta function in this way too, in order to allow for the particular case $a=0$, but that would not have been the usual definition). Here we are implicitly assuming that the expressions in brackets never vanish, for any value of n_1, n_2. The two functions are related, provided we allow for the analytic continuation of the Barnes zeta function to negative values of the parameters r_1, r_2 (eventually, to complex values in general, since we are going to allow for the possibility $r_1, r_2 \in \mathbb{C}$, in what follows). In fact:

$$\begin{aligned}\zeta(s;a|\vec{r}) &= \zeta_B(s;a|\vec{r}) + \zeta_B(s;a|-\vec{r}) + \zeta_B(s;a|(r_1,-r_2)) + \zeta_B(s;a|(-r_1,r_2)) \\ &\quad - \sum_{i=1}^{2} r_i^{-s}\zeta_H(s,a/r_i) + (-r_i)^{-s}\zeta_H(s,-a/r_i) \ ,\end{aligned} \tag{3.3}$$

being ζ_H the Hurwitz zeta function.

[2] The reader should be warned that this $\zeta-$trace is actually no trace in the usual sense. In particular, it is highly non-linear, as often explained by the author elsewhere [25]. Some colleagues are unaware of this fact, which has lead to important mistakes and erroneous conclusions too often.

3.1 Asymptotic expansion method

We work, in a first approach, mainly at the level of asymptotic expansions. However, in the second part of this section we shall deal all the time with analytic, absolutely convergent series. Remember the asymptotic expansion for the Hurwitz zeta function

$$\zeta_H(s,a) \quad \sim \quad \frac{1}{s-1}\,a^{1-s} + \frac{1}{2}\,a^{-s} + \sum_{k=2}^{\infty} \frac{B_k}{k!}\,\frac{\Gamma(s+k-1)}{\Gamma(s)}\,a^{1-s-k}, \qquad a \to \infty \tag{3.4}$$

where the B_k are Bernoulli numbers, and the asymptotic expansion of its first derivative [28, 8, 9]

$$\begin{aligned} \zeta_H'(s,a) \quad &\sim \quad -\frac{a^{1-s}}{(s-1)^2} - \zeta_H(s,a)\,\log a \\ &+\sum_{k=2}^{\infty} \frac{B_k}{k!}\left[\psi(s+k-1)-\psi(s)\right]\frac{\Gamma(s+k-1)}{\Gamma(s)}\,a^{1-s-k}, \qquad a\to\infty. \end{aligned} \tag{3.5}$$

As starting point we will take the Barnes zeta function (1.1). It can be easily written as an infinite sum of Hurwitz zeta functions (absolutely convergent, for Re s conveniently big):

$$\begin{aligned} \zeta_B\,(s;a|\vec{r}) \quad &= \quad r_2^{-s}\sum_{n=0}^{\infty}\zeta_H\left(s,\frac{a}{r_2}+\frac{r_1}{r_2}n\right) \sim \frac{r_1^{1-s}r_2^{-1}}{s-1}\zeta_H(s-1,a/r_1) + \frac{r_1^{-s}}{2}\zeta_H(s,a/r_1) \\ &+\frac{s\,B_2}{2}r_1^{-1-s}r_2\zeta_H(s+1,a/r_1) + \mathcal{O}(s). \end{aligned} \tag{3.6}$$

Performing at this instance the analytic continuation to $s=0$:

$$\begin{aligned} \zeta_B\,(0;a|\vec{r}) \quad &= \quad -\frac{r_1}{r_2}\zeta_H(-1,a/r_1) + \frac{1}{2}\zeta_H(0,a/r_1) + \frac{B_2 r_2}{2r_1} = \frac{r_1}{2r_2}B_2(a/r_1) + \frac{1}{2}\left(\frac{1}{2}-\frac{a}{r_1}\right) + \frac{r_2}{12r_1} \\ &= \quad \frac{a^2}{2r_1r_2} - \frac{a}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{r_1}{12r_2} + \frac{r_2}{12r_1} + \frac{1}{4}, \end{aligned} \tag{3.7}$$

we obtain the well known result for the Barnes zeta function at $s=0$ in terms of generalized Bernoulli polynomials (see, e.g., Ref. [27]).

A similar procedure can be used for obtaining the derivative of the Barnes zeta function, namely to take the derivative of the series in the first line of Eq. (0.5) (which is, again, absolutely convergent, for Re s big enough), to use then appropriate expansions for the derivatives of the Hurwitz zeta functions appearing there, and, finally, to analytically continue in s to the desired value, say $s=0$. We write the successive steps, without further comment:

$$\zeta_B'\,(s;a|\vec{r}) \quad = \quad -\log(r_2)\;\zeta_B\,(s;a|\vec{r}) + r_2^{-s}\sum_{n=0}^{\infty}\zeta_H'\left(s,\frac{a}{r_2}+\frac{r_1}{r_2}n\right). \tag{3.8}$$

The analytic continuation to $s=0$ yields the asymptotic series expansion ($0 < r_2 \le r_1$):

$$\begin{aligned} \sum_{n=0}^{\infty}\zeta_H'\left(0,\frac{a}{r_2}+\frac{r_1}{r_2}n\right) \quad &\sim \quad \frac{r_1}{2r_2}B_2(a/r_1)\left(1-\log\frac{r_1}{r_2}\right) - \frac{r_2}{12r_1}\left(\log\frac{r_1}{r_2}+\psi(a/r_1)\right) \\ &+\left(\frac{1}{4}-\frac{a}{2r_1}\right)\log\frac{r_2}{r_1} + \frac{1}{2}\log\Gamma\left(\frac{a}{r_1}\right) - \frac{1}{4}\log(2\pi) \\ &-\frac{r_1}{r_2}\zeta_H'(-1,a/r_1) + \sum_{k=3}^{\infty}\frac{B_k}{k(k-1)}\left(\frac{r_1}{r_2}\right)^{1-k}\zeta_H(k-1,a/r_1). \end{aligned} \tag{3.9}$$

For the derivative at $s=0$ of the Barnes zf in dimension two, we obtain the asymptotic series

$$\begin{aligned} \zeta_B'\,(0;a|\vec{r}) \quad &\sim \quad -\zeta_B\,(0;a|\vec{r})\,\log r_1 + \frac{r_1}{2r_2}B_2(a/r_1) + \frac{1}{2}\log\Gamma(a/r_1) - \frac{1}{4}\log(2\pi) - \frac{r_2}{12r_1}\psi(a/r_1) \\ &-\frac{r_1}{r_2}\zeta_H'(-1,a/r_1) + \sum_{k=1}^{\infty}\frac{B_{2k+2}}{(2k+1)(2k+2)}\left(\frac{r_2}{r_1}\right)^{2k+1}\zeta_H(2k+1,a/r_1), \end{aligned} \tag{3.10}$$

in which the derivative of the Hurwitz zeta function at $s = -1$ can be written, in either of the two ways [8, 9]:

$$\begin{aligned} \zeta_H'(-1,a) &= \frac{a(a-1)}{2} - \frac{a}{2}\log(2\pi) + \frac{1}{12} + \int \log\Gamma(a) \\ &\sim \frac{B_2(a)}{2} \quad \log a - \frac{1}{2} \quad - \frac{a}{4} + \frac{1}{8} - \sum_{k=1}^{\infty} \frac{B_{2k+2}\, a^{-2k}}{2k(2k+1)(2k+2)}. \end{aligned} \tag{3.11}$$

This will require a specific determination of the logarithm. Taking the second approach,

$$\begin{aligned} \zeta_B'\,(0;a|\vec{r}) \quad \sim \quad & \frac{3a^2}{4r_1r_2} - \frac{a}{2r_2} - \quad \frac{a^2}{2r_1r_2} - \frac{a}{2r_2} + \frac{r_1}{12r_2} \quad \log a \\ & + \quad \frac{a}{2r_1} - \frac{r_2}{12r_1} - \frac{1}{4} \quad \log r_1 - \frac{r_2}{12r_1}\psi(a/r_1) + \frac{1}{2}\log\Gamma \quad \frac{a}{r_1} \\ & -\frac{1}{4}\log(2\pi) + \frac{r_1}{r_2}\sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)} \quad \frac{r_1}{a}^{2k} \\ & + \sum_{k=1}^{\infty} \frac{B_{2k+2}}{(2k+1)(2k+2)} \quad \frac{r_2}{r_1}^{2k+1} \zeta_H(2k+1, a/r_1), \qquad r_1, r_2 < a. \end{aligned} \tag{3.12}$$

Observe also that the initial symmetry has been here explicitly broken (from the beginning) by the assumption $|r_2| \leq |r_1|$, that has been crucial in obtaining the final formulas. This symmetry seems impossible to recover at the end.

3.2 A rigorous procedure

A different method starts from Hermite's formula for the celebrated Hurwitz zeta function:

$$\zeta_H(s,a) = \frac{a^{1-s}}{s-1} + \frac{a^{-s}}{2} + 2\int_0^{\infty} dy \; \frac{(a^2+y^2)^{-s/2}\sin[s\,\mathrm{arctg}(y/a)]}{e^{2\pi y}-1}. \tag{3.13}$$

This expression exhibits the singularity structure of $\zeta_H(s,a)$ explicitly, the integral being an analytic function of s, $\forall s \in \mathbf{C}$ (it is uniformly convergent for $|s| \leq R$, for any $R > 0$). Calling this integral $I(s,a)$, it is immediate that $I(s,a) \to 0, s \to 0, \forall a$. For the derivative, we obtain,

$$\begin{aligned} \zeta_H'(s,a) = -\frac{a^{1-s}}{(s-1)^2} - \quad \frac{a^{1-s}}{s-1} + \frac{a^{-s}}{2} \quad \log a - \int_0^{\infty} dy \; \frac{(a^2+y^2)^{-s/2}\log(a^2+y^2)\sin[s\,\mathrm{arctg}(y/a)]}{e^{2\pi y}-1} \\ +2\int_0^{\infty} dy \; \frac{(a^2+y^2)^{-s/2}\,\mathrm{arctg}(y/a)\cos[s\,\mathrm{arctg}(y/a)]}{e^{2\pi y}-1}. \end{aligned} \tag{3.14}$$

Calling the new integrals $I_1(s,a)$ and $I_2(s,a)$, respectively, we get

$$I_1(s,a) \to 0, \quad I_2(s,a) \to 2a\int_0^{\infty} dx \; \frac{\mathrm{arctg}\, x}{e^{2\pi a x}-1} = \frac{1}{12a} \quad 1 + \mathcal{O}(a^{-2}) \quad , \qquad s \to 0, \tag{3.15}$$

and it follows that

$$\zeta_H'(0,a) \sim -a + \quad a - \frac{1}{2} \quad \log a + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!\,\zeta(2k+2)}{(2\pi a)^{2k+1}}. \tag{3.16}$$

Recall that the Barnes zeta function is written in terms of the Hurwitz zeta function, as given by the first of Eqs. (0.5) above. To obtain the derivative of the Barnes zeta function, however, we will proceed as before: we first work in terms of a general s in order to be able to analytically continue the expression to the desired *Umgebung* of $s = 0$, expand then in Taylor series around $s = 0$, and finally take the derivative at the point $s = 0$. Being more precise, we first expand $\zeta_H(s,a)$ around $s = 0$, what yields

$$\zeta_B\,(s;a|\vec{r}) = \frac{r_1^{-s}}{s-1}\frac{r_1}{r_2}\zeta_H(s-1,a/r_1) + \frac{r_1^{-s}}{2}\zeta_H(s,a/r_1) + \frac{r_1^{-s}}{12}\frac{r_2}{r_1}s\zeta_H(1+s,a/r_1) + R(s;a|\vec{r}), \quad (3.17)$$

where the remainder term can be expressed in the following alternative ways (other possibilities may exist, too)

$$R(s;a|\vec{r}) \begin{cases} = 2s\dfrac{r_1^{1-s}}{r_2}\displaystyle\sum_{n=0}^{\infty}(n+a/r_1)^{1-s}\left[\int_0^\infty \frac{\operatorname{arctg} x\, dx}{\exp\left[2\pi(r_1/r_2)x\,(n+a/r_1)\right]-1} \right. \\ \qquad\qquad\qquad\qquad \left. -\dfrac{1}{24}\left(\dfrac{a}{r_2}+\dfrac{r_1}{r_2}n\right)^{-2}\right] + \mathcal{O}(s^2), \\ \sim \dfrac{sr_1^{-s}}{\pi}\displaystyle\sum_{k=1}^{\infty}(-1)^k(2k)!\,\zeta(2k+2)\left(\frac{r_2}{2\pi r_1}\right)^{2k+1}\zeta_H(s+2k+1,a/r_1) + \mathcal{O}(s^2). \end{cases} \quad (3.18)$$

From here it is immediate to obtain, at $s = 0$, the formula (cf. Eq. (3.7)):

$$\zeta_B\,(0;a|\vec{r}) = \frac{r_1}{2r_2}B_2(a/r_1) + \frac{1}{2}\left(\frac{1}{2} - \frac{a}{r_1}\right) + \frac{r_2}{12r_1}. \quad (3.19)$$

And from Eqs. (3.17), (3.18), taking the derivative with respect to s, at $s = 0$, we finally arrive to the very simple expression

$$\begin{aligned} \zeta_B'\,(0;a|\vec{r}) \quad &= \quad -\zeta_B\,(0;a|\vec{r})\,\log r_1 + \frac{r_1}{2r_2}B_2(a/r_1) + \frac{1}{2}\log\Gamma(a/r_1) - \frac{1}{4}\log(2\pi) \\ &\quad -\frac{r_2}{12r_1}\psi(a/r_1) - \frac{r_1}{r_2}\zeta_H'(-1,a/r_1) + R(a|\vec{r}). \end{aligned} \quad (3.20)$$

Observe that, here, analyticity has been rigorously preserved at every step of the calculation. Those in Eq. (3.18) are two possible expressions for the remainder term $R(a|\vec{r})$. The first of them is valid for $0 < r_2 \leq r_1$. It is very quickly convergent in this region and, therefore, very appropriate for numerical computation.

Concerning this extreme, we note that, calling

$$f(a,b) \equiv 2\sum_{n=0}^{\infty}(a+bn)\left[\int_0^\infty \frac{\operatorname{arctg} x\, dx}{\exp\left[2\pi(a+bn)x\right]-1} - \frac{1}{24}(a+bn)^{-2}\right], \quad (3.21)$$

one gets immediately (even with a pocket calculator)

$$\begin{aligned} &f(1,1) = -0.0028, \qquad f(1,10) = -0.0023, \qquad f(10,1) = -1.5\times 10^{-5} \\ &f(i,1) = 3.6\times 10^{-4} + i\,0.084, \qquad f(10\,i,1) = -2\times 10^{-6} + i\,0.008, \\ &f(10,i) = -3.3\times 10^{-5} + i\,1.4\times 10^{-5}. \end{aligned} \quad (3.22)$$

Observe the very quick convergence of the integral expression, even for imaginary values of the argument. The second form of the remainder is an explicit asymptotic series for $r_2 << 2\pi r_1$. It coincides term by term with the asymptotic expansion obtained before. In fact, this second representation is exactly the same of the preceding subsection (cf. Eq. (0.9)). Notice once more that the initial symmetry under interchange of these two parameters has been explicitly broken by this assumption that permits the whole calculation to be carried out. Moreover, by treating apart (if necessary) a finite number of first terms in the original expression, it is easy to artificially increase the value of a and with this considerably increase the speed and accuracy of the numerical results.

3.3 Analytic continuation to arbitrary values of r_1, r_2

Taking as starting point for the analytic continuation the formula:

$$\sum_{n_1,n_2=0}^{\infty} (a + r_1 n_1 + r_2 n_2)^{-s} = \frac{1}{\Gamma(s)} \sum_{n_1,n_2=0}^{\infty} \int_0^\infty dt\, t^{s-1} e^{-(a+r_1n_1+r_2n_2)t}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-at}}{(1-e^{-r_1 t})(1-e^{-r_2 t})}, \qquad a, r_1, r_2 > 0, \quad \text{Re } s > 2, \qquad (3.23)$$

it is plain that it remains valid without change as long as the real parts of a, r_1, r_2 are all positive. It is also clear that in the complex planes corresponding to the parameters r_1, r_2, as soon as any of them gets purely imaginary, say $r_1 = |r_1|e^{i\pi/2} = i|r_1|$, than an infinite number of poles are created in the integrand, with

$$\sum_{r_1-poles} \text{Res}[\ ] = \sum_{k=1}^{\infty} \frac{2\pi k}{|r_1|}^{s-1} e^{-2\pi k a/|r_1|} \; 1 - e^{-2\pi k r_2/|r_1|}{}^{-1} \equiv \Delta_1, \qquad (3.24)$$

which is an absolutely convergent series that will provide an additional contribution, $2\pi\Delta_1$, in the analytic continuation process when, equivalently, the integration path for the integrand variable t (originally over the positive real axis) is deformed beyond the positive imaginary axis, in order to reach the possibility of setting Re $r_1 < 0$ in the integral formula for the new integration path. Complementing this procedure —namely, just for *one* of the two parameters r_i— with a formula for the analytic continuation of the Hurwitz zeta function $\zeta_H(s,a)$ in the parameter a, we are done.[3]

Thus, the final formula for the analytic continuation in the parameter r_1 has the form:

$$\sum_{n_1,n_2=0}^{\infty} (a + r_1 n_1 + r_2 n_2)^{-s} = \frac{1}{\Gamma(s)} \int_{\mathcal{R}} d\tau\, \tau^{s-1} \frac{e^{-a\tau}}{(1-e^{-r_1 \tau})(1-e^{-r_2 \tau})} + 2\pi\Delta_1,$$

$$\text{Re } a, \text{Re } r_2 > 0, \text{Re } r_1 < 0, \qquad (3.25)$$

being R a straight-line in the second quadrant of the r_1-complex plane, starting at the origin.

3.4 Barnes' related zeta function in two dimensions

For the other, Barnes' related, two-dimensional zeta function considered at the begining, Eq. (1.2), we get

$$\zeta'(0; a|\vec{r}) = \sum_{\alpha=1}^{4} \zeta_B'(0; a|\vec{r}_\alpha) + \sum_{i=1}^{2} \quad \frac{1}{2} - \frac{a}{r_i} \quad \log r_i + \quad \frac{1}{2} + \frac{a}{r_i} \quad \log(-r_i)$$

$$- \log\Gamma \quad \frac{a}{r_i} \quad - \log\Gamma \quad \frac{a}{-r_i} \quad + \log(2\pi) \qquad (3.26)$$

where $\vec{r}_1 = (r_1, r_2)$, $\vec{r}_2 = (r_1, -r_2)$, $\vec{r}_3 = (-r_1, r_2)$, $\vec{r}_4 = (-r_1, -r_2)$. Note that this evaluation involves an analytic continuation on the values of r_1 and r_2 —which are strictly positive in the case of the Barnes zeta function— to negative and, in general, complex values $r_1, r_2 \in \mathbb{C}$. In this way we try to elude the unsurmountable problems that a direct interpretation of the sum in Eq. (1.2) poses. In particular, for $r_1 = r_2$ it develops an infinite number of zero and constant modes for an infinite number of constants, what renders a direct interpretation of the series extremely problematic. After the analytic continuation performed here from the Barnes zeta function —together with a natural transportation of the negative signs of the indices n_i to the parameters r_i— this limit is obtained in the final expressions, although with some severe restrictions (see below).

[3]This last can be read off from the usual tables, with rigorous prescriptions and domains of validity.

After some long but straightforward computation, we obtain

$$\zeta'(0;a|\vec{r}) = -\log \Gamma \frac{a}{r_2} \Gamma \frac{a}{-r_2} + \frac{a\pi i}{r_2} - \log \frac{a}{r_2} \pm i\frac{\pi}{2} + \log 2\pi. \tag{3.27}$$

Important considerations are here in order. (i) Observe, to begin with, that the last one is a *finite* expression and not an asymptotic series expansion as for the case of the Barnes zeta function. In fact, it has happened that the terms of the asymptotic expansions canceled one by one when performing the sum over all the $\vec{r}$'s. (ii) Second, note that the breaking of the initial symmetry under interchange of r_1 and r_2 in dealing with the analytic continuation (valid for $r_2 \leq r_1$), has finally resulted in the disappearance of one of the two parameters from the end result. Remarkable is the fact, however, that the final formula actually preserves the modular invariance of the inital one, under the parameter change on the other variable, here r_2: $a \longrightarrow a + kr_2$, $k \in Z$, since it transforms as $\zeta'(0;a|\vec{r}) \longrightarrow \zeta'(0;a|\vec{r}) + 2\pi k$, so that the modular symmetry of a with respect to r_2 is preserved. What has happened to the r_1 dependence? It has disappeared completely, the reason for this being the following. The formula for the Barnes zeta function for positive r_1 and r_2 obtained in the preceding section cannot be analytically continued to both r_1 and r_2 negative, because, for any value of s (and a, r_1, r_2 fixed), once we break the symmetry and do the analytic continuation say in r_2, then the analytic continuation in r_1 is restricted to the only possibility for n_1 of being zero. In fact, in other words, when we try to pull out of the r_1 real axis, for any value of $n_1 \neq 0$ we encounter a pole of the initial zeta function, for Im r_1 *arbitrarily small* —as these poles form a dense set in the complex r_1-plane— taking n_2 big enough. Now, it is immediate to see that one can interprete the final formula (0.13) as the true analytic continuation for the restriction $n_1 = 0$, namely: $\sum_{-\infty}^{\infty} \zeta'(0;a|(0,r_2)) = -\zeta'(0;a|(0,r_2)) = (0.13)$.

Let us quote also the following result, that we have obtained in two different ways: by analytically continuing in the factor 1 of n_1 to -1 or, alternatively, on a and i to $-a$ and $-i$, simultaneously. We get

$$\begin{aligned}\sum_{n_1=-\infty}^{\infty} \sum_{n_2=0}^{\infty} (a+n_1+i\,n_2)^{-s} \sim \\ ai(1-\log a) - \frac{i\pi a}{2} - \frac{\pi}{12} - \frac{\pi a}{2} - \frac{i}{12}\left[\psi(a) - \psi(-a)\right] + \frac{1}{2}\log\left[\Gamma(a)\Gamma(-a)\right] \\ - \log\Gamma(-ai) + i\sum_{k=1}^{\infty} \frac{(-1)^k B_{2k+2}}{(2k+1)(2k+2)}\left[\zeta_H(2k+1,a) - \zeta_H(2k+1,-a)\right].\end{aligned} \tag{3.28}$$

Finally, these formulas give us immediately the values of the determinants of the corresponding operators (say A) in each case [29]: $\det_\zeta A(a|\vec{r}) = \exp\left[-\zeta_A{}'(0;a|\vec{r})\right]$. This issue will be analyzed in the next section.

3.5 Application to the multidimensional harmonic oscillator: determinants and a multiplicative anomaly

Let us consider the harmonic oscillator in d dimensions, with angular frequencies $(\omega_1, \dots, \omega_d)$. The eigenvalues read $\lambda_{\bar{n}} = \bar{n} \cdot \bar{\omega} + b$, $\bar{n} \equiv (n_1, ..., n_d)$, $\bar{\omega} \equiv (\omega_1, ..., \omega_d)$, $b = \frac{1}{2}\sum_{k=1}^{d} \omega_k$, and the related zeta function is the Barnes one, $\zeta_d(s, b|\bar{\omega})$, whose poles are to be found at the points $s = k$ $(k = d, d-1, \dots, 1)$. Their corresponding residua can be expressed in terms of generalized Bernoulli polynomials $B_{d-k}^{(d)}(b|\bar{\omega})$, defined by

$$\frac{t^d e^{-at}}{\prod_{i=1}^{d}(1-e^{-b_i t})} = \frac{1}{\prod_{i=1}^{d} b_i} \sum_{n=0}^{\infty} B_n^{(d)}(a|b_i)\frac{(-t)^n}{n!}. \tag{3.29}$$

The residua of the Barnes zeta function are

$$\text{Res}\,\zeta_d(k, b|\bar{\omega}) = \frac{(-1)^{d+k}}{(k-1)!(d-k)!\prod_{j=1}^{d}\omega_j} B_{d-k}^{(d)}(b|\bar{\omega}), \quad k = d, d-1, \dots. \tag{3.30}$$

The issue will be now the calculation, through the zeta function, of the determinant of the harmonic oscillator Hamiltonian in any number, d, of dimensions. In QFT the zeta function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations reduce basically (as we already said) to the computation of determinants. The problem of the noncommutative or multiplicative anomaly appears here (see, for instance [30]).

Using the formulas of the preceding sections, we are able to obtain the determinant of the harmonic oscillator Hamiltonian in d dimensions. We will consider here explicitly the case $d = 3$.

Now, being V is a constant potential, we obtain

$$a(H, H_V) = \frac{(-1)^d}{2\prod_{j=1}^d \omega_j} \sum_{k=1}^{[d/2]} \frac{[\gamma + \psi(d-2k)]\, B_{2k}^{(d)}(b|\bar{\omega})}{(2k)!\,(d-2k)!} V^{2k}. \tag{3.31}$$

Here the generalized Bernoulli polynomials of odd order vanish. In spite of the manifold being non-compact, we confirm the validity of the Wodzicki formula. On the other hand, the remaining generalized Bernoulli polynomials are *never* zero, in fact

$$B_0^{(d)}(b|\bar{\omega}) = 1, \quad B_2^{(d)}(b|\bar{\omega}) = -\frac{1}{12}\sum_{i=1}^d \omega_i^2, \quad B_4^{(d)}(b|\bar{\omega}) = \frac{1}{24}\left[\frac{7}{10}\sum_{i=1}^d \omega_i^4 + \sum_{i<j}\omega_i^2\omega_j^2\right],$$
$$B_6^{(d)}(b|\bar{\omega}) = -\frac{5}{96}\left[\frac{31}{70}\sum_{i=1}^d \omega_i^6 + \frac{7}{10}\sum_{i\neq j}\omega_i^4\omega_j^2 + \sum_{i<j<k}\omega_i^2\omega_j^2\omega_k^2\right], \quad \ldots \tag{3.32}$$

Thus, the anomaly does not vanish for d odd or $d = 2$, whatever the frequencies ω_i be, and only *even* powers of the potential V appear.

4 Last considerations

We should also recall here the important applications of zeta function methods to quantum vacuum Physics, the Casimir effect, and its possible influence in modern cosmology (the dark energy issue, for a very short list of references see [31]). It has been nice to collaborate with Sergei Odintsov in many of these problems during a very long time. Sergei came to Barcelona as a visiting scholar in 1992. In fact he was the first one in Catalonia to be financed through the brand new Ministerial Program for foreign visitors (and I of course had the honor to be the first host in this program too!). Thus, the Catalan Television TV3 send a team to the Physics Department, 647 Diagonal Ave, and both of us felt important for one hour (this was not so trivial for theoretical physicists in those days, only Olympiad participants were bound for glory in Barcelona 92). After some years in Mexico and Colombia, Sergei has come back as an ICREA Professor to our new Institute, ICE/CSIC, and to the beautiful Bellaterra UAB campus, that he seems to like. Let me finish by saying that I have Sergei in high regard, both as as a scientist and also as a person and that I wish him success in the years to come.

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Extension of the EGS theorem to metric and Palatini $f(R)$ gravity

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Abstract

By using the equivalence between metric and Palatini $f(R)$ (or "modified") gravities with $\omega = 0, -3/2$ Brans-Dicke theories, it is shown that the Ehlers-Geren-Sachs theorem of general relativity is extended to modified gravity. In the case of metric $f(R)$ gravity studied before, this agrees with previous literature.

1 Introduction

Through an exceptional number of publications on the subject, Sergei Odintsov has contributed to advance alternative theories of gravity and cosmology motivated by quantum corrections to the classical Einstein-Hilbert action. Among these are scalar-tensor and $f(R)$ gravitational theories and, therefore, it seems appropriate to include in this volume dedicated to Sergei a small contribution to cosmology in these theories.[1]

In relativistic cosmology, the identification of our universe with a Friedmann-Lemaitre-Robertson-Walker (FLRW) space relies on the observations of spatial homogeneity and isotropy around us. The strongest support for this assumption, which lies at the core of relativistic cosmology (in both Einstein's theory of general relativity and in alternative gravitational theories) comes from the observation of the high degree of isotropy of the cosmic microwave background (CMB), supplemented by the assumption that isotropy would be observed from any spatial point in the universe (the Copernican principle — such an assumption would be hard to check). The fact that a spacetime in which a family of observers exists who see the CMB isotropic around them can be identified with a FLRW space is far from trivial and, from the mathematical point of view, constitutes a kinematical characterization of FLRW spaces known as the Ehlers-Geren-Sachs (hereafter EGS) theorem [2]. Usually, the vanishing of acceleration, shear, and vorticity,

$$\dot{u}^a \equiv u^c \nabla_c u^a = 0 \, , \qquad \sigma_{ab} = 0 \, , \qquad \omega_{ab} = 0 \, , \tag{1.1}$$

for a congruence of "typical" observers with four-velocity u^a is taken to imply that the spacetime is of the FLRW type, *i.e.*, with line element

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x^k)dx^i dx^j \, , \tag{1.2}$$

where γ_{ij} is a constant curvature 3-metric. However, this is guaranteed only if *i*) matter is described by a perfect fluid, and *ii*) the Einstein equations are imposed. These conditions enforce the vanishing of the Weyl tensor.

[1] A streamlined version of the discussion presented here was reported in [1].

In its original version, the EGS theorem states that, if a congruence of timelike, freely falling observers in a dust-dominated (*i.e.*, with vanishing pressure P) spacetime sees an isotropic radiation field, then (assuming that isotropy holds about every point) the spacetime is spatially homogeneous and isotropic and, therefore, a FLRW one. The original EGS theorem was generalized to an arbitrary perfect fluid that is geodesic and barotropic and with observers that are geodesics and irrotational [3, 4]. Moreover, an "almost EGS theorem" has been proved: spacetimes that are close to satisfying the EGS conditions are close to FLRW spaces in an appropriate sense[2] [6, 7]. Perhaps it is not exaggerated to regard the EGS theorem and its generalizations as a cornerstone of relativistic cosmology motivating the use of the standard Big Bang model.

Since the discovery of the EGS theorem in 1968, theoretical cosmology has expanded considerably to include the possibility that general relativity may have to be augmented by adding quantum corrections which take the form of extra terms in the Einstein-Hilbert action[3] $S_{EH} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,R + S^m$. Here we are interested in theories described by an action of the form

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f(R) + S^m\ , \tag{1.3}$$

where $f(R)$ is an arbitrary (but twice differentiable) function of its argument. While $f(R)$ gravity has a long history ([9] - see [10] for an historical review), quadratic corrections to general relativity , *i.e.*, $f(R) = R + \alpha R^2$, were introduced early on as semiclassical corrections or counterterms to renormalize general relativity [11]: such corrections, which are motivated also by string theories [12], are obviously important at large curvatures, *e.g.*, in the very early universe (in which they have been used to propel inflation in Starobinski's scenario [11]) and near black holes, in which non-linear choices of the function $f(R)$ may cure the problem of the central singularity [14]. More recently, "modified" or "$f(R)$" gravity has seen a new lease on life after the 1998 discovery of the acceleration of the cosmic expansion using type Ia supernovae [1]. While one possibility is to explain the present acceleration of the universe by postulating a mysterious form of dark energy with exotic properties ($P \simeq -\rho$, where ρ is the energy density), it has been proposed that perhaps we are seeing the first deviations from Einstein's gravity on very large scales. The prototypical modification of the Einstein-Hilbert action consisted of the choice $f(R) = R - \mu^4/R$, where $\mu \simeq H_0^{-1}$ is a mass scale of the order of the present value of the Hubble parameter, *i.e.,* extremely small on particle physics scales [16, 17]. While this particular model is in gross violation of the Solar System observational constraints on the parametrized-post-Newtonian parameter γ [18] and is subject to a violent instability [19, 20, 21], choices of the function $f(R)$ that satisfy the experimental constraints and provide the correct cosmological dynamics abound in the literature ([22] — see [23, 1] for reviews). Three versions of modified gravity exist: the first is *metric $f(R)$ gravity*, in which the action (1.3) is varied with respect to the metric, and provides the fourth order field equations

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - \left[\nabla_a\nabla_b - g_{ab}\Box\right] f'(R) = \kappa\, T_{ab}, \tag{1.4}$$

where, as usual, $T_{ab} = \frac{-2}{\sqrt{-g}}\,\frac{\delta S_M}{\delta g^{ab}}$, and a prime denotes differentiation with respect to R.

The second version of the theory is *Palatini $f(R)$ gravity*, in which the metric g_{ab} and the connection Γ^a_{bc} are considered as independent quantities (*i.e.*, the connection is not identified with the metric connection $\left\{{a \atop bc}\right\}$ of g_{ab}), the Ricci tensor $\mathcal{R}_{ab}$ is constructed out this non-metric connection, and $\mathcal{R} \equiv g^{ab}\mathcal{R}_{ab}$ [24]. The Palatini action is

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f(\mathcal{R}) + S^m\ , \tag{1.5}$$

in which the matter part of the action does not depend explicitly on the (non-metric) connection. Variation with respect to the metric and the connection Γ^a_{bc} provides the second order field equations

$$f'(\mathcal{R})\mathcal{R}_{(ab)} - \frac{1}{2}f(\mathcal{R})g_{ab} = \kappa\, G\, T_{ab}\ , \tag{1.6}$$

$$\bar{\nabla}_c \ \sqrt{-g}f'(\mathcal{R})g^{ab} = 0\ , \tag{1.7}$$

[2]For a discussion of inhomogeneous or anisotropic cosmological models admitting an isotropic radiation field, see [5].

[3]Here R is the Ricci curvature of the metric g_{ab}, which has determinant g and $\kappa \equiv 8\pi G$, G is Newton's constant, and S^m denotes the matter part of the action. We follow the notations of Ref. [3].

respectively, where $\tilde{\nabla}_a$ denotes the covariant derivative operator of Γ^a_{bc}. Little is still known about a third version (*metric-affine gravity*), in which the matter action is allowed to depend explicitly on the (non-metric) connection [25], and which will not be considered here.

2 The EGS theorem in $f(R)$ gravity

Since the possibility that our present universe is described by some modification of general relativity is now taken rather seriously, and $f(R)$ gravity is at least a convenient toy model (if not a serious candidate), it is natural to ask whether the basic result that allows one to identify the observed universe with a FLRW space, the EGS theorem, survives in these theories. The first investigations [26, 7] gave an affirmative answer for the theory

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \quad R + \alpha R^2 + \beta R_{ab} R^{ab} \quad + S^m \tag{2.1}$$

in the metric formalism,[4] subject to the additional condition (which does not appear in general relativity) that the perfect fluid filling the universe obeys a barotropic equation of state $P = P(\rho)$ with $dP/d\rho \neq 0$, which implies that surfaces of constant P and surfaces of constant ρ coincide. Subsequently, the validity of the EGS theorem was extended to general *metric* $f(R)$ gravity in Ref. [27]. Here we extend this result to *Palatini* modified gravity, and we provide an independent proof also for the metric verson of these theories. The result is straightforward because it builds on the results of Ref. [28] that extend the validity of the EGS theorem to scalar-tensor theories of gravity described by the action [29]

$$S_{ST} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \quad \psi R - \frac{\omega(\psi)}{\psi} \nabla^c \psi \nabla_c \psi - V(\psi) \quad + S^m \ , \tag{2.2}$$

where the coupling function $\omega(\psi)$ generalizes the constant Brans-Dicke parameter [10]. Now, it is well-known that metric or Palatini $f(R)$ gravity can be seen as a Brans-Dicke theory [31, 32]. In the metric formalism, with the introduction of an extra field ϕ, the action (1.3) can be rewritten as

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, [\psi(\phi) R - V(\phi)] + S^{(m)} \tag{2.3}$$

when $f''(R) \neq 0$, where ϕ is defined by

$$\psi(\phi) = f'(\phi) \tag{2.4}$$

and

$$V(\phi) = \phi f'(\phi) - f(\phi) \ . \tag{2.5}$$

The action (2.3) reduces to (1.3) trivially if $\phi = R$ and, vice-versa, variation of (2.3) with respect to ϕ yields

$$(R - \phi) \, f''(R) = 0 \ , \tag{2.6}$$

which implies that $\phi = R$ if $f'' \neq 0$. The action can now be seen as a Brans-Dicke action with Brans-Dicke parameter $\omega = 0$ if the field $\psi \equiv f'(\phi) = f'(R)$ is used instead of ϕ as the independent Brans-Dicke-like field:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, [\psi R - U(\psi)] + S^{(m)} \ , \tag{2.7}$$

where

$$U(\psi) = V(\phi(\psi)) = \psi\phi(\psi) - f(\phi(\psi)) \tag{2.8}$$

(this is called "O'Hanlon theory" or "massive dilaton gravity" [33, 32]).

In the Palatini formalism, by introducing the metric $h_{ab} \equiv f'(\tilde{R}) \, g_{ab}$ conformally related to g_{ab} and the scalar $\phi \equiv f'(\mathcal{R})$, and using the transformation property of the Ricci scalar [3], one obtains

$$\mathcal{R} = R + \frac{3}{2\phi} \nabla^c \phi \nabla_c \phi - \frac{3}{2} \Box \phi \ , \tag{2.9}$$

[4]Due to the fact that the Gauss-Bonnet expression $R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ gives rise to a topological invariant, it is unnecessary to include Riemann-squared terms in the action (2.1).

and the action is equivalent to

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \;\; f(\chi) + f'(\chi) \left(\mathcal{R} - \chi \right) \;\; + S^{(m)} \tag{2.10}$$

if $f'' \neq 0$ with $\chi = \tilde{R}$. By redefining χ through $\phi = f'(\chi)$, it is

$$f(\chi) + f'(\chi) \left(\mathcal{R} - \chi \right) = \phi \mathcal{R} - \phi \chi(\phi) + f\left(\chi(\phi) \right) = \phi \mathcal{R} + \frac{3}{2} \nabla^c \phi \nabla_c \phi - V(\phi) - 3 \Box \phi \tag{2.11}$$

using eq. (2.9), where

$$V(\phi) = \phi \chi(\phi) - f(\chi(\phi)) \, . \tag{2.12}$$

Apart from a boundary term, this yields

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \;\; \phi \mathcal{R} + \frac{3}{2\phi} \nabla^c \phi \nabla_c \phi - V(\phi) \;\; + S^{(m)} \tag{2.13}$$

which describes a Brans-Dicke theory with parameter $\omega = -3/2$, seldom considered in the literature [34] until the recent attempts to model the present-day cosmic acceleration.

Now, since the EGS theorem has been proved to hold for scalar-tensor gravity [28], it is straightforward to conclude that its validity is extended to metric modified gravity. This quick proof is consistent with the results obtained in [27] for general metric $f(R)$ gravity with a more direct approach, and with the findings of [26, 7] for quadratic $f(R)$. The validity of the EGS theorem is then extended to Palatini $f(R)$ gravity, which was not considered before in the EGS context. This is not entirely trivial when one considers the different order of the field equations with respect to metric $f(R)$ theories (second order instead of fourth), and the different physics described.

3 Conclusions

The equivalence between metric and Palatini $f(R)$ theories and $\omega = 0, -3/2$ Brans-Dicke theories allows for a straightforward proof of the EGS theorem for modified gravity, which relies on the previous work [28] extending the validity of this theorem to scalar-tensor gravity. By contrast, the direct approach of [27] appears a bit cumbersome.

In addition to providing a different approach to the EGS theorem for metric $f(R)$ gravity, we provide a straightforward proof of its validity for *Palatini* $f(R)$ gravity, which was not considered before in this context. Although evidence is now accumulating that Palatini modified gravity is not physically viable for various reasons (see [35, 36, 37, 38, 1] for a discussion of the various theoretical aspects involved), it may still be useful as a toy model to analyze mathematical and physical features of generalized gravity theories.

Given the fact that the EGS theorem extends to Lagrangian densities of the form $\mathcal{L} = R + \alpha R^2 + \beta R_{ab} R^{ab} + calL^m$ [26, 7], one wonders if it is actually valid for more general theories of the form $\mathcal{L} = f \;\; R, R_{ab} R^{ab}, R_{abcd} R^{abcd} \;\;$, which are motivated by low-energy string corrections to general relativity and are not equivalent to a simple scalar-tensor theory. This possibility will be examined elsewhere.

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The induced Cosmological Constant as a tool for exploring geometries

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Abstract

The cosmological constant induced by quantum fluctuation of the graviton on a given background is considered as a tool for building a spectrum of different geometries. In particular, we apply the method to the Schwarzschild background with positive and negative mass parameter. In this way, we put on the same level of comparison the related naked singularity ($-M$) and the positive mass wormhole. We use the Wheeler-De Witt equation as a basic equation to perform such an analysis regarded as a Sturm-Liouville problem . The cosmological constant is considered as the associated eigenvalue. The used method to study such a problem is a variational approach with Gaussian trial wave functionals. We approximate the equation to one loop in a Schwarzschild background. A zeta function regularization is involved to handle with divergences. The regularization is closely related to the subtraction procedure appearing in the computation of Casimir energy in a curved background. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation.

1 Introduction

In 1969, Penrose suggested that there might be a sort of "cosmic censor" that forbids naked singularities from forming[1], namely singularities that are visible to distant observers. Although there is no proof of such conjecture, naked singularities and the cosmic censorship are still a source of interest. A simple and particularly interesting example of naked singularities is the negative mass Schwarzschild spacetime. This is simply obtained by the Schwarzschild solution

$$ds^2 = -\left(1-\frac{2MG}{r}\right)dt^2+\left(1-\frac{2MG}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \tag{1.1}$$

replacing M with $-M$. This simple substitution gives rise to a naked singularity not protected by a horizon. An immediate consequence of a negative Schwarzschild mass is that if one were to place two bodies initially at rest, one with a negative mass and the other with a positive mass, both will accelerate in the same direction going from the negative mass to the positive one. Furthermore, if the two masses are of the same magnitude, they will uniformly accelerate forever. This feature leads to the problem of stability of such a geometry discussed by Gibbons, Hartnoll and Ishibashi[2] and

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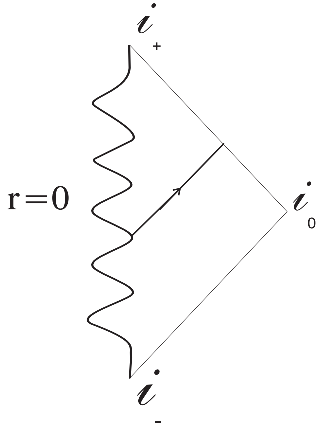

Figure 1: Carter Penrose diagram representing a naked singularity.

Gleiser and Dotti[3]. However, even if the stability issue is an open debate which seems to incline more to the unstable behavior than to the stable one[4], in this paper we wish to study the relation between the Schwarzschild solution for positive and negative masses with the induced cosmological constant. A cosmological constant can be considered "*induced*" when it appears as a consequence of quantum fluctuations. Since, apparently the Schwarzschild solution, naked singularities and the induced cosmological constant appear to be disconnected, it urges to establish a point of contact. We claim that such a link is in the Wheeler-De Witt equation (WDW)[5]. This equation can be simply obtained starting by the Einstein field equations without matter fields in four dimensions

$$G_{\mu\nu} + \Lambda_c g_{\mu\nu} = 0, \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and Λ_c is the cosmological constant. By introducing a time-like unit vector u^μ, we get

$$G_{\mu\nu} u^\mu u^\mu = \Lambda_c. \tag{1.3}$$

This is simply the Hamiltonian constraint written in terms of equation of motion. Indeed, if we multiply by $\sqrt{g}/(2\kappa)$ Eq.(1.3), we obtain ($\kappa = 8\pi G$)

$$\frac{\sqrt{g}}{2\kappa} G_{\mu\nu} u^\mu u^\mu = \frac{\sqrt{g}}{2\kappa} R + \frac{2\kappa}{\sqrt{g}} \quad \frac{\pi^2}{2} - \pi^{\mu\nu}\pi_{\mu\nu} \quad = \frac{\sqrt{g}}{2\kappa}\Lambda_c. \tag{1.4}$$

Here R is the scalar curvature in three dimensions and

$$\frac{\sqrt{g}}{2\kappa} G_{\mu\nu} u^\mu u^\mu = -\mathcal{H}. \tag{1.5}$$

If we multiply both sides of Eq.(1.4) by $\Psi\left[g_{ij}\right]$, we can re-cast the equation in the following form

$$\frac{2\kappa}{\sqrt{g}} G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}\left(R - 2\Lambda_c\right) \quad \Psi\left[g_{ij}\right] = 0. \tag{1.6}$$

This is known as the Wheeler-DeWitt equation with a cosmological term. Eq.(1.6) together with

$$-2\nabla_i \pi^{ij}\Psi\left[g_{ij}\right] = 0, \tag{1.7}$$

describe the *wave function of the universe*. The WDW equation represents invariance under *time* reparametrization in an operatorial form, while Eq.(1.7) represents invariance under diffeomorphism. G_{ijkl} is the *supermetric* defined as

$$G_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}). \tag{1.8}$$

Note that the WDW equation can be cast into the form

$$\frac{2\kappa}{\sqrt{g}}G_{ijkl}\pi^{ij}\pi^{kl}-\frac{\sqrt{g}}{2\kappa}R\quad\Psi\left[g_{ij}\right]=-\sqrt{g}\frac{\Lambda_c}{\kappa}\Psi\left[g_{ij}\right], \tag{1.9}$$

which formally looks like an eigenvalue equation. In this paper, we wish to use the induced cosmological constant argument evaluated to one loop in the different backgrounds as a tool to establish which kind of background induce the larger cosmological constant or, in other words, the larger Zero Point Energy (ZPE). In particular, we will compute the graviton ZPE propagating on the Schwarzschild background which positive and negative mass (naked singularity). This choice is dictated by considering that the Schwarzschild solution represents the only non-trivial static spherical symmetric solution of the Vacuum Einstein equations. Therefore, in this context the ZPE can be attributed only to quantum fluctuations. An example of this method applied in a completely different context without a cosmological term is in Refs.[6, 7], where the ZPE graviton contribution computed on different metrics is compared. In practice, we desire to compute

$$\Delta\Lambda_c=\Lambda_c^S-\Lambda_c^N\gtreqless 0, \tag{1.10}$$

where $\Lambda_c^{S,N}$ are the induced cosmological constant computed in the different backgrounds. Moreover, the Schwarzschild solution for both masses, namely $\pm M$ is asymptotically flat. Therefore we are comparing backgrounds with the same asymptotically behavior. Nevertheless, in Eq.(1.6), surface terms never come into play because $\mathcal{H}$ as well Λ_c/κ are energy densities and surface terms are related to the energy (e.g. ADM mass) and not to the energy density. We want to point up that we are neither discussing the problem of forming the naked singularity nor a transition during a gravitational collapse, rather the singularity is considered already existing. The semi-classical procedure followed in this work relies heavily on the formalism outlined in Ref.[8], where the graviton one-loop contribution in a Schwarzschild background was computed, through a variational approach with Gaussian trial wave functionals. A zeta function regularization is used to deal with the divergences, and a renormalization procedure is introduced, where the finite one loop is considered as a self-consistent source for traversable wormholes. Rather than reproducing the formalism, we shall refer the reader to [16] for details, when necessary. The rest of the paper is structured as follows, in section 2, we show how to apply the variational approach to the Wheeler-De Witt equation and we give some of the basic rules to solve such an equation approximated to second order in metric perturbation, in section 3, we analyze the spin-2 operator or the operator acting on transverse traceless tensors specified for the Schwarzschild metric with $\pm M$, in section 4 we use the zeta function to regularize the divergences coming from the evaluation of the ZPE for TT tensors and we write the renormalization group equation. We summarize and conclude in section 5.

2 The cosmological constant as an eigenvalue of the Wheeler De Witt Equation

In this section we shall consider the formalism outlined in detail in Ref.[8], where the graviton one-loop contribution in a Schwarzschild background is used. We refer the reader to Ref.[8] for details. The WDW equation (1.6), written as an eigenvalue equation, can be cast into the form

$$\hat{\Lambda}_\Sigma\Psi\left[g_{ij}\right]=-\frac{\Lambda_c}{\kappa}\Psi\left[g_{ij}\right], \tag{2.1}$$

where

$$\hat{\Lambda}_\Sigma=\frac{2\kappa}{\sqrt{g}}G_{ijkl}\pi^{ij}\pi^{kl}-\frac{\sqrt{g}}{2\kappa}R. \tag{2.2}$$

We, now multiply Eq.(2.1) by $\Psi^*\left[g_{ij}\right]$ and we functionally integrate over the three spatial metric g_{ij}, then after an integration over the hypersurface Σ, one can formally re-write the WDW equation as

$$\frac{1}{V}\frac{\int\mathcal{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi\left[g_{ij}\right]}{\int\mathcal{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\Psi\left[g_{ij}\right]}=\frac{1}{V}\frac{\left\langle\Psi\left|\int_\Sigma d^3x\hat{\Lambda}_\Sigma\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle}=-\frac{\Lambda_c}{\kappa}. \tag{2.3}$$

The formal eigenvalue equation is a simple manipulation of Eq.(1.6). However, we gain more information if we consider a separation of the spatial part of the metric into a background term, $\bar{g}_{ij}$, and a perturbation, h_{ij},

$$g_{ij}=\bar{g}_{ij}+h_{ij}. \tag{2.4}$$

The perturbation can be decomposed in a canonical way to give[9, 10, 11, 12]

$$h_{ij} = \frac{1}{3}\left(h + 2\nabla\cdot\xi\right)g_{ij} + \left(L\xi\right)_{ij} + h_{ij}^{\perp} \tag{2.5}$$

where the operator L maps ξ_i into symmetric tracefree tensors

$$\left(L\xi\right)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \frac{2}{3}g_{ij}\left(\nabla\cdot\xi\right) \tag{2.6}$$

and

$$g^{ij}h_{ij}^{\perp} = 0, \qquad \nabla^{i}h_{ij}^{\perp} = 0. \tag{2.7}$$

It is immediate to recognize that the trace element

$$\sigma = h + 2\left(\nabla\cdot\xi\right) \tag{2.8}$$

is gauge invariant. We write the trial wave functional as

$$\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right] = \mathcal{N}\Psi\left[h_{ij}^{\perp}\left(\overrightarrow{x}\right)\right]\Psi\left[h_{ij}^{\|}\left(\overrightarrow{x}\right)\right]\Psi\left[\sigma\left(\overrightarrow{x}\right)\right], \tag{2.9}$$

where

$$\begin{aligned}
\Psi\ h_{ij}^{\perp}\left(\overrightarrow{x}\right) &= \exp\left\{-\tfrac{1}{4}\ hK^{-1}h\ {}_{x,y}^{\perp}\right\} \\
\Psi\left[h_{ij}^{\|}\left(\overrightarrow{x}\right)\right] &= \exp\left\{-\tfrac{1}{4}\ \left(L\xi\right)K^{-1}\left(L\xi\right)\ {}_{x,y}^{\|}\right\} \ . \\
\Psi\left[\sigma\left(\overrightarrow{x}\right)\right] &= \exp\left\{-\tfrac{1}{4}\ \sigma K^{-1}\sigma\ {}_{x,y}^{Trace}\right\}
\end{aligned} \tag{2.10}$$

The symbol "$\perp$" denotes the transverse-traceless tensor (TT) (spin 2) of the perturbation, while the symbol "$\|$" denotes the longitudinal part (spin 1) of the perturbation. Finally, the symbol "*trace*" denotes the scalar part of the perturbation. $\mathcal{N}$ is a normalization factor, $\langle\cdot,\cdot\rangle_{x,y}$ denotes space integration and K^{-1} is the inverse "*propagator*". We will fix our attention to the TT tensor sector of the perturbation representing the graviton and the scalar sector. Therefore, representation (2.9) reduces to

$$\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right] = \mathcal{N}\exp\ -\frac{1}{4}\ hK^{-1}h\ {}_{x,y}^{\perp}\ \ \exp\ -\frac{1}{4}\ \sigma K^{-1}\sigma\ {}_{x,y}^{Trace}\ . \tag{2.11}$$

Actually there is no reason to neglect longitudinal perturbations. However, following the analysis of Mazur and Mottola of Ref.[11] on the perturbation decomposition, we can discover that the relevant components can be restricted to the TT modes and to the trace modes. Moreover, for certain backgrounds TT tensors can be a source of instability as shown in Refs.[13]. Even the trace part can be regarded as a source of instability. Indeed this is usually termed *conformal* instability. The appearance of an instability on the TT modes is known as non conformal instability. This means that does not exist a gauge choice that can eliminate negative modes. Since the wave functional (2.11) separates the degrees of freedom, we assume that

$$-\frac{\Lambda_c}{\kappa} = -\frac{\Lambda_c^{\perp}}{\kappa} - \frac{\Lambda_c^{trace}}{\kappa}, \tag{2.12}$$

then Eq.(2.3) becomes

$$\frac{1}{V}\frac{\left\langle\Psi\ \hat{\Lambda}_{\Sigma}^{\perp}\ \Psi\right\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda_c^{\perp}}{\kappa} \tag{2.13}$$

and

$$\frac{1}{V}\frac{\left\langle\Psi\ \hat{\Lambda}_{\Sigma}^{trace}\ \Psi\right\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda_c^{trace}}{\kappa} \tag{2.14}$$

3 The transverse traceless (TT) spin 2 operator and the W.K.B. approximation

Extracting the TT tensor contribution from Eq.(2.13), we get to one loop

$$\frac{\Lambda_c^{\perp}}{\kappa}(\lambda_i) = -\frac{1}{4}\sum_{\tau} \sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)} \quad . \tag{3.1}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$. To further proceed, we need to compute $\omega_i^2(\tau)$ $(i = 1, 2)$. To this purpose we write the background metric in the following way

$$ds^2 = -N^2(r)\,dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 \quad d\theta^2 + \sin^2\theta d\phi^2 \quad , \tag{3.2}$$

with a generic $b(r)$, to keep the discussion on a general ground, when possible. $N(r)$ is the "*lapse function*" playing the role of the "*redshift function*", while $b(r)$ is termed "*shape function*". The Spin-two operator for this metric is

$$\triangle_2 h^{TT}{}_i^{\ j} := - \quad \triangle_T h^{TT}{}_i^{\ j} + 2 \quad Rh^{TT}{}_i^{\ j}, \tag{3.3}$$

where the transverse-traceless (TT) tensor for the quantum fluctuation is obtained with the help of Eq.(2.5). Thus

$$- \quad \triangle_T h^{TT}{}_i^{\ j} = -\triangle_S \quad h^{TT}{}_i^{\ j} + \frac{6}{r^2} \quad 1 - \frac{b(r)}{r} \quad . \tag{3.4}$$

$\triangle_S$ is the scalar curved Laplacian, whose form is

$$\triangle_S = \quad 1 - \frac{b(r)}{r} \quad \frac{d^2}{dr^2} + \quad \frac{4r - b'(r)\,r - 3b(r)}{2r^2} \quad \frac{d}{dr} - \frac{L^2}{r^2} \tag{3.5}$$

and R_j^a is the mixed Ricci tensor whose components are:

$$R_i^a = \quad \frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} \quad , \tag{3.6}$$

This implies that the scalar curvature is traceless. We are therefore led to study the following eigenvalue equation

$$\triangle_2 h^{TT}{}_i^{\ j} = \omega^2 h_j^i \tag{3.7}$$

where ω^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity[15]. In particular, our choice for the three-dimensional gravitational perturbation is represented by its even-parity form

$$h_{ij}^{even}(r,\vartheta,\phi) = diag\left[H(r) \quad 1 - \frac{b(r)}{r}^{\ -1}, r^2K(r), r^2\sin^2\vartheta K(r)\right] Y_{lm}(\vartheta,\phi)\,. \tag{3.8}$$

For a generic value of the angular momentum L, representation (3.8) joined to Eq.(3.4) lead to the following system of PDE's

$$\begin{cases} -\triangle_S + \frac{6}{r^2} \quad 1 - \frac{b(r)}{r} \quad + 2 \quad \frac{b'(r)}{r^2} - \frac{b(r)}{r^3} \quad H(r) = \omega_{1,l}^2 H(r) \\ \\ -\triangle_S + \frac{6}{r^2} \quad 1 - \frac{b(r)}{r} \quad + 2 \quad \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} \quad K(r) = \omega_{2,l}^2 K(r) \end{cases} , \tag{3.9}$$

Defining reduced fields

$$H(r) = \frac{f_1(r)}{r}; \qquad K(r) = \frac{f_2(r)}{r}, \tag{3.10}$$

and passing to the proper geodesic coordinate

$$dx = \pm \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}}, \tag{3.11}$$

the system (3.9) becomes

$$\begin{cases} \left[-\frac{d^2}{dx^2} + V_1(r)\right] f_1(x) = \omega_{1,l}^2 f_1(x) \\ \\ \left[-\frac{d^2}{dx^2} + V_2(r)\right] f_2(x) = \omega_{2,l}^2 f_2(x) \end{cases} \tag{3.12}$$

with

$$\begin{cases} V_1(r) = \frac{l(l+1)}{r^2} + U_1(r) \\ \\ V_2(r) = \frac{l(l+1)}{r^2} + U_2(r) \end{cases}, \tag{3.13}$$

where we have defined $r \equiv r(x)$ and

$$\begin{cases} U_1(r) = \frac{6}{r^2} \quad 1 - \frac{b(r)}{r} \quad + \quad \frac{3}{2r^2} b'(r) - \frac{3}{2r^3} b(r) \\ U_2(r) = \frac{6}{r^2} \quad 1 - \frac{b(r)}{r} \quad + \quad \frac{1}{2r^2} b'(r) + \frac{3}{2r^3} b(r) \end{cases}. \tag{3.14}$$

Note that the coordinate x is appropriate only for Schwarzschild. Nevertheless, we find convenient use the same variable even in the case of the naked singularity. We choose

$$b(r) = \quad \begin{matrix} 2MG & \text{for Schwarzschild} \\ 2\bar{M}G & \text{for a naked singularity} \quad \bar{M} = -|M| \end{matrix} \quad . \tag{3.15}$$

We find that

$$b(r_t) = r_t \tag{3.16}$$

only for Schwarzschild. r_t is termed the throat and $r \in [r_t, +\infty)$. Of course, for the negative Schwarzschild mass, $r \in (0, +\infty)$. The potentials of the Lichnerowicz operator 3.3 simplify into

$$\begin{cases} U_1(r) = m_1^2(r) = \frac{6}{r^2} \quad 1 - \frac{2MG}{r} \quad - \frac{3MG}{r^3} \\ \\ U_2(r) = m_1^2(r) = \frac{6}{r^2} \quad 1 - \frac{2MG}{r} \quad + \frac{3MG}{r^3} \end{cases}, \tag{3.17}$$

for the Schwarzschild case and

$$\begin{cases} \bar{U}_1(r) = \bar{m}_1^2(r) = \frac{6}{r^2} + \frac{15\bar{M}G}{r^3} \\ \\ \bar{U}_2(r) = \bar{m}_1^2(r) = \frac{6}{r^2} + \frac{9\bar{M}G}{r^3} \end{cases}. \tag{3.18}$$

for the naked singularity. In the Schwarzschild case, we get

$$\begin{cases} m_1^2(r) \geq 0 \qquad \text{when } r \geq \frac{5MG}{2} \\ m_1^2(r) < 0 \qquad \text{when } 2MG \leq r < \frac{5MG}{2} \\ \\ m_2^2(r) > 0 \ \forall r \in [2MG, +\infty) \end{cases}. \tag{3.19}$$

The functions $U_1(r)$ and $U_2(r)$ play the rôle of two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$, respectively. In order to use the WKB approximation, we define two r-dependent radial wave numbers $k_1(r, l, \omega_{1,nl})$ and $k_2(r, l, \omega_{2,nl})$

$$\begin{cases} k_1^2(r, l, \omega_{1,nl}) = \omega_{1,nl}^2 - \frac{l(l+1)}{r^2} - m_1^2(r) \\ \\ k_2^2(r, l, \omega_{2,nl}) = \omega_{2,nl}^2 - \frac{l(l+1)}{r^2} - m_2^2(r) \end{cases} \tag{3.20}$$

for $r \geq \frac{5MG}{2}$. When $2MG \leq r < \frac{5MG}{2}$, $k_1^2(r, l, \omega_{1,nl})$ becomes

$$k_1^2(r, l, \omega_{1,nl}) = \omega_{1,nl}^2 - \frac{l(l+1)}{r^2} + m_1^2(r). \tag{3.21}$$

4 One loop energy Regularization and Renormalization

The total regularized one loop energy density for the graviton is

$$\rho(\varepsilon) = \rho_1(\varepsilon) + \rho_2(\varepsilon)\,, \tag{4.1}$$

where the energy densities, $\rho_i(\varepsilon)$ (with $i = 1, 2$), are defined as

$$\begin{aligned}\rho_i(\varepsilon) &= \frac{1}{4\pi}\mu^{2\varepsilon}\int_{\sqrt{m_i^2(r)}}^{\infty} d\omega_i \, \frac{\omega_i^2}{\left[\omega_i^2 - m_i^2(r)\right]^{\varepsilon-1/2}} \\ &= -\frac{m_i^4(r)}{64\pi^2}\;\; \frac{1}{\varepsilon} + \ln\;\; \frac{\mu^2}{m_i^2\,(r)} \;\; + 2\ln 2 - \frac{1}{2}\;\; .\end{aligned} \tag{4.2}$$

The two r-dependent effective masses $m_1^2\,(r)$ and $m_2^2\,(r)$ can be cast in the following form

$$\begin{cases} m_1^2\,(r) = m_L^2\,(r) + m_{1,S}^2\,(r) \\ \\ m_2^2\,(r) = m_L^2\,(r) + m_{2,S}^2\,(r) \end{cases}, \tag{4.3}$$

where

$$m_L^2\,(r) = \frac{6}{r^2}\;\; 1 - \frac{b\,(r)}{r} \tag{4.4}$$

and

$$\begin{aligned} m_{1,S}^2\,(r) &= \tfrac{3}{2r^2}b'\,(r) - \tfrac{3}{2r^3}b\,(r) \\ m_{2,S}^2\,(r) &= \tfrac{1}{2r^2}b'\,(r) + \tfrac{3}{2r^3}b\,(r) \end{aligned}\quad . \tag{4.5}$$

Essentially for the problem we are investigating, the term containing $m_L^2\,(r)$ is a long range term and will be discarded in this analysis. The zeta function regularization method has been used to determine the energy densities, ρ_i. It is interesting to note that this method is identical to the subtraction procedure of the Casimir energy computation, where the zero point energy in different backgrounds with the same asymptotic properties is involved. In this context, the additional mass parameter μ has been introduced to restore the correct dimension for the regularized quantities. Note that this arbitrary mass scale appears in any regularization scheme. Eq.(3.1) for the energy density becomes

$$\frac{\Lambda_c}{8\pi G} = \rho_1(\varepsilon) + \rho_2(\varepsilon)\,. \tag{4.6}$$

Taking into account Eq.(4.2), Eq.(2.3) yields the following relationship

$$\frac{\Lambda_c}{8\pi G} = \sum_{i=1}^{2}\frac{m_i^4(r)}{64\pi^2}\;\; \frac{1}{\varepsilon} + \ln\;\; \frac{4\mu^2}{m_i^2(r)\sqrt{e}} \quad . \tag{4.7}$$

Thus, the renormalization is performed via the absorption of the divergent part into the re-definition of the bare classical constant Λ_c

$$\Lambda_c \to \Lambda_{0,c} + \Lambda^{div}, \tag{4.8}$$

where

$$\Lambda^{div} = \frac{G}{32\pi\varepsilon}\;\; m_1^4\,(r) + m_2^4\,(r)\;\; . \tag{4.9}$$

The remaining finite value for the cosmological constant reads

$$\frac{\Lambda_{0,c}}{8\pi G} = \sum_{i=1}^{2}\frac{m_i^4(r)}{64\pi^2}\ln\;\; \frac{4\mu^2}{m_i^2(r)\sqrt{e}} \;\; = \rho_{eff}^{TT}\,(\mu, r)\,. \tag{4.10}$$

The quantity in Eq.(4.10) depends on the arbitrary mass scale μ. It is appropriate to use the renormalization group equation to eliminate such a dependence. To this aim, we impose that[16]

$$\frac{1}{8\pi G}\mu\frac{\partial\Lambda_{0,c}\,(\mu)}{\partial\mu} = \mu\frac{d}{d\mu}\rho_{eff}^{TT}\,(\mu, r)\,. \tag{4.11}$$

Solving it we find that the renormalized constant Λ_0^{TT} should be treated as a running one in the sense that it varies provided that the scale μ is changing

$$\Lambda_{0,c}(\mu, r) = \Lambda_{0,c}(\mu_0, r) + \frac{G}{16\pi} \quad m_1^4(r) + m_2^4(r) \quad \ln \frac{\mu}{\mu_0}. \tag{4.12}$$

Substituting Eq.(4.12) into Eq.(4.10) we find

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \sum_{i=1}^{2} \frac{m_i^4(r)}{64\pi^2} \ln \quad \frac{4\mu_0^2}{m_i^2(r)\sqrt{e}} \quad . \tag{4.13}$$

Eq.(4.13) is the expression we shall use to evaluate both the geometries.

4.1 The Schwarzschild metric

The Schwarzschild background is simply described by the choice $b(r) = 2MG$. In terms of the induced cosmological constant of Eq.(4.13), we get

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \frac{1}{64\pi^2} \sum_{i=1}^{2} \quad \frac{3MG}{r^3} \quad ^2 \ln \quad \frac{4r^3\mu_0^2}{3MG\sqrt{e}} \quad , \tag{4.14}$$

where we have used the assumption that $m_{i,L}^2(r)$ can be neglected. We know that an extremum appears, maximizing the induced cosmological constant for

$$\frac{3MG\sqrt{e}}{4r^3\mu_0^2} = \frac{1}{\sqrt{e}} \tag{4.15}$$

and leading to

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \frac{\mu_0^4}{4e^2\pi^2} \tag{4.16}$$

or

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \quad \frac{3MG}{r^3} \quad ^2 \frac{1}{64\pi^2} \qquad r \in \quad r_t, \frac{5}{4} r_t \quad . \tag{4.17}$$

Therefore, it appears that there exists a bound for $\Lambda_{0,c}$

$$\frac{9}{256\pi^2 r_t^4} \leq \frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} \leq \frac{225}{4096\pi^2 r_t^4} \tag{4.18}$$

4.2 The Naked Schwarzschild metric

The energy densities of Eq.(4.2) can be used also for the negative Schwarzschild mass. The only change is in the range of integration of the energy integral which can be extended to $\omega = 0$. The final result does not change, then in Eq.(4.13), we can substitute M with $\bar{M}$. Although the equation formally maintains the same expression, the throat is no more there. This means that a piece of $m_{i,L}^2(r)$ cannot be neglected and the two effective masses become

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} + \frac{15\bar{M}G}{r^3} \\ \\ m_2^2(r) = \frac{6}{r^2} + \frac{9\bar{M}G}{r^3} \end{cases} . \tag{4.19}$$

Since the effective mass grows approaching the singularity, we approximate them close to $r = 0$. Thus, we get

$$\begin{cases} m_1^2(r) =\simeq \frac{15\bar{M}G}{r^3} \\ \\ m_2^2(r) \simeq \frac{9\bar{M}G}{r^3} \end{cases} . \tag{4.20}$$

In this case, Eq.(4.13) yields

$$\frac{\Lambda_{0,c}^{naked}(\mu_0, r)}{8\pi G} = \frac{1}{64\pi^2} \left[\quad \frac{15\bar{M}G}{r^3} \quad ^2 \ln \quad \frac{4r^3\mu_0^2}{15\bar{M}G\sqrt{e}} \quad + \quad \frac{9\bar{M}G}{r^3} \quad ^2 \ln \quad \frac{4r^3\mu_0^2}{9\bar{M}G\sqrt{e}} \quad \right]. \tag{4.21}$$

In order to find an extremum, it is convenient to define the following dimensionless quantity

$$\frac{9\bar{M}G\sqrt{e}}{4r^3\mu_0^2} = x, \tag{4.22}$$

then Eq.(4.21) becomes

$$\frac{\Lambda_{0,c}^{naked}(\mu_0, r)}{8\pi G} = -\frac{\mu_0^4}{4e\pi^2}\quad x^2\ln x + \frac{25}{9}x^2\ln\quad\frac{5x}{3}\qquad. \tag{4.23}$$

We find a solution when

$$\bar{x} = \frac{1}{\sqrt{e}}\quad\frac{3}{5}^{\frac{25}{34}} \simeq 0.417 \tag{4.24}$$

corresponding to a value of

$$\frac{\Lambda_{0,c}^{naked}(\mu_0, r)}{8\pi G} = \frac{\mu_0^4}{4e^2\pi^2}\frac{17}{75}5^{\left(\frac{9}{17}\right)}3^{\left(\frac{8}{17}\right)} \simeq 0.328\frac{\mu_0^4}{4e^2\pi^2} = 0.328\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G}. \tag{4.25}$$

This means that

$$\frac{\Lambda_{0,c}^{naked}(\mu_0, r)}{\Lambda_{0,c}(\mu_0, r)} = 0.328 < 1. \tag{4.26}$$

Finally, we spend few words on the trace part contribution, which essentially confirms what has been found in Ref.[8]. Repeating the same procedure for the trace operator, we find for both values of the Schwarzschild mass ($\pm M$), that the only consistent value of finding extrema is that of vanishing M. This happens because solutions (4.15, 4.24) create a constraint on M, r and μ_0 which cannot be simultaneously satisfied for the graviton and for the trace term.

5 Summary and Conclusions

In Ref.[8], we considered how to extract information on the cosmological constant using the Wheeler-De Witt equation. In this paper, even if we have applied the same formalism, we have looked at the reversed idea, namely the induced cosmological constant represents a certain amount of energy density which varies with the choice of the underlying background. It is quite natural of thinking to an arrangement of the various induced constants in such a way to have a classification system which looks like to a spectrum of geometries. Note that this method, in principle can be extended beyond the Schwarzschild sector, in such a way to include all the spherically symmetric metrics. Extensions to modified theories of gravity have also been studied only for positive Schwarzschild mass[17]. It is interesting also to note that it is possible to use such method, represented by Eq.(2.3), not only for the induced cosmological constant, but even for electric or magnetic charges, simply by replacing

$$\frac{1}{V}\frac{\int \mathcal{D}[g_{ij}]\,\Psi^*[g_{ij}]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}]\,\Psi^*[g_{ij}]\,\Psi[g_{ij}]} = \frac{1}{V}\frac{\left\langle\Psi\ \ \int_\Sigma d^3x\hat{\Lambda}_\Sigma\ \ \Psi\right\rangle}{\langle\Psi|\Psi\rangle} = -\frac{\Lambda_c}{\kappa}. \tag{5.1}$$

$$\hat{\Lambda}_\Sigma \to \hat{Q}_\Sigma \tag{5.2}$$

and

$$-\frac{\Lambda_c}{\kappa} \to -\frac{1}{2}\int_\Sigma d^3x\sqrt{{}^3g}\rho_e \tag{5.3}$$

leading to the following eigenvalue equation[18]

$$\frac{\left\langle\Psi\ \ \int_\Sigma d^3x\hat{Q}_\Sigma\ \ \Psi\right\rangle}{\langle\Psi|\Psi\rangle} = -\frac{1}{2}\int_\Sigma d^3x\sqrt{{}^3g}\rho_e. \tag{5.4}$$

$\hat{Q}_\Sigma$ is the charge operator containing only the gravitational field. Coming back to the result (4.26), we can conclude that the Schwarzschild naked singularity has a lower value of ZPE compared to the positive Schwarzschild mass. This means that, even if the order of magnitude is practically the same, the naked singularity is less favored with respect to the Schwarzschild wormhole. A further progress could be the study of the unstable sector in our formalism to better understand the behavior of the naked Schwarzschild background. Indeed, we have studied the spectrum in a W.K.B. approximation with the following condition $k_i^2(r, l, \omega_i) \geq 0$, $i = 1, 2$. Thus to complete the analysis, we need to consider the possible existence of nonconformal unstable modes, like the ones discovered in Refs.[13]. If such an instability appears, this does not mean that we have to reject the solution. In fact in Ref.[19], we have shown how to cure such a problem. In that context, a model of "space-time foam" has been introduced in a large N wormhole approach reproducing a correct decreasing of the cosmological constant and simultaneously a stabilization of the system under examination. It could be interesting in the context of the multi-gravity to examine what happens for a large number of naked singularities.

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On the onset of the dark energy era

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Abstract

The occurrence of the scaling accelerated phase after matter dominance has been shown to be rather problematic for all existing dark energy and modified gravity models. In this paper we consider a cosmic scenario where both the matter particles and scalar field are associated with sub-quantum potentials which make the effective mass associated with the matter particles to vanish at the coincidence time, so that a cosmic system where matter dominance phase followed by accelerating expansion is allowed.

A recent paper by Amendola, Quartin, Tsujikawa and Waga (hereafter denoted as AQTW) [1] has put all existing models for dark energy in an apparent very serious trouble. Actually, if the result obtained by AQTW would be confirmed with full generality, then these authors have claimed that the whole paradigm of dark energy ought to be abandoned (See however the results in Ref. [12], e.g). Such as it happens with other aspects of the current accelerating cosmology, the problem is to some extend reminiscent of the difficulty initially confronted by earliest inflationary accelerating models [2] which could not smoothly connect with the following Friedmann-Robertson-Walker decelerating evolution [3]. As it is well-known, such a difficulty was solved by invoking the new inflationary scenario [4]. In fact, the problem recently posed for dark energy can be formulated by saying that a previous decelerating matter-dominated era cannot be followed by an accelerating universe dominated by dark energy and it is in this sense that it can be somehow regarded as the time-reversed version of the early inflationary exit difficulty. In more technical terms what AQTW have shown is that it is impossible to find a sequence of matter and scaling acceleration for any scaling Lagrangian which can be approximated as a polynomials because a scaling Lagrangian is always singular in the phase space so that either the matter-dominated era is prevented or the region with a viable matter is isolated from that where the scaling acceleration occurs. Ways out from this problem required assuming either a sudden emergence of dark energy domination or a cyclic occurrence of dark energy, both assumptions being quite hard to explain and implement. In this paper we however consider a dark energy model where such problems are no longer present due to some sort of quantum characteristics which can be assigned to particles and radiation in that model.

We start with an action integral that contains all the ingredients of our model. Such an action is a generalization of the one used by AQTW which contains a time-dependent coupling between dark energy and matter and leads to a general Lagrangian that admits scaling solutions formally the same as those derived in Ref. [1]. Setting the Planck mass unity, our Lorentzian action reads

$$S = \int d^4x\sqrt{-g}\left[R + p(X,\phi)\right] + S_m\left[\psi_i, \xi, m_i(V_{SQ}), \phi, g_{\mu\nu}\right] + ST\left(K, \psi_i, \xi\right), \tag{0.1}$$

where g is the determinant of the four-metric, p is a generically non-canonical general Lagrangian for the dark-energy scalar field ϕ with kinetic term $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, formally the same as the one used in Ref. [1], S_m corresponds to the Lagrangian for the matter fields ψ_i, each with mass m_i, which is going to depend on a sub-quantum potential V_{SQ} in a way that will be made clear in what follows, so as on the time-dependent coupling ξ of the matter field to the dark energy field ϕ. The term ST denotes the surface term which generally depends on the trace on the second fundamental form K, the matter fields ψ_i and the time-dependent coupling $\xi(t)$ between ψ_i and ϕ for the following reasons.

We first of all point out that in the theory being considered the coupling between the matter and the scalar fields can generally be regarded to be equivalent to a coupling between the matter fields and gravity plus a set of potential energy terms for the matter fields. In fact, if we restrict ourselves to this kind of theories, a scalar field ϕ can always be mathematically expressed in terms of the scalar curvature R [5]. More precisely, for the scaling accelerating phase we shall consider a sub-quantum dark energy model (see Refs. [7] and [8]) in which the Lagrangian for the field ϕ vanishes in the classical limit where the sub-quantum potential is made zero; i.e. we take $p = L = -V(\phi)\left(E(x,k) - \sqrt{1-\dot{\phi}^2}\right)$, where $V(\phi)$ is the potential energy and $E(x,k)$ is the elliptic integral of the second kind, with $x = \arcsin\sqrt{1-\dot{\phi}^2}$ and $k = \sqrt{1 - V_{SQ}^2/V(\phi)^2}$, and the overhead dot $\dot{}$ means derivative with respect to time. Using then a potential energy density for ϕ and the sub-quantum medium [note that the sub-quantum potential energy density becomes constant [8] (see later on)], we have for the energy density and pressure, $\rho \propto X(HV_{SQ}/\dot{H})^2 = p(X)/w(t)$, with $H \propto \phi V_{SQ} + H_0$, $\dot{H} \propto \sqrt{2X}V_{SQ}$, where H_0 is constant. For the resulting field theory to be finite, the condition that $2X = 1$ (i.e. $\phi = C_1 + t$) had to be satisfied [8], and from the Friedmann equation, the scale factor ought to be given by $a(t) \propto \exp\left(C_2 t + C_3 t^2\right)$, with C_1, C_2 and C_3 being constants. It follows then that for at least a flat space-time, we generally have $R \propto 1 + \alpha\phi^2$ (where α is another constant and we have re-scaled time) in that type of theories, and hence the matter fields - scalar field couplings, which can be generally taken to be proportional to $\phi^2\psi_i^2$, turn out to yield $\xi R\psi_i^2 - K_0\psi_i^2$, with K_0 again a given constant. The first term of this expression corresponds to a coupling between matter fields and gravity which requires an extra surface term, and the second one ought to be interpreted as a potential energy term for the matter fields $V_i \equiv V(\psi_i) \propto \psi_i^2$. In this way, for a general theory that satisfied the latter requirement, the action integral (1) should be re-written as

$$S = \int d^4x\sqrt{-g}\left(R(1-\xi\psi_i^2) + p(X,\phi)\right) + S_m\left[\psi_i, V_i, m_i(V_{SQ}), g_{\mu\nu}\right] - 2\int d^3x\sqrt{-h}\mathrm{Tr}K(1-\xi\psi_i^2), \tag{0.2}$$

in which h is the determinant of the three-metric induced on the boundary surface and it can be noticed that the scalar field ϕ is no longer involved at the matter Lagrangian. We specialize now in the minisuperspace that corresponds to a flat Friedmann-Robertson-Walker metric in conformal time $\eta = \int dt/a(t)$

$$ds^2 = -a(\eta)\left(-d\eta^2 + a(\eta)^2 d\mathrm{x}^2\right), \tag{0.3}$$

with $a(\eta)$ the scale factor. In this case, if we assume a time-dependence of the coupling such that it reached the value $\xi(\eta_c) = 1/6$ at the coincidence time η_c and choose suitable values for the arbitrary constants entering the above definition of R in terms of ϕ^2, then the action at that coincidence time would reduce to

$$S = \frac{1}{2}\int d\eta\left[a'^2 - \sum_i(\chi_i'^2 - \chi_i^2) + a^4\left(p(X,\phi) + \sum_i m_i(V_{SQ})^2\right)\right], \tag{0.4}$$

where the prime $'$ denotes derivative with respect to conformal time η and $X = \frac{1}{2a^2}(\phi')^2$. Clearly, the fields χ_i would then behave like though if they formed a collection of conformal radiation fields were it not by the presence of the nonzero mass terms m_i^2 also at the coincidence time. If for some physical cause the latter mass terms could all be made to vanish at the coincidence time, then all matter fields would behave like though they were a collection of radiation fields filling the universe at around the coincidence time and there would not be the disruption of the evolution from a matter-dominated era to a stable accelerated scaling solution of the kind pointed out by AQTW, but the system smoothly entered the accelerated regime after a given brief interlude where the matter fields behave like pure radiation. In what follows we shall show that in the sub-quantum scenario considered above such a possibility can actually be implemented.

At the end of the day, any physical system always shows the actual quantum nature of its own. One of the most surprising implications tough by dark energy and phantom energy scenarios is that the universal system is not exception on that at any time or value of the scale factor. Thus, we shall look at the particles making up the matter fields in the universe as satisfying the Klein-Gordon wave equation [6] for a Bohmian quasi-classical wave function [7] $\Psi_i = R_i\exp(iS_i/\hbar)$, where we have restored an explicit Planck constant, R_i is the probability amplitude for the given particle to occupy a certain position within the whole homogeneous and isotropic space-time of the universe, as expressed in terms of relativistic coordinates, and S_i is the corresponding classical action also defined in terms of relativistic coordinates.

Taking the real part of the expression resulting from applying the Klein-Gordon equation to the wave function Ψ_i, and defining the classical energy as $E_i = \partial_i S/\partial t$ and the classical momentum as $p_i = \nabla S_i$, one can then derive the modified Hamilton-Jacobi equation

$$E_i^2 - p_i^2 + V_{SQi}^2 = m_{0i}^2, \tag{0.5}$$

where V_{SQi} is the relativistic version of the so-called sub-quantum potential [7] which is here given by

$$V_{SQi} = \hbar\sqrt{\frac{\nabla^2 R_i - \ddot{R}_i}{R_i}}, \tag{0.6}$$

that should also satisfy the continuity equation (i.e. the probability conservation law) for the probability flux, $J = \hbar\,\mathrm{Im}(\Psi^*\nabla\Psi)/(m\mathrm{V})$ (with $\mathrm{V} \propto a^3$ the volume), stemming from the imaginary part of the expression that results by applying the Klein-Gordon equation to the wave equation Ψ. Thus, if the particles are assumed to move locally according to some causal laws [7], then the classical expressions for E_i and p_i will be locally satisfied. Therefore we can now interpret the cosmology resulting from the above formulae as a classical description with an extra sub-quantum potential, and average Eq. (5) with a probability weighting function for which we take $P_i = |R_i|^2$, so that

$$\int\int\int dx^3 P_i \left(E_i^2 - p_i^2 + V_{SQi}^2\right) = \langle E_i^2\rangle_{\mathrm{av}} - \langle p_i^2\rangle_{\mathrm{av}} + \langle V_{SQi}^2\rangle_{\mathrm{av}} = \langle m_{0i}^2\rangle_{\mathrm{av}}, \tag{0.7}$$

with the averaged quantities coinciding with the corresponding classical quantities and the averaged total sub-quantum potential squared being given by $\langle V_{SQ}^2\rangle_{\mathrm{av}} = \hbar^2\left(\langle\nabla^2 P\rangle_{\mathrm{av}} - \langle\ddot{P}\rangle_{\mathrm{av}}\right)$.

It is worth noticing that in the above scenario the velocity of the matter particles should be defined to be given by

$$\langle v_i\rangle_{\mathrm{av}} = \frac{\langle p_i^2\rangle_{\mathrm{av}}^{1/2}}{\left(\langle p_i^2\rangle_{\mathrm{av}} + \langle m_{0i}^2\rangle_{\mathrm{av}} - \langle V_{SQi}^2\rangle_{\mathrm{av}}\right)^{1/2}}. \tag{0.8}$$

It follows that in the presence of a sub-quantum potential, a particle with nonzero rest mass $m_{0i} \neq 0$ can behave like though if was a particle moving at the speed of light (i.e. a radiation massless particle) provided $\langle m_{0i}^2\rangle_{\mathrm{av}} = \langle V_{SQi}^2\rangle_{\mathrm{av}}$. Thus, if we introduce an effective particle rest mass $m_{0i}^{\mathrm{eff}} = \sqrt{\langle m_{0i}^2\rangle_{\mathrm{av}} - \langle V_{SQi}^2\rangle_{\mathrm{av}}}$, then we get that the speed of light again corresponds to a zero effective rest mass. It has been noticed [8], moreover, that in the cosmological context the averaged sub-quantum potential defined for all existing radiation in the universe should be regarded as the cosmic stuff expressible in terms of a scalar field ϕ that would actually make up our scaling dark-energy solution. At the coincidence time, that idea should actually extend in the present formalism to also encompass in an incoherent way, together with the averaged sub-quantum potential for CMB radiation, the averaged sub-quantum potential for matter particles, as a source of dark energy. On the other hand, it has been pointed out as well [8] that the sub-quantum potential ought to depend on the scale factor $a(t)$ in such a way that it steadily increases with time, being the sub-quantum energy density satisfying the above continuity equation what keeps constant along the whole cosmic evolution.

Assuming the mass m_i appearing in the action (4) to be an effective particle mass, it turns out that the onset of dark energy dominance would then be precisely at the coincidence time when $\langle V_{SQi}^2\rangle_{\mathrm{av}} \equiv \langle V_{SQi}(a)^2\rangle_{\mathrm{av}}$ reached a value which equals $\langle m_{0i}^2\rangle_{\mathrm{av}}$ and all the matter fields behaved in this way like a collection of radiation fields which are actually irrelevant to the issue of the incompatibility of the previous eras with a posterior stable accelerated current regime. In this case, the era of matter dominance can be smoothly followed by the current accelerated expansion where all matter fields would effectively behave like though if they cosmologically were tachyons. This interpretation would ultimately amount to the unification of dark matter and dark energy, as the dark energy model being dealt here with is nothing but a somehow quantized version of tachyon dark energy [9], so that one should expect both effective tachyon matter and tachyon dark energy to finally decay to dark matter, so providing a consistent solution to the cosmic coincidence problem.

Now, from our action integral (4) one can derive the equation of motion for the field ϕ; that is (See also Refs. [10] and [11])

$$\ddot{\phi}\left(p_X + 2Xp_{XX}\right) + 3Hp_X\dot{\phi} + 2Xp_{X\rho} - p_\phi = \frac{\delta S}{a^3\delta\phi}, \tag{0.9}$$

where we have restored the cosmic time t, using the notation of Refs. [1], [10] and [11], so that a suffix X or ϕ denotes a partial derivative with respect to X or ϕ, respectively, and now the last coupling term is time-dependent. Note that if we confine ourselves to the theory where $a(t)$ accelerates in an exponential fashion and $\dot{\phi}^2 = 1$ then the first term of this equation would vanish. Anyway, in terms of the energy density ρ for the scalar field ϕ the above general equation becomes formally the same as that which was derived in Ref. [1]

$$\frac{d\rho}{dN} + 3(1+w)\rho = -Q\rho_m \frac{d\phi}{dN}, \tag{0.10}$$

with ρ_m the energy density for the matter field, $N = \ln a$, and $Q = -\frac{1}{a^3\rho_m}\frac{\delta S_m}{\delta\phi}$. We can then derive the condition for the existence of scaling solutions for time-dependent coupling which, as generally the latter two equations are formally identical to those derived by AQTW, is the same as that was obtained by these authors. Hence, we have the generalized master equation for p [1]

$$1 + \frac{2dQ(\phi)}{\lambda Q^2 d\phi}\ \frac{\partial \ln p}{\partial \ln X} - \frac{\partial \ln p}{\lambda Q \partial\phi} = 1, \tag{0.11}$$

whose solution was already obtained by AQTW [1] to be:

$$p(X,\phi) = XQ(\phi)^2 g \quad XQ(\phi)^2 e^{\lambda\kappa(\phi)} \tag{0.12}$$

where g is an arbitrary function, λ is a given function of the parameters of the equations of state for matter and ϕ and the energy density for ϕ, being $\kappa = \int^{\phi} Q(\xi)d\xi$ (see Ref. [1]). In the phase space we then have an equation-of-state effective parameter for the system $w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2} = gx^2 + z^2/3$, with H the Hubble parameter and x and z respectively being $x = \dot{\phi}/(\sqrt{6}H)$ and $z = \sqrt{\rho_{\text{rad}}/(3H^2)}$. At the coincidence time where we have just radiation ($z \neq 0$ and $\rho_m = \rho_{\text{rad}}$) the effective equation of state is [1] $w_{\text{eff}} = 1/3$. Hence at the coincidence time interval we can only have radiation, neither matter or accelerated expansion domination, just the unique condition that would allow the subsequent onset of the accelerated expansion era where conformal invariance of the field χ no longer holds.

Thus, it appears that in the considered model a previous matter-dominated phase can be evolved first into a radiation phase at a physical regular coincidence short stage which is then destroyed to be finally followed by the required new, independent phase of current accelerating expansion. This conclusion can be more directly drawn if one notices that there is no way by which the general form of the Lagrangian (12) can accommodate the Lagrangian final form $L \equiv p = f(a,\dot{a})\dot{\phi}^2 V_{SQ}^2$ which characterizes sub-quantum dark energy models whose pressure p vanishes in the limit $V_{SQ} \to 0$. It thus appears that at least these models can be taken to be counter examples to the general conclusion that current dark-energy and modified gravity models (see however Ref. [12]) are incompatible with the existence of a previous matter-dominated phase, as suggested in Ref. [1].

We finally notice, moreover, that the kind of sub-quantum dark energy theory providing the above counter example is one which shows no classical analog (i.e. the Lagrangian, energy density and pressure are all zero in the classical limit $\hbar \to 0$) and is thereby most economical of all. Thus, the above conclusion can also be stated by saying that, classically, a previous phase of matter dominance is always compatible with the ulterior emergence of a dominating phase made up of "nothing". In this way, similarly to as the abrupt, unphysical exit of the old inflationary problem was circumvented by introducing [4] a scalar field potential with a flat plateau leading to a "slow-rollover" phase transition, the abrupt disruption of the scaling phase after matter dominance can be also avoided by simply considering a vanishing scalar field potential that smooths the transition and ultimately makes it to work.

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Reconstructing the f(R) gravity from the holographic principle

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Abstract

An holographic f(R) gravity model of dark energy is proposed. The correspondence between the f(R) geometrical effective energy density with the holographic density, allows the reconstruction of the f(R) gravity in flat FRW background in the Einstein frame. The proposed infrared cut-off for the holographic energy density depends on two parameters which are fit using the luminosity versus redshift data, allowing a suitable reconstruction in two representative cases of the EoS parameter: for $\omega > -1$ and $\omega < -1$.
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1 Introduction

Many astrophysical data, such as observations of large scale structure [1], searches for type Ia supernovae [2], and measurements of the cosmic microwave background anisotropy [3], all indicate that the expansion of the universe is undergoing cosmic acceleration at the present time, due to some kind of negative-pressure form of matter known as dark energy (for a review see[13],[5]). Although the cosmological observations suggest that dark energy component is about 2/3 of the total content of the universe, the nature of the dark energy as well as its cosmological origin remain unknown at present. Among other approaches related with a variety of scalar fields (see [13]), a very promising approach to dark energy is related with the modified theories of gravity known as $f(R)$ gravity, in which dark energy emerges from the modification of geometry [6],[7],[8],[9],[10]. Modified gravity gives a natural unification of the early-time inflation and late-time acceleration thanks to different role of gravitational terms relevant at small and at large curvature and may naturally describe the transition from deceleration to acceleration in the cosmological dynamics. Among others, modified gravity is one of the areas of the gravitational science in which S. Odintsov has contributed to develop thanks to a great number of pioneer investigations. Thus, the connection of modified gravity theories with M-theory was suggested in [6] and some explicit asymptotic examples were shown. In [7],[8],[11],[12],[13], a modified gravity model with positive and negative powers of curvature and $\ln R$ terms were proposed, to describe different unification scenarios of early time inflation with the late-time cosmic acceleration. This unification was also studied with a Gauss-Bonnet invariant dependent function as proposed in [14], [15]. The one-loop quantization approach to $f(R)$ gravity was investigated in [16]. The asymptotic behavior of a wide variety of $f(R)$ models, for different cosmological epochs in the FRW background has been presented in [17] and a consistency with experimental tests has been studied in [17],[18],[19],[20]. A comprehensive work on reconstruction of different modified

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$f(R)$ models, has been done trough the works [21],[22],[23],[24], [25], where it was explicitly demonstrated which versions of above theories may be reconstructed from the known universe expansion history (see also [10], [26], [27] for $f(R)$ reconstruction). Other generalizations like a non-local model of f(R) gravity, non-minimal coupling of f(R) gravity with matter, with Maxwell theory and non-minimal Yang-Mills coupling were discussed in [28, 29, 30, 31] respectively. The structure of future singularities in models of $f(R)$ gravity consistent with local tests, have been worked in [32], [33], and the constraining of some $f(R)$ models according to VIRGO, LIGO and LISA experiments have been investigated in [34].
Another way to the solution of the dark energy problem within the fundamental theory framework, is related with some facts of the quantum gravity theory, known as the holographic principle. Applied to the dark energy issue, this principle establishes an infrared cut-off for the so called holographic energy density, related with cosmological scales (discussion and references are given in [35],[36]). An infrared cut-off given by the future event horizon was proposed in [35] and a generalized holographic infrared cut-off depending on local and non-local quantities was proposed in [37]. Viewing the modified $f(R)$ gravity models as an effective description of the underlying theory of dark energy, and considering the holographic vacuum energy scenario as pointing in the direction of the underlying theory of dark energy, it is interesting to study how the $f(R)$ gravity can describe the holographic energy density as an effective theory. The reconstruction of $f(R)$ theories under different conditions has been presented in [21, 22, 23, 24, 25, 26, 27] and the holographic reconstruction of $f(R)$ gravity have been discussed in [27] with the infrared cut-off given by the future event horizon. In this contribution, an holographic $f(R)$ reconstruction using the new infrared cut-off given in [36, 38], is presented.

2 The holographic model

Let us start with the holographic dark energy density given by [36, 38]

$$\rho_\Lambda = 3M_p^2 \quad \alpha H^2 + \beta \dot{H} \tag{2.1}$$

where $H = \dot{a}/a$ is the Hubble parameter and α, β are constants which must satisfy the restrictions imposed by the current observational data. This kind of density may appear as the simplest case of more general $f(H, \dot{H})$ holographic density in the FRW background. Note that this proposal avoids conflict with the causality. Here the constants α and β will be fixed according to the luminosity versus redshift observational data as given in [39], based on [40] data. As we will focus on late time reconstruction, we consider that the holographic dark energy density gives the dominant contribution to the Friedmann equation and neglect the contribution from matter and radiation, thus the Friedmann equation becomes simpler

$$H^2 = \frac{1}{3M_p^2}\rho_\Lambda = \alpha H^2 + \beta \dot{H} \tag{2.2}$$

which gives a power-law solution [38]

$$H = \frac{\beta}{\alpha - 1}\frac{1}{t} \tag{2.3}$$

with constant equation of state parameter given by

$$\omega_\Lambda = -1 + \frac{2}{3}\frac{\alpha - 1}{\beta} \tag{2.4}$$

In terms of the redshift parameter we can write H in Eq. (2.3) as follows

$$H = H_0 \left(1 + z\right)^{(\alpha-1)/\beta} \tag{2.5}$$

where $H_0 = \frac{\beta}{\alpha-1}\frac{1}{t_0}$ and t_0 is the present time ($z = 0$). In the flat FRW background the luminosity distance L_d can be written as [13]

$$d_L = (1 + z)\int_0^z \frac{dz}{H(z)} \tag{2.6}$$

where we used the light speed $c = 1$. With $H(z)$ given by (2.5) the luminosity distance takes the explicit form from (2.6)

$$d_L(z) = \frac{1+z}{H_0}\frac{\beta}{\beta-\alpha+1}\left[(1+z)^{(\beta-\alpha+1)/\beta}-1\right] \tag{2.7}$$

based on this result we can plot the luminosity distance L_d in terms of the redshift, and tune the parameters α and β in such a way that may fit the data reconstructed plot as shown in [5] and [39]. The plot for three representative values of the α and β parameters is given in Fig. 1

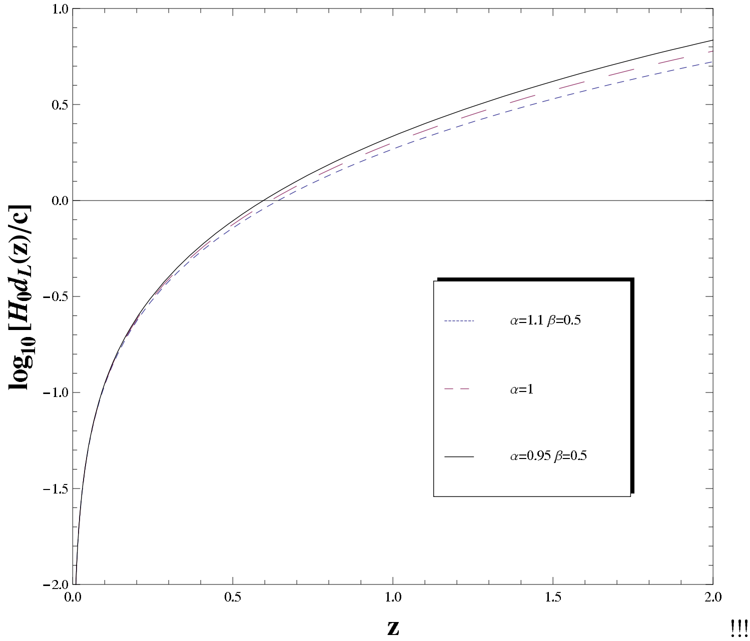

Figure 1: The luminosity distance $\log_{10}(H_0 d_L(z))$ versus redshift. The three curves show $H_0 d_L$ in logarithmic scale for $\Omega_{m0} = 0$, $\Omega_{\Lambda 0} = 1$, $\alpha = 1.1, 1, 0.95$ and $\beta = 0.5$. The $\alpha = 1$ curve corresponds to the de Sitter solution.

3 Late time reconstruction of f(R)

Let's start with the action for f(R) gravity with matter

$$S = \int d^4x\sqrt{-g}\;\; \frac{1}{2\kappa^2}f(R) + \mathcal{L}_m \tag{3.1}$$

where $f(R)$ is a generic function of the Ricci scalar curvature R, and $\mathcal{L}_m$ is the matter fields Lagrangian density. Variation with respect to the metric and setting aside the complications with the variations of $f(R)$-type actions leads to the field equation [33],[41]

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\, f'(R) = \kappa^2 T^{(m)}_{\mu\nu} \tag{3.2}$$

where ∇_μ is the covariant derivative, prime denotes the derivative with respect to R and $T^{(m)}_{\mu\nu}$ is the stress-energy tensor for the standard matter source. Interpreting this field equations in the form of Einstein equations with an effective stress-energy tensor composed of curvature terms moved to the right hand side, Eq. (3.2) can be written as (hereafter we will take $\kappa^2 = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{T^{(m)}_{\mu\nu}}{f'(R)} + T^{(R)}_{\mu\nu} \tag{3.3}$$

with $T^{(R)}_{\mu\nu}$ given by

$$T^{(R)}_{\mu\nu} = \frac{1}{f'(R)} \quad \frac{1}{2} \quad f(R) - Rf'(R) \quad g_{\mu\nu} + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) \tag{3.4}$$

where $T^{(R)}_{\mu\nu}$ may be defined as an effective stress-energy tensor for some kind of curvature fluid. This allows to put Eq. (3.4) in the form of a perfect fluid energy-momentum tensor, which will turn out to be useful in our reconstruction scheme. Note also that in this case the matter term becomes coupled to the geometry through $f'(R)$.
Assuming flat FRW background, the modified Friedmann equations take the form

$$H^2 = \frac{1}{3f'(R)} \quad \rho_m + \frac{Rf'(R) - f(R)}{2} - 3H\dot{R}f''(R) \tag{3.5}$$

in this way we can define an effective curvature-density

$$\rho_R = \frac{1}{f'(R)} \quad \frac{Rf'(R) - f(R)}{2} - 3H\dot{R}f''(R) \tag{3.6}$$

The space-space components of (3.3) lead to the other Einstein equation

$$2\dot{H} + 3H^2 = -\frac{1}{f'(R)} \quad p_m + \ddot{R}f''(R) + \dot{R}^2 f'''(R) + 2H\dot{R}f''(R) + \frac{1}{2} \quad f(R) - Rf'(R) \tag{3.7}$$

which allows the identification of the effective curvature-pressure

$$p_R = \frac{1}{f'(R)} \quad \ddot{R}f''(R) + \dot{R}^2 f'''(R) + 2H\dot{R}f''(R) + \frac{1}{2} \quad f(R) - Rf'(R) \tag{3.8}$$

Hence, the curvature terms in the r.h.s. of Eqs. (3.6) and (3.8) can be viewed as an effective curvature-fluid with equation of state parameter expressed as

$$\omega_R = -1 + 2\frac{\ddot{R}f''(R) + \dot{R}^2 f'''(R) - H\dot{R}f''(R)}{Rf'(R) - f(R) - 6H\dot{R}f''(R)} \tag{3.9}$$

defining the function $\Phi(R) = f'(R)$ [41] and combining Eqs. (3.5) and (3.6) it follows that the EoS parameter for the curvature-fluid (3.9) can be written as

$$\omega_R = -1 + \frac{\ddot{\Phi} - H\dot{\Phi}}{3H^2\Phi - \rho_m} \tag{3.10}$$

where the last term ρ_m is the pressureless dark matter contribution. According to the holographic correspondence this EoS parameter should be equal to the one given by Eq. (2.4). This conduces to the following equation for the function Φ

$$\ddot{\Phi} - H\dot{\Phi} - 2\frac{\alpha - 1}{\beta}\Phi H^2 = -\frac{2}{3}\frac{\alpha - 1}{\beta}\rho_m \tag{3.11}$$

It should be noted here that the cosmological evolution takes place in the Einstein frame as is clear from Eq. (3.3), and therefore only in this frame the solution (2.3) makes sense as this solution is obtained via solving the standard Einstein equations.
Changing the variable from cosmic time t to redshift z, the matter density ρ_m becomes

$$\rho_m = 3H_0^2 \Omega_{m0}(1+z)^3 \tag{3.12}$$

here we assumed that ρ_m is conserved separately. After this change of variable, the equation (3.11) transforms to

$$(1+z)^2 H^2 \frac{d^2\Phi}{dz^2} + (1+z)^2 H \frac{dH}{dz}\frac{d\Phi}{dz} - 2\frac{\alpha - 1}{\beta}\Phi = -2\frac{\alpha - 1}{\beta} H_0^2 \Omega_{m0}(1+z)^3 \tag{3.13}$$

The general solution to this equation is given by

$$\begin{aligned}\Phi(z) =&\frac{2p\Omega_{m0}}{9p-6p^2-2}\,(1+z)^{3-2/p}+\Phi_0^+\,(1+z)^{(p-1+\sqrt{(p-1)^2+8p})/2p}\\&+\Phi_0^-\,(1+z)^{(p-1-\sqrt{(p-1)^2+8p})/2p}\end{aligned}\tag{3.14}$$

where $p=\frac{\beta}{\alpha-1}$ and the constants $\Phi_0^\pm$ will be determined using local solar system consistency. To determine $f(R)$, we first reconstruct the form of $f(R)$ from (3.14) as a function of the redshift z. From $\Phi(R)=f'(R)$ it follows that

$$\frac{df[R(Z)]}{dz}=\Phi(z)\frac{dR}{dz}\tag{3.15}$$

Expressing the scalar curvature $R=6(\dot{H}+2H^2)$ in terms of z through the equation (3.12) and replacing Φ from (3.14), the Eq. (3.15) can be integrated with respect to z, yielding

$$\begin{aligned}f(z) =&C_1(p)H_0^2\Omega_{m0}(1+z)^3+C_2^+(p)H_0^2\Phi_0^+\,(1+z)^{(p+3+\sqrt{(p-1)^2+8p})/2p}\\&+C_2^-(p)H_0^2\Phi_0^-\,(1+z)^{(p+3-\sqrt{(p-1)^2+8p})/2p}\end{aligned}\tag{3.16}$$

where

$$C_1(p)=\frac{8(1-2p)}{6p^3-9p^2+2p},\;C_2^\pm(p)=\frac{24(2p-1)}{p(p+3\pm\sqrt{(p-1)^2+8p})}\tag{3.17}$$

writing the variable $(1+z)$ in terms of R using Eq. (3.12), we can obtain $f(R)$ explicitly in terms of R

$$\begin{aligned}f(R) =&\left(\frac{p}{6H_0^2(2p-1)}\right)^{3p/2}H_0^2\Omega_{m0}C_1(p)R^{3p/2}\\&+\left(\frac{p}{6H_0^2(2p-1)}\right)^{(p+3+\sqrt{(p-1)^2+8p})/4}H_0^2\Phi_0^+C_2^+(p)R^{(p+3+\sqrt{(p-1)^2+8p})/4}\\&+\left(\frac{p}{6H_0^2(2p-1)}\right)^{(p+3-\sqrt{(p-1)^2+8p})/4}H_0^2\Phi_0^-C_2^-(p)R^{(p+3-\sqrt{(p-1)^2+8p})/4}\end{aligned}\tag{3.18}$$

The constants of integration $\Phi_0^\pm$ can be determined using a local consistency conditions with the aim to not affect by measurement errors the reconstructed $f(R)$. This conditions are $f'(R)_{z=0}=1$ (to recover the actual value of the Newtonian constant from Eq. (3.3) at $z=0$) and $f''(R)_{z=0}=0$ (consistency with the weak field approximation), which translates into (see [42],[26])

$$\left.\frac{df}{dz}\right|_{z=0}=\left.\frac{dR}{dz}\right|_{z=0},\quad\left.\frac{d^2f}{dz^2}\right|_{z=0}=\left.\frac{d^2R}{dz^2}\right|_{z=0}\tag{3.19}$$

then, using this conditions the constants $\Phi_0^\pm$ become

$$\begin{aligned}\Phi_0^\pm =&3\Big[\mp 8(2p^2-5p+2)\mp 4(2p-1)(3-p\mp\sqrt{p^2+6p+1})\\&+p^2C_1(p)\Omega_{m0}(3-5p\mp\sqrt{p^2+6p+1})\Big]/\\&\Big[pC_2^\pm(p)\sqrt{p^2+6p+1}(p+3\pm\sqrt{p^2+6p+1})\Big]\end{aligned}\tag{3.20}$$

Note that this reconstruction was made in the low redshift region, where the power law expansion resulting from the restriction of the holographic model to the dark energy dominance (see (2.2)), is an acceptable approximation according to the data, as shown in Fig. 1. Note also that our model is approximate as is obtained from reconstruction scheme and it is valid within the adopted approximation. It is not so easy to recover more realistic non-linear f(R) gravity which gives theory 3.18 as the approximation.

4 Discussion

In this work we showed a reconstruction of the late time cosmological $f(R)$ gravity dynamics, based on the holographic energy density cut-off proposed in [36],[38]. An analytical expression for the $f(R)$ model compatible with observational data at low redshift has been found. The reconstructed Lagrangian is of a power-law type and contains only positive curvature powers for $\alpha > 1, \beta > 0$ (quintessence phase) and negative curvature powers for $\alpha < 1, \beta > 0$ (phantom phase), except in the region of parameters $-3-\sqrt{8} < \beta/(\alpha-1) < \sqrt{8-3}$, which is forbidden in this model as the power becomes complex. The $\alpha < 1, \beta < 0$ can also be considered, giving similar to the case $\alpha > 1, \beta > 0$ results. From equations (2.2) and (2.4) it follows that for $\alpha = 1$ we obtain the de Sitter solution with $\omega_\Lambda = -1$. The last two terms in (3.18) correspond to the homogeneous solution of (3.11) with $\Omega_{m0} = 0$, which describe a purely geometrical-dominance model of dark energy . For the specific case of $\alpha = 1.1, \beta = 0.5$ we expect that the term with lower power $R^{2-\sqrt{7}/4}$ dominates. The matter dominance epoch occurs for $p = 2/3$, giving the Einstein term plus corrections (this corrections may be caused due to the extrapolation to high redshift). Is important to note that the holographic dark energy density used here was obtained within the Einstein frame and not as the solution to the modified $f(R)$ gravity. Hence the reconstructed $f(R)$ theory effectively describes the holographic dark energy in Einstein gravity. Nevertheless, Fig. 1 shows that the proposed solution (2.3) is a good starting point, at least at the phenomenological level, to reconstruct the $f(R)$ model at low redshift independent of the theory.
In summary, we have studied how the modified $f(R)$ gravity model can be used to describe the holographic energy density as an effective theory at low redshift, which conduces to the reconstruction in a direct and unambiguous way. However, this reconstruction is settled at the phenomenological level, and the theoretical root of the holographic density still to be investigated.

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Four- and eight-fermion interactions in a space-time with non-trivial topology

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Abstract

The phase structure of an eight-fermion interaction model is investigated in a compact space-time with non-trivial topology, $M^{D-1} \otimes S^1$. The phase boundary dividing the symmetric and the broken phase is shown as a function of the space-time dimensions. It is found that the eight-fermion interaction does not modify the phase boundary for the chiral symmetry breaking.

1 Introduction

It is expected that a higher symmetry may be realized at the early universe. Some of the symmetry is spontaneously broken to yield a theory with a lower level symmetry in the current universe. The mechanism of the symmetry breaking may be tested in critical phenomena in the early universe. One of the possible mechanisms to induce the spontaneous symmetry breaking is found in the dynamics of quarks and gluons. In QCD the expectation value of the composite operator constructed by a quark and an anti-quark field develops a non-vanishing value and the chiral symmetry is eventually broken [1].

Y. Nambu and G. Jona-Lasinio introduced a four-fermion interaction model as a prototype model to describe the dynamical mechanism of the symmetry breaking. The model is regards as a low energy effective theory of a fundamental theory at high energy and extend to include another type interactions, higher dimensional operators and so on [2, 3, 4]. Much work has been done to study the phase structure of the models under extreme situation at the early universe, inside dense stars, or heavy ion collisions.

If the fundamental theory is described by the superstring theory, it is defined in ten space-time dimensions. Thus six space-time dimensions should be compactified to realize four space-time dimensions at low energy. The compactified space-time may have a non-trivial topology. The finite size and the topological effects from the compactified space-time may play an important role in the symmetry breaking at very early universe. It is also expected that the critical phenomena for the symmetry breaking may explain how to stabilize the compactification scale.

The finite size and the topological effects are investigated in the four-fermion interaction model with one compactified dimension, S^1, [5, 6, 7] the torus universe [8, 9, 10, 11, 12], and D-dimensional sphere, S^D, [13, 14]. In the case of the anti-periodic boundary condition the finite size effect raises the ground state energy and the broken symmetry is restored for a sufficiently small compact direction. The finite temperature effect is classified as the similar type effect. The fermion fields which possess the periodic boundary condition have an opposite effect. The symmetry breaking is enhanced by the finite size effect.

In the present paper we consider a model with a four- and an eight-fermion interaction and study a contribution from the higher dimensional operator to the finite size and the topological effects on the symmetry breaking. In Sec. 2 we introduce the model with scalar type four- and eight-fermion interactions in Minkowski space-time, M^D, and evaluate the effective potential. In Sec. 3 we consider the model in $S^1 \otimes M^{D-1}$ and discuss the influence of the finite-size and the non-trivial topology to the symmetry breaking. Solving the gap equations, we calculate the critical size of the compactified dimension. The critical temperature is also mentioned. In Sec. 4 we give concluding remarks.

2 Eight-fermion interaction model in Minkowski space-time, M^D

In this section we introduce the model with scalar type four- and eight-fermion interactions and study the features of the model in arbitrary space-time dimensions $2 < D < 4$. We start with the simple model with N components of fermions described by the Lagrangian [4],

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i i\gamma^\mu \partial_\mu \psi_i + \frac{G_1}{N} \left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2 + \frac{G_2}{N} \left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^4, \tag{2.1}$$

where the index i represents flavor of the fermion field ψ, N is the number of the flavors, G_1 and G_2 the coupling constants for the four- and the eight-fermion interactions, respectively.

The Lagrangian (2.1) has a global $SU(N)$ flavor symmetry under the transformation,

$$\psi \to e^{i \sum_a g_a T^a} \psi, \tag{2.2}$$

where T^a is the generators of the $SU(N)$ symmetry. The Lagrangian (2.1) is also invariant under a discrete Z_2 chiral transformation,

$$\psi \to \gamma_5 \psi. \tag{2.3}$$

This discrete chiral symmetry prevents the Lagrangian from having mass terms. If the composite operator $\bar{\psi}\psi$ develops a non-vanishing value, fermion mass terms are generated and the discrete chiral symmetry is dynamically broken.

In practical calculations it is more convenient to introduce auxiliary fields, σ_1, σ_2, and evaluate the equivalent Lagrangian

$$\mathcal{L}_{aux} = \bar{\psi} \left(i\gamma^\mu \partial_\mu - \sigma \right) \psi - \frac{N\sigma_1^2}{4G_1} - \frac{N\sigma_2^2}{4G_2}, \tag{2.4}$$

where σ is defined by

$$\sigma \equiv \sigma_1 \sqrt{1 + \frac{N\sigma_2}{G_1}}\ . \tag{2.5}$$

It is easily found that the original Lagrangian (2.1) can be reproduced by replacing σ_1 and σ_2 by the solutions of the equations of motion,

$$\sigma_1 = -\frac{2G_1}{N} \sqrt{1 + \frac{N\sigma_2}{G_1}}\, \bar{\psi}\psi, \quad |\sigma_2| = \mathrm{sgn}(G_1) \frac{2G_2}{N} (\bar{\psi}\psi)^2. \tag{2.6}$$

To find the ground state of the model we observe the minimum of the effective potential for the auxiliary fields σ_1 and σ_2. In the leading order of the $1/N$ expansion the effective potential is given by

$$V_0(\sigma_1, \sigma_2) = \frac{\sigma_1^2}{4G_1} + \frac{\sigma_2^2}{4G_2} + i\mathrm{tr} \ln \frac{i\gamma^\mu \partial_\mu - \sigma}{i\gamma^\mu \partial_\mu}, \tag{2.7}$$

where we normalize the effective potential so that $V(0,0) = 0$.

In the D-dimensional Minkowski space-time, M^D, we rewrite the second term on the right hand side of Eq.(2.7) by using the Schwinger proper time method and obtain

$$V_0 = \frac{\sigma_1^2}{4G_1} + \frac{\sigma_2^2}{4G_2} - i\mathrm{tr} \int_0^\sigma s ds \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - s^2}. \tag{2.8}$$

We can perform the integration and get

$$
\begin{aligned}
V_0 &= \frac{\sigma_1^2}{4G_1} + \frac{\sigma_2^2}{4G_2} + \frac{\mathrm{tr}1}{2(4\pi)^{D/2}}\Gamma \quad -\frac{D}{2} \quad (\sigma^2)^{D/2} \\
&= \frac{\sigma_1^2}{4G_1} + \frac{\sigma_2^2}{4G_2} + \frac{\mathrm{tr}1}{2(4\pi)^{D/2}}\Gamma \quad -\frac{D}{2} \quad ({\sigma_1}^2)^{D/2} \quad 1 + \quad \frac{N\sigma_2}{G_1} \quad ^{D/2} .
\end{aligned} \tag{2.9}
$$

Since the first and the third terms on the right hand side of Eq.(2.9) disappear for $\sigma_1 = 0$, we should set a positive value for the coupling constant G_2 to obtain a stable ground state. At the limit $\sigma_2 \to 0$ the effective potential (2.9) coincides with the one for the four-fermion interaction model. If the space-time dimensions are more than four, the main contribution comes from the third term on the right hand side of Eq.(2.9) for a large σ_1 and σ_2. Thus the effective potential can not be bounded below and there is no stable ground state. For $6 < D < 8$ a stable ground state can be realized.

To find the minimum of the effective potential, V_0, we differentiate the effective potential in terms of σ_1 and σ_2 and get

$$
\frac{\partial V_0}{\partial \sigma_1} = \sigma_1 \left[\frac{1}{2G_1} - \frac{\mathrm{tr}1}{(4\pi)^{D/2}}\Gamma \quad 1 - \frac{D}{2} \quad ({\sigma_1}^2)^{D/2-1} \quad 1 + \quad \frac{N\sigma_2}{G_1} \quad ^{D/2} \right], \tag{2.10}
$$

$$
\frac{\partial V_0}{\partial \sigma_2} = \frac{\sigma_2}{2G_2} - \frac{\mathrm{sgn}(\sigma_2)\mathrm{tr}1}{2(4\pi)^{D/2}}\Gamma \quad 1 - \frac{D}{2} \quad \frac{N}{|G_1|}({\sigma_1}^2)^{D/2} \quad 1 + \quad \frac{N\sigma_2}{G_1} \quad ^{D/2-1} . \tag{2.11}
$$

For $2 < D < 4$ and $6 < D < 8$ the right hand side of Eq.(2.11) has a negative value for $\sigma_2 < 0$ and a positive value for $\sigma_2 > 0$. It implies that the effective potential monotonically decreases for $\sigma_2 < 0$ and increases for $\sigma_2 > 0$. Therefore the minimum of the effective potential is found at $\sigma_2 = 0$ and the expectation value for σ_1 is equal to that for the four-fermion interaction model,

$$
\langle \sigma_1 \rangle = \pm \quad \frac{1}{2G_1} \frac{(4\pi)^{D/2}}{\mathrm{tr}1\Gamma\,(1 - D/2)} \quad ^{1/(D-2)} . \tag{2.12}
$$

We normalize the mass scale by an arbitrary renormalization scale μ and numerically evaluate the effective potential (2.9). We observe only the symmetric phase for a positive G_1. The discrete chiral symmetry is broken for $G_1 < 0$. In Figs.1 and 2 a typical behavior of the effective potential, $V_0(\sigma_1, \sigma_2)$, is drawn for fixed space-time dimension D less than four. As is shown in Fig.1, the state $\sigma_1 = \sigma_2 = 0$ is unstable for a negative four-fermion coupling G_1. Thus the composite operator, $\bar{\psi}\psi$ develops a non-vanishing value and the discrete chiral symmetry is dynamically broken. This result exactly reproduce the one obtained in Ref.[15]. In Fig.2 it is shown that the eight-fermion interaction enhances the symmetry breaking. We also observe that the expectation value for the auxiliary field, σ_2, is vanishing. It is consistent with the behavior of Eq.(2.11) discussed above.

3 Eight-fermion interaction model in $M^{D-1} \otimes S^1$

If the space-time has a compact direction, the effective potential is modified by finite size and topological effect. Here we focus on the finite size and topological effect and consider a simple flat space-time with non-trivial topology. We assume that one of the space direction is one dimensional sphere, S^1. In $M^{D-1} \otimes S^1$ the effective potential is given by

$$
V(\sigma_1, \sigma_2) = \frac{\sigma_1^2}{4G_1} + \frac{\sigma_2^2}{4G_2} - i\mathrm{tr} \int_0^{\sigma} sds \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{k^2 - s^2}, \tag{3.1}
$$

where L is the size of the S^1 direction, k_1 is replaced as

$$
k_1 \to \frac{(2n + \delta_{p,1})\pi}{L}, \tag{3.2}
$$

and $\delta_{p,1}$ is fixed by the boundary condition, i.e. $\delta_{p,1} = 0$ for the fermion fields with the periodic boundary condition and $\delta_{p,1} = 1$ for the fermion fields with the anti-periodic boundary condition [5, 7].

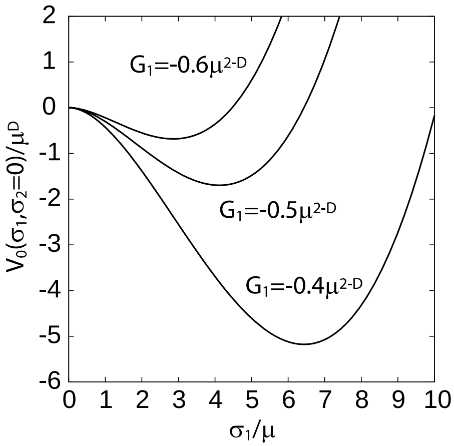

Figure 1: Behavior of the effective potential (2.9) at $\sigma_2 = 0$ for $N = 3$, $G_2 = 0.5\mu^{4-3D}$, $D = 2.5$.

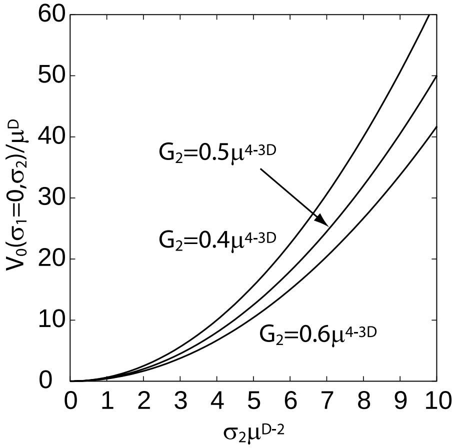

Figure 2: Behavior of the effective potential (2.9) at $\sigma_1 = 0$ for $N = 3$, $G_1 = -0.5\mu^{2-D}$, $D = 2.5$.

Performing the integration in Eq.(3.1), the effective potential reads

$$\begin{aligned} V(\sigma_1, \sigma_2) &= \frac{\sigma_1^2}{4G_1} + \frac{\sigma_2^2}{4G_2} + \frac{\mathrm{tr}1}{2(4\pi)^{(D-1)/2}}\Gamma \quad \frac{1-D}{2} \\ &\times \frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\quad \frac{(2n+\delta_{p,1})\pi}{L} \quad {}^2 + \sigma^2 \right]^{(D-1)/2} - V(0,0). \end{aligned} \tag{3.3}$$

This expression is convenient for analytically evaluating the behavior of the effective potential. Only the second term on the right hand side of Eq.(3.3), remains for $\sigma_1 = 0$, we again set a positive value for the coupling constant G_2 to obtain a stable ground state.

We can also perform the summation in Eq.(3.1) and get

$$\begin{aligned} V(\sigma_1, \sigma_2) &= V_0(\sigma_1, \sigma_2) - \frac{\mathrm{tr}1}{(4\pi)^{(D-1)/2}} \frac{2}{\Gamma \quad \frac{D-1}{2}} \\ &\times \frac{1}{L}\int_0^\infty k^{D-2} dk \ln \frac{1-(-1)^{\delta_{p,1}}\exp \quad -L\sqrt{k^2+\sigma^2}}{1-(-1)^{\delta_{p,1}}\exp\left(-L|k|\right)}. \end{aligned} \tag{3.4}$$

These expressions is suitable for the numerical analysis of the effective potential.

To find the phase structure of the model we want to find the ground state of the model in $S^1 \otimes M^D$. The ground state is found by observing the minimum of the effective potential. The potential is not always analytic at $\sigma_2 = 0$. Hence, the minimum satisfies the equations,

$$\begin{aligned} \frac{\partial V}{\partial \sigma_1} &= \frac{\sigma_1}{2G_1} - \frac{\mathrm{tr}1}{(4\pi)^{(D-1)/2}}\Gamma \quad \frac{3-D}{2} \quad \sigma_1 \quad 1 + \frac{N\sigma_2}{G_1} \\ &\times \frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\quad \frac{(2n+\delta_{p,1})\pi}{L} \quad {}^2 + \sigma^2 \right]^{(D-3)/2} = 0, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \frac{\partial V}{\partial \sigma_2} &= \frac{\sigma_2}{2G_2} - \frac{\mathrm{sgn}(\sigma_2)\mathrm{tr}1}{2(4\pi)^{(D-1)/2}}\Gamma \quad \frac{3-D}{2} \quad \frac{N{\sigma_1}^2}{|G_1|} \\ &\times \frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\quad \frac{(2n+\delta_{p,1})\pi}{L} \quad {}^2 + \sigma^2 \right]^{(D-3)/2} = 0, \end{aligned} \tag{3.6}$$

or $\sigma_2 = 0$. In the latter case, $\sigma_2 = 0$, the effective potential coincides with the one for the four-fermion interaction model and the minimum of the effective potential is found by solving Eq.(3.5) at $\sigma_2 = 0$.

Non-vanishing expectation values for σ_1 and σ_2 are given by the solution of the gap equations,

$$\begin{aligned}\frac{1}{2G_1} &= \frac{\mathrm{tr}1}{(4\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right)\left(1+\frac{N\langle\sigma_2\rangle}{G_1}\right)\\ &\times\frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\left(\frac{(2n+\delta_{p,1})\pi}{L}\right)^2+\langle\sigma\rangle^2\right]^{(D-3)/2},\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\frac{1}{2G_2} &= \frac{\mathrm{tr}1}{2(4\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right)\frac{N\langle\sigma_1\rangle^2}{|G_1\langle\sigma_2\rangle|}\\ &\times\frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\left(\frac{(2n+\delta_{p,1})\pi}{L}\right)^2+\langle\sigma\rangle^2\right]^{(D-3)/2}.\end{aligned}\tag{3.8}$$

Eliminating the summation from these gap equations, we obtain a relationship between $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ [4],

$$\frac{N}{2G_1|G_1|}\langle\sigma_1\rangle^2 = \frac{|\langle\sigma_2\rangle|}{G_2}\left(1+N\frac{\langle\sigma_2\rangle}{G_1}\right).\tag{3.9}$$

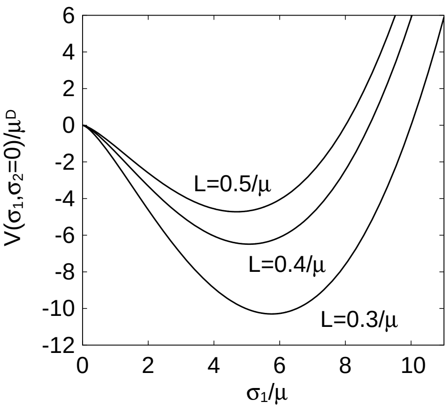

Figure 3: Behavior of the effective potential (3.4) for fermion fields with the periodic boundary condition, $\delta_{p,1} = 0$, at $\sigma_2 = 0$ for $N = 3$, $G_1 = -0.5\mu^{2-D}$, $G_2 = 0.5\mu^{4-3D}$, $D = 2.5$.

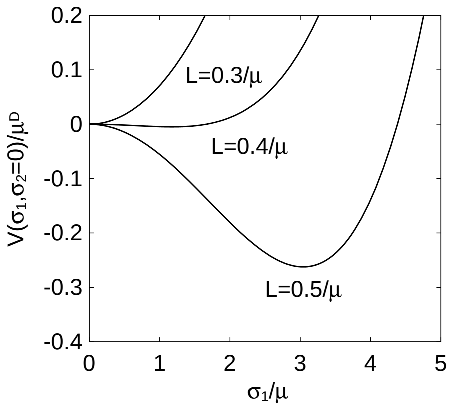

Figure 4: Behavior of the effective potential (3.4) for fermion fields with the anti-periodic boundary condition, $\delta_{p,1} = 1$, at $\sigma_2 = 0$ for $N = 3$, $G_1 = -0.5\mu^{2-D}$, $G_2 = 0.5\mu^{4-3D}$, $D = 2.5$.

If we set a negative value to G_1, Eq.(3.9) has no solution and the minimum is given by Eq.(3.8) at $\sigma_2 = 0$. We numerically show the effective potential at $\sigma_2 = 0$ as a function of σ_1 in Figs.3 and 4 for fermion fields with the periodic and the anti-periodic boundary condition, respectively. As is shown in Fig.3, the symmetry breaking is enhanced by the finite size effect and only the broken phase can be realized for the fermion fields with the periodic boundary condition. In Fig.4 it is clearly seen that the finite size effect restores the broken symmetry through the second order phase transition for the fermion fields with the anti-periodic boundary condition. These results exactly reproduce those for the four-fermion interaction model.

We also numerically evaluate the effective potential for a positive G_1. In this case we observe only the symmetric phase for the fermion fields with the anti-periodic boundary condition. On the other hand, we find the broken phase for the fermion fields with the periodic boundary condition. The chiral symmetry is dynamically broken for a sufficiently small L through the second order phase transition. In Fig.5 we draw the effective potential at $\sigma_2 = 0$. We also draw the effective potential under the relationship (3.9) in Fig.6. Comparing Fig.5 with 6, we find that the absolute minimum

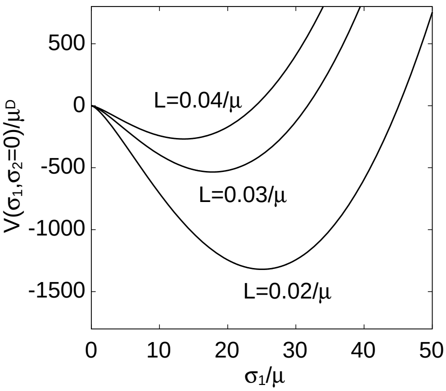

Figure 5: Behavior of the effective potential (3.4) for fermion fields with the periodic boundary condition, $\delta_{p,1} = 0$, at $\sigma_2 = 0$, $N = 3$, $G_1 = 0.5\mu^{2-D}$, $G_2 = 0.5\mu^{4-3D}$, $D = 2.5$.

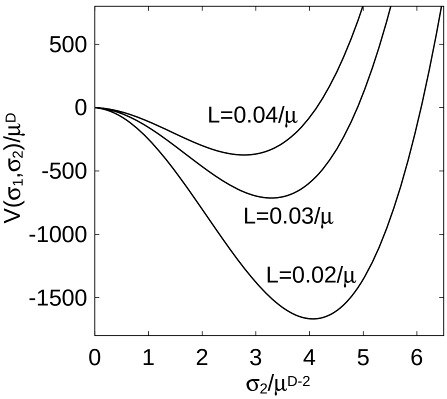

Figure 6: Behavior of the effective potential (3.4) for fermion fields with the periodic boundary condition , $\delta_{p,1} = 0$, under the relationship (3.9) at $N = 3$, $G_1 = 0.5\mu^{2-D}$, $G_2 = 0.5\mu^{4-3D}$, $D = 2.5$.

of the effective potential satisfies the relationship (3.9). It implies that the eight-fermion interaction enhances the symmetry breaking.

Since the phase transition is of the second order, the critical length, L_{cr}, is obtained by taking the limit $\langle\sigma_1\rangle \to 0$ and $\langle\sigma_2\rangle \to 0$ in Eqs. (3.7) and (3.8). We can perform the summation in these equations at the limit and get

$$L_{cr} = 2\pi\left[\frac{2\mathrm{tr}1 G_1}{\pi(4\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right)\zeta(3-D)\right]^{1/(D-2)}, \tag{3.10}$$

for fermion fields with the periodic boundary condition and

$$L_{cr} = 2\pi\left[\frac{2\mathrm{tr}1 G_1}{\pi(4\pi)^{(D-1)/2}}(2^{3-D}-1)\Gamma\left(\frac{3-D}{2}\right)\zeta(3-D)\right]^{1/(D-2)}, \tag{3.11}$$

for fermion fields with the anti-periodic boundary condition. Since the critical length, L_{cr}, depends only on the four-fermion coupling G_1 and the space-time dimensions D, the eight-fermion coupling has no contribution to the phase boundary. The critical length coincides with the four-fermion interaction model [7]. The model for fermion fields with the anti-periodic boundary condition in $S^1 \otimes M^{D-1}$ is equivalent to the model at finite temperature, $T = 1/L$. Thus the critical temperature of our model is given by the inverse of Eq.(3.11).

We plot the critical length of the compactified dimension for a (3.10) and (3.11), as a function of the space-time dimensions in Figs.7 and 8, respectively. Blow three dimensions Eq.(3.11) has no real solution for a positive G_1. Thus the critical length is divergent at the limit, $D \to 3$, in Fig.7. Because of the topological effect from the fermion fields with the periodic boundary condition only the broken phase can be realized for $2 < D < 3$. To find a finite critical length in Fig.8 at the limit $D \to 2$ we have to renormalize the four-fermion coupling, G_1. The renormalization of the coupling is discussed in Ref.[15].

4 Conclusion

We have investigated the model with the scalar type four- and eight fermion interactions in Minkowski space-time, M^D, and a space-time with non-trivial topology, $S^1 \otimes R^{D-1}$. Evaluating the effective potential, we have shown the finite size and topological effects on the phase structure of the chiral symmetry breaking.

In D-dimensional Minkowski space-time, $M^D(2 < D < 4)$, the minimum of the effective potential is equivalent to the one for the four-fermion interaction model. The discrete chiral symmetry is

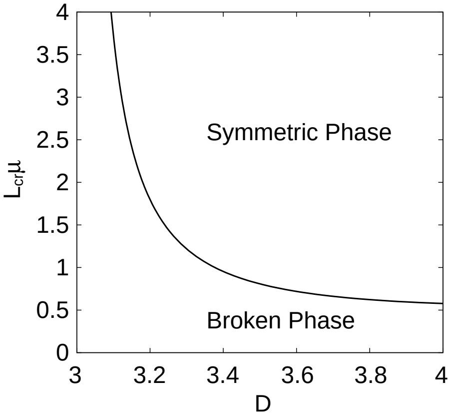

Figure 7: Phase diagram for fermion fields with the periodic boundary condition at $G_1 = 0.5\mu^{2-D}$

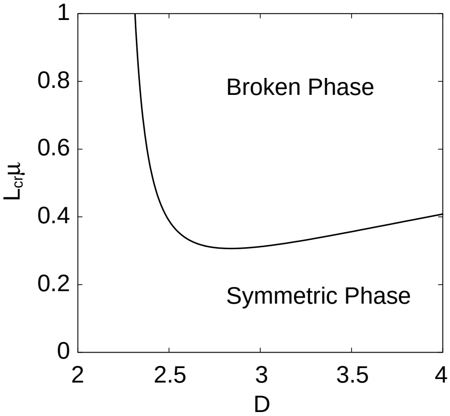

Figure 8: Phase diagram for fermion fields with the anti-periodic boundary condition at $G_1 = -0.5\mu^{2-D}$

dynamically broken for a negative four-fermion coupling, G_1. For $2 < D < 4$ the four-fermion interaction is marginal but the eight-fermion interaction is irrelevant. Since the irrelevant operator has only a small contribution at low energy, the phase structure for the model coincides with the one for the four-fermion interaction model in M^D.

In $S^1 \otimes R^{D-1}$ the opposite finite size effects are observed between the fermion fields with the periodic and the anti-periodic boundary conditions. Under the periodic boundary condition the symmetry breaking is enhanced and only the broken phase can be realize for a negative G_1. It is found that the discrete chiral symmetry is dynamically broken through the second order phase transition by the finite size effect for a positive G_1. In this case it is observed that the eight-fermion interaction also enhances the symmetry breaking. In the case of the anti-periodic boundary condition the symmetry breaking is suppressed. Only the symmetric phase can be realized for a positive G_1. For a negative G_1 the broken symmetry is restored as the length of the compactified space decreases. The phase transition is of the second order.

It should be noted the regularization dependence of the result. We can regard the model in D dimensions as the regularized model of the low energy effective theory in four-dimensions and compare the result with the one with cut-off regularization [4]. Since the four- and eight- fermion interactions are non-renoramalizable in four dimensions, some of physical behaviors depend on the regularization procedure. The result obtained in the present paper may have difference from the one obtained in the other regularization procedure. In the cut-off regularized model, the eight-fermion interaction has non-negligible effect on the phase structure even in M^4.

One is interested in applying the result to the model with extra dimensions. If the space-time dimensions are no less than four, both the four- and eight-fermion interactions are irrelevant. It is expected that the eight-fermion interaction has non-negligible contributions. It would be interesting also to study such effects in supersymmetric models in curved space-time [16]. One will consider the problem further and hope to publish reports in these directions.

Acknowledgements

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Disformal quintessence

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Abstract

After a brief review of the roles of disformal relations in cosmology, a canonic scalar field minimally coupled to a disformally modified metric generated by the field itself is considered. Causality and stability conditions are derived for such a field. Cosmological effects are studied and it is shown that the modification could trigger an acceleration after a scaling matter era, thus possibly alleviating the coincidence problem. Connections to other models are pointed out.

1 Introduction

Quantum effects in curved spacetime are known to introduce geometric corrections to the Einstein-Hilbert action. Thus a de Sitter era for the early universe and the resolution of the initial singularity were predicted [1] from the semiclassical considerations of the spacetime itself. In the past few years, with the advent of the dark energy paradigm, the possibility has been contemplated that the present acceleration of the universe could also be derived from a more fundamental gravity theory superseding general relativity at large scales [2, 3]. This attractive prospect of unified inflation and dark energy (with perhaps dark matter) without ad hoc exotic matter sources has received a lot of attention [4, 5]. The prototype models are the nonlinear or $f(R)$ gravity among more general scalar-tensor theories [6, 7]. As is well known [8, 9], these models are related to coupled scalar field matter by conformal transformation [10, 11].

One may ask whether relations of different forms could be considered as a consistent framework for gravity theories and their cosmological applications. It has been argued that a general Finslerian relation between matter and gravity geometries is restricted, by requiring causality, validity of weak equivalence principle and in particular covariance in its strictest sense, to a certain generalisation of the conformal relation [12]. Let us therefore consider the class of so called disformal transformations, which may depend on a scalar field ϕ,

$$\bar{g}_{\alpha\beta} = A(\phi)g_{\alpha\beta} + B(\phi)\phi_{,\alpha}\phi_{,\beta}. \tag{1.1}$$

If $B = 0$, the relation reduces to the conformal transformation which preserves the angles and the light cones. Any nonzero B then causes a disformal modification which results in difference of the causal structures of the two Riemannian geometries. This feature has been exploited to construct variable speed of light cosmology to cope without a usual inflation [13, 14]. On the other hand, an interestingly short inflation model with $A = -B = 1$, has been considered in detail [15]. The disformal property is crucial in producing lensing phenomena in relativistic MOND models [16, 17] and is naturally present in bimetric or bigravity theories [18, 19]. It appears also within the so called Palatini formalism [20, 21] if the action includes Ricci tensor squared terms [23, 22].

The main aim of the present paper is to further explore the cosmological possibilities of the relation (1.1) and in particular see how the disformal property could be used to unify dark energy within this framework. Previously, most authors couple matter to the disformal metric $\bar{g}_{\mu\nu}$, while gravity is given by the Einstein-Hilbert action of $g_{\mu\nu}$. The cosmological expansion is then effectively sourced by matter, whose energy density ρ and the pressure p are given by (dot denoting the time derivative)

$$\rho = \left(1 - \frac{B}{A}\dot{\phi}^2\right)^{-\frac{1}{2}}\bar{\rho}, \quad p = \left(1 - \frac{B}{A}\dot{\phi}^2\right)^{\frac{1}{2}}\bar{p}. \tag{1.2}$$

In usual coupled quintessence scenario [24], the cosmon field ϕ [25] has a conformal coupling to matter. The field should become energetically dominating recently to drive the acceleration. Now, we could consider a disformal extension of the usual coupled quintessence scenarios based on purely conformal (dilatation) symmetry considerations [26]. Then we would consider dark matter living in the metric disformed by a nonzero B. One notes that if the "bare" pressure vanishes, the coupling does not generate effective pressure. Then, as B/A grows, the physically effective energy density just dilutes faster, and one does not expect new effects if the ρ was not dynamically significant already earlier. Moreover, we have checked that viewing cosmology as a dynamical system, one does not find new fixed points by taking into account B/A: all fixed points require $B = 0$. Thus a disformal coupling between dark matter and dark energy does not seem helpful to the coincidence problem.

Instead, we then consider the case that only the field itself is coupled to the disformal metric. In the simplest case, we have the Einstein-Hilbert action for gravity coupled minimally to the matter sector and a canonical scalar field coupled to the barred metric. This seems to be a minimal set-up employing the relation (1.1), in the sense that all other matter than ϕ is minimally coupled to standard general relativity. The only unusual feature is then an effective self-coupling of the field ϕ. Explicitly, we write

$$S = \int d^4x \left[\sqrt{-g}\left(\frac{1}{2\kappa^2}R + \mathcal{L}_m\right) - \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + V(\phi)\right)\right], \tag{1.3}$$

where $\mathcal{L}_m$ is the Lagrangian density for matter, and $\kappa^2 = 8\pi G$. In the following, we denote

$$I \equiv g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}. \tag{1.4}$$

The scalar field has a canonical energy momentum tensor in the disformal metric and is covariantly conserved with respect to this *barred* metric, $\bar{\nabla}^\mu \bar{T}^{(\phi)}_{\mu\nu} = 0$. However, the physical metric is $g_{\mu\nu}$, and to avoid confusion, we rather write everything in terms of this unbarred metric and its Levi-Civita connection. The field couples also, through the relation (1.1) to the physical metric. One may thus associate an effective energy momentum tensor with the field. It can be shown to have the form

$$T^{(\phi)}_{\mu\nu} = \frac{1}{\sqrt{A(A+IB)}}\left[\left(\frac{2A+IB}{2(A+IB)} + BV\right)\phi_{,\mu}\phi_{,\nu} - \left(\frac{1}{2}I + (A+IB)V\right)g_{\mu\nu}\right]. \tag{1.5}$$

This tensor is also covariantly conserved, but with respect to the *unbarred* metric, consistently with the generalised Bianchi identity [27]. It is easy to see that this energy-momentum tensor can be put in the perfect-fluid form $T^{(\phi)}_{\mu\nu} = \rho u_\mu u_\nu + p\,(g_{\mu\nu} + u_\mu u_\nu)$ where the four-velocity is $u_\mu = \phi_{,\mu}/\sqrt{-I}$. Thus the field cannot generate anisotropic stress. The form (1.5) makes also transparent that the model, in general, cannot be reduced to k-essence [28]. However, when B is zero, the model can be reduced to canonic scalar field theory by the redefinition $\phi \to \int d\phi/\sqrt{A}$ (with just the potential possibly changing its form from the original). Therefore a scalar field in a purely conformal metric is not essentially new model.

2 Cosmology

We then consider homogeneous and isotropic solutions cosmology with the action (4.4). Consider the flat Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{2.1}$$

The expressions for the energy density and the pressure of the scalar field, which can be deduced from (1.5), have then, consistently with (1.2), the rather simple forms,

$$\rho^{(\phi)} = \frac{1}{\sqrt{1-\frac{B}{A}\dot{\phi}^2}} \quad \frac{\dot{\phi}^2}{2(A-B\dot{\phi}^2)} + V \tag{2.2}$$

$$p^{(\phi)} = \sqrt{1-\frac{B}{A}\dot{\phi}^2} \quad \frac{\dot{\phi}^2}{2(A-B\dot{\phi}^2)} - V \tag{2.3}$$

The Friedmann equation gives the Hubble rate

$$H^2 = \frac{8\pi G}{3} \quad \rho^{(m)} + \rho^{(\phi)} \quad , \tag{2.4}$$

and its derivative can be found from

$$2\dot{H} + 3H^2 = -\frac{8\pi G}{3} \quad p^{(m)} + p^{(\phi)} \quad , \tag{2.5}$$

where $\rho^{(m)}$ and $p^{(m)}$ are the energy density and pressure of the minimally coupled matter.

The interpretation is that the scalar field ϕ now lives in the metric

$$d\bar{s}^2 = -(A(t) - B(t)\dot{\phi}^2)dt^2 + A(t)a^2(t)(dx^2 + dy^2 + dy^2). \tag{2.6}$$

Now A is just the usual conformal factor. Let us thus consider what happens when the disformal factor B is significant. Evidently, in cosmology the function B corresponds to a distortion of the lapse function for the barred metric. As B grows $B \gg A$, the time experienced by the field begins to elapse slower (as measured by our unbarred clocks). As things happen faster in the matter frame, where distances and time intervals are measured with respect to the "physical", unbarred metric, the disformal field ϕ begins to look frozen when B is large enough. This can be seen from the formulas (2.3, 2.3), where the kinetic terms are suppressed by the square of the ratio of peculiar times as measured using the different metrics. However, the feature will always disappear when the field is a constant, and is thus a purely dynamical effect requiring some rolling of the scalar field.

This suggests the following cosmological application. The cosmon field is well known to have a so called tracking property, which guarantees it exhibits a constant ratio of the total energy density of the universe regardless of the initial conditions [25]. Hence, one may have a scalar field present during the whole evolution of the universe without fine-tuning. However, the field cannot explain the observed acceleration if it stays on the attractor. To toss the field off the attractor, one may reshape the potential [29, 30]. A theoretically motivated possibility is to couple the field nonminimally to matter [24], but this is known to lead to an instability at the linear level [31] (see also [32, 33] but however [34, 35]) and possible problems with quantum loop corrections [36]. However, considering neutrino coupling allows to link the acceleration scale with the neutrino mass [37]. Mechanisms based on nonminimal gravity couplings are employed in extended quintessence [38], Gauss-Bonnet quintessence [39] and in more general models with nonlinear functions of curvature invariants [40, 41].

Here we instead let the disformal effect freeze the scalar field. Thus we consider the possibility that the lapse distortion B redshifts the kinetic energy away, thus stopping the field and triggering a potential dominated era. Clearly, the potential of the field then provides an effective cosmological constant and the universe evolves into future de Sitter stage. To look into this in more detail, let us specialise to the simple case

$$A = 1, \quad B = B_0 e^{\beta\kappa\phi}, \quad V = V_0 e^{\lambda\kappa\phi}. \tag{2.7}$$

We set the purely conformal factor to unity to focus on the novel features. We choose the exponential forms for the disformal factor and the potential motivated by of high-energy physics considerations and convenience. Now, assuming initially $B \ll 1$, the tracking stage is characterised by

$$\kappa\phi = \frac{3(1+w_m)}{\lambda} + constant. \tag{2.8}$$

Then the field mimics the background component with the equation of state w_m. By Eq.(2.4), the Hubble parameter drops as $H^2 \sim a^{-3(1+w_m)}$. The disformal factor grows as $B \sim a^{3(1+w_m)\beta/\lambda}$.

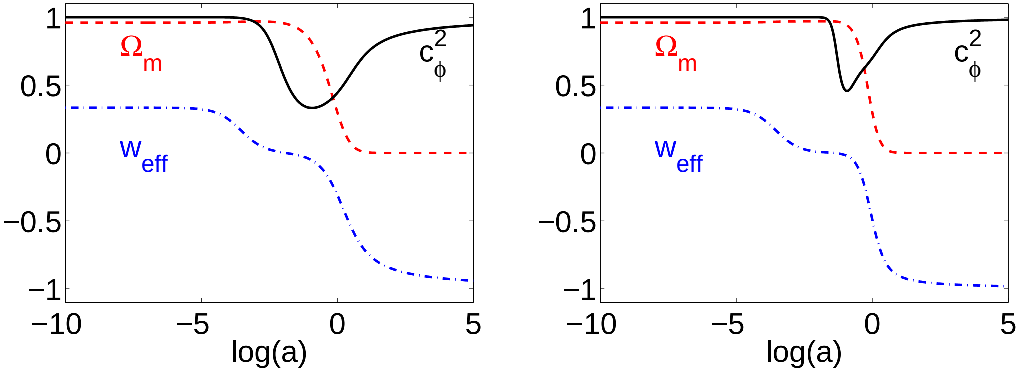

Figure 1: Evolution of the model (2.7) with two examples having $\Omega_m = 0.3$ today. In the left panel we have $\beta = 15$ and in the right panel $\beta = 25$. The blue dash-dotted lines show the effective equation of state of the universe, and the red dashed lines show the relative amount of matter Ω_m as a function of the logarithm of the scale factor. The black solid line is the square of the sound speed of perturbations of the scalar field. We used $\lambda = 10$ for the potential and for the radiation density $\Omega_r \approx 8.5 \cdot 10^{-5}$.

Therefore, the importance of the term $B\dot{\phi}^2$ is growing, and the tracking regime will eventually be interrupted iff $\beta > \lambda$. A similar condition exists for Gauss-Bonnet dark energy proposed in Ref. [42]: iff the slope of the Gauss-Bonnet coupling is steeper than that of the potential, dark energy domination occurs [43]. Numerical examples of the evolution in the present models are shown in Figure 1. There, as usual $\Omega_m = \kappa^2\rho_m/(3H^2)$ is the fractional matter density, and w_{eff} is the effective pressure per density of the total matter content. The quantity c_ϕ^2 is discussed in the next section.

3 Fluctuations in the field

The propagation of the perturbations of the field is characterised by the sound speed squared, $c_\phi^2 = \delta p/\delta\rho|_{T_{oi}=0}$ [44]. This is an important determinant of the physical properties of the effective fluid; one requires $c_\phi^2 < 1$ in order to eliminate the possibility of superluminal information exhange via excitations of the field, and in addition one sees that if $c_\phi^2 < 0$ the perturbations are unstable and may blow up too fast. The sound speed by definition depends on the background. Here we consider the FRW background as it includes Minkowski and de Sitter as particular limits. The sound speed is evaluated as the ratio of pressure and density perturbation in the comoving frame, and therefore no gauge ambiguity arises. Consider the so called total matter gauge, where the metric perturbations are parameterised by two longitudinal scalar potentials and the matter velocity vanishes. As we argued above, even the nonstandard scalar field does not generate anisotropic stresses, and its velocity field is proportional to the field perturbation. It follows that the mentioned longitudinal metric potentials are equal and the field ϕ is smooth in this frame. With these observations, it becomes straightforward to obtain from (1.5) that

$$c_\phi^2 = \frac{-\dot{\phi}^2 B + 2ABV - 2\dot{\phi}^2 B^2 V + 2A \quad 1 - \dot{\phi}^2 \frac{B}{A}}{\dot{\phi}^2 B + 2ABV - 2\dot{\phi}^2 B^2 V + 2A}. \tag{3.1}$$

One notes that when B vanishes the sound speed squared becomes identically unity. It is a well known property of canonic fields that their perturbations always propagate with the speed of light. When B is not identically zero, one gets nontrivial constraints on the model by requiring causality and stability.

In the scenario (2.7) the sound speed deviates from unity at the present and during the matter dominated epoch when the disformal effect is freezing the field. The sound speed decreases to a smaller positive value and then evolves to back to unity. Two examples of this evolution are shown in Figure 1. This confirms that the model is free of instabilities and that causality violations do not occur. Compared to standard quintessence, one expects the disformal field to cluster somewhat more. The effective sound speed dips during the critical era of structure formation, and thus the inhomogeneities forming in the field are dissipated less efficiently. The possibility to exploit this feature to observationally distinguish the disformal from standard quintessence is left to future studies.

To end, let us remark that flipping the sign of B in this model would make the sound speed of the field tachyonic. In fact, rather than acceleration, a singularity would occur. This is of the type II in the classification of Ref. [45]. The sudden future singularity is of the generalised, higher order nature [46] in the sense that the second derivative of the Hubble rate diverges. Recently similar finite-time singularities have been found to possibly occur in $f(R)$ gravity models [48, 47] and the nonlocally corrected gravity [49, 50] proposed in [51, 52]. However, we consider here only the nonsingular case of positive B.

4 Conclusion

Modified gravity could unify inflation and the dark sector in cosmology. The standard frameworks for this pursuit, the so called nonlinear gravity as a class of scalar-tensor theories, and quintessence models are connected by conformal transformation. Here we investigated the possibility to use the disformal generalisation (1.1) as the fundamental relation between the two metrics. For the purpose of constructing a minimal set-up where this is possible, we considered the case where only the scalar field itself lives in the disformally related metric. Then all usual matter respects the equivalence principle. The disformal effect, a new type of effective self-interaction, is only seen when the field is dynamical.

This can have interesting consequences for cosmological, rolling scalar fields. In particular, the disformal modification can freeze a tracking quintessence field in such a way that an accelerating de Sitter era follows a scaling matter era. For an exponential potential with the slope λ and an exponential disformal factor with the slope β the necessary condition for acceleration becomes very simply $\beta > \lambda$. The relative scale of the potential and the disformal factor is then determined by requiring the observed relative abundance of matter today. This mechanism resembles the scenario where the string-motivated exponential Gauss-Bonnet coupling of the Nojiri-Odintsov-Sasaki modulus triggers the acceleration.

To determine the causality and stability of the propagating perturbations in the field, we derived an expression for the sound speed squared (3.1). Requiring consistency limits the possible solutions. These nontrivial conditions are generically satisfied by the scenarios described above. This seems, together with the previous considerations of gravitational models of inflation and dark matter mentioned in the introduction, to suggest that a subtle modification of the relation of matter and gravitational geometries through the disformal transformation could also have a role in the resolution of present cosmological problems.

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Conformally-flat Stackel spaces in Brans-Dicke theory

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Abstract

The classification problem for conformally-flat space-times that admit a separation of variables in the Hamilton-Jacobi equation of the scalar-tensor Brans-Dicke theory of gravity is examined. The field equations of the scalar-tensor theory of Brans and Dicke for conformally-flat Stackel space-times of type (1.1) are solved. An explicit form of the metric tensor and scalar field is obtained.

1 Introduction.

Conformally-flat spaces are widely used in cosmology as simple but sufficiently realistic models of space-time. In particular, they include the models of de Sitter and Freedman-Robertson-Walker. On the other hand, the available experimental data evidence that the theory of general relativity (GRT) is insufficient for a construction of adequate models. This fact stimulates a search for alternative theories of gravity. The gravity theory of Brans-Dicke (BDT) [1, 2] is one of the first scalar-tensor theories of gravity. At present, scalar-tensor theories are of interest as low-energy approximations of quantum field theories. BDT differs from GRT by the form of its field equations; however, the influence of gravity on physical systems in BDT and GRT is equivalent, and is determined by the metric tensor. In both BDT and GRT, test particles move along geodesic lines, and therefore in BDT one is interested in Stackel spaces (SS), which, by definition, admit integration of the equations of motion for test particles in the form of Hamilton-Jacobi, according to the method of a complete separation of variables. In finding exact solutions of the field equations in BDT, Stackel spaces, however, are more important than they are in GRT, because in these spaces there is a possibility of finding a general solution of the scalar equation that enters the set of field equations of BDT. In this article, we examine conformally-flat spaces that admit a complete separation of variables in the Hamilton-Jacobi equation of the Brans-Dicke theory. Conformally-flat metrices of this kind, the Stackel metrices of type (1.1), (2.1) and (3.1), have been found in [3].

2 General information on conformal SS of type (1.1).

Let us recall that Stackel spaces are such Riemann spaces that admit an integration of the equations of geodesic lines by the method of a complete separation of variables in the Hamilton-Jacobi equation. A more detailed description of Stackel spaces and their classification can be found, e.g., in [4].

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We recall that conformally-flat spaces are such spaces whose metrices satisfy the equation $C^i_{jkl} = 0$, where C^i_{jkl} is the Weyl tensor.

Let us present the metrices of the conformal SS of type (1.1) obtained in [3], namely,

$$ds^2 = \Delta^2(2W^{(1)}(x^2,x^3)dx_0dx_1 + G(x^2,x^3){dx^1}^2 - \varepsilon_2 W^{(2)}(x^0,x^3){dx^2}^2 - \varepsilon_3 W^{(3)}(x^0,x^2){dx^3}^2), \quad (2.1)$$

where

$$W^{(2)}(x^0,x^3) = t_0(x^0) + t_3(x^3),\ W^{(3)}(x^0,x^2) = t_2(x^2) - t_0(x^0),$$
$$G(x^2,x^3) = g_2(x^2)W^{(2)} + g_3(x^3)W^{(3)},\ \varepsilon = \pm 1,$$
$$W^{(1)}(x^2,x^3) = t_2(x^2) + t_3(x^3) = W^{(2)}(x^0,x^3) + W^{(3)}(x^0,x^2),\ \Delta = \Delta(x^0,x^1,x^2,x^3).$$

The functions $t_0(x^0)$, $t_2(x^2)$, $t_3(x^3)$, $g_2(x^2)$, $g_3(x^3)$ are arbitrary functions of their arguments; Δ is an arbitrary function of all of its arguments. Here and elsewhere, we assume that constants are denoted by lowercase Greek characters with a subscript or without it; functions of one argument are denoted by lowercase Latin characters with a subscript that denotes the variable; functions of several variables are denoted by uppercase Latin and Greek characters. An exceptional case is given by field components, denoted by uppercase Greek characters with an index that stands for a variable.

The problem of selecting the conformally-flat metrices from the class of conformal SS leads to the condition

$$t_0 t_2' = t_0 t_3' = 0,$$

which implies the presence of two classes of conformally flat Stackel metrices:

Class A. $t_0 = 0$, **ClassB.** $t_2' = t_3' = 0$,

where the metrics can be presented in the same form for both classes:

$$ds^2 = \Delta^2(2dx^0dx^1 + G(x^2,x^3){dx^1}^2 + T{dx^2}^2 + (1-T){dx^3}^2), \quad (2.2)$$

with the functions and being determined separately for each class.

Class A.

$$G(x^2,x^3) = \rho \quad \frac{1}{t_2} - \frac{1}{t_3} \quad ,\ T = \frac{t_3}{t_2+t_3}, \quad (2.3)$$

$$t_2'' = \frac{1}{4}{t_2}^2(\mu {t_2}^2 + \nu t_2 + \kappa),\ \ t_3'' = \frac{1}{4}{t_3}^2(-\mu {t_3}^2 + \nu t_3 - \kappa).$$

Class B.

$$G = - \quad \frac{\mu {x^2}^2}{2} + \mu_1 x^2 + \mu_2 \quad t_0 - \quad \frac{\nu {x^3}^2}{2} + \nu_1 x^3 \quad (1-t_0) \quad , \quad (2.4)$$

$$T = t_0,\ \ {t_0'}^2 = {t_0}^2(1-t_0)^2(2\mu t_0 + 2\nu(1-t_0) - \kappa t_0(1-t_0)).$$

The condition $g = det\ g^{ij} < 0$, imposed on the metrices (2), implies

$$0 < T < 1,$$

or, equivalently, for the functions t_2, t_3,

$$t_2 t_3 > 0.$$

3 Field equations of Brans-Dicke theory.

The field equations of the Brans-Dicke theory in the case of vacuum have the form [1]

$$\phi_{;\alpha}^{\ ;\alpha} = 0$$
$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\tfrac{\omega}{\phi^2}(g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha} - \phi_{;\mu}\phi_{;\nu}) - \tfrac{1}{\phi}(g_{\mu\nu}\phi_{;\alpha}^{\ ;\alpha} - \phi_{;\mu\nu}).$$

Here, ϕ is a scalar field; ω is a constant. A separation of variables in the scalar equation implies the following representation of the function ϕ:

$$\phi = \prod_{i=1}^{n} \phi_i(x^i) \quad (3.1)$$

(a multiplicative separation of variables). The form of equations (5) can be simplified, if one excludes the scalar curvature and introduces, instead of ϕ, a new variable,

$$\Phi = ln|\phi| = \sum_{i=1}^{n} \Phi_i(x^i), \tag{3.2}$$

which admits a separation of variables of additive type.

Then, equations (5) acquire the form

$$\begin{gathered}\Phi_{;\alpha}^{;\alpha} + \Phi^{;\alpha}\Phi_{;\alpha} = 0\\ R_{\mu\nu} = (\omega+1)\Phi_{;\mu}\Phi_{;\nu} + \Phi_{;\mu\nu}\end{gathered}$$

We have the following equations $R_{11},\ R_{12},\ R_{13}$ for the metrics (2):

$$\frac{2\Delta_{,11}}{\Delta} = -\frac{2\Delta_{,1}\Phi_1'}{\Delta} - \omega\Phi_1'^{\,2} - \Phi_1'',$$

$$\frac{2\Delta_{,1a}}{\Delta} = -\frac{2\Delta_{,a}\Phi_1'}{\Delta} - \frac{2\Delta_{,1}\Phi_a'}{\Delta} - \omega\Phi_1'\Phi_a', \quad a = 2, 3.$$

If one subjects these equations to the following transformation of the unknown function Δ,

$$\Delta \to \Delta e^{\Phi/2},$$

then the equations take a simplified form:

$$\Delta_{,1p} = \omega\Delta\Phi_1'\Phi_p', \ p = 1, 2, 3. \tag{3.3}$$

The mentioned change literally coincides with a conformal transformation examined in [5]; however, in the case under consideration we deal with an arbitrary function , and therefore, the presented change only signifies an expedient choice of the form of equations. Integrating the subset of equation (9) allows one to obtain a solution of the entire set of equations.

4 Summary of results

While searching for solutions, one encounters subclasses to which one applies an indication of vanishing, or non-vanishing, of various components of the scalar field in the additive presentation (7); thus the results are listed in the following classification table:

Class A:

Form of scalar field	$\Phi_1 = 0$	$\Phi_1 \neq 0$
$\Phi_2 = \Phi_3 = 0$	Subclass A3	Subclass A2
$\Phi_2 \neq 0, \Phi_3 = 0$	-	-
$\Phi_2 \neq 0, \Phi_3 \neq 0$	-	Subclass A1

Class B:

Form of scalar field	$\Phi_1 = 0$	$\Phi_1 \neq 0$
$\Phi_2 = \Phi_3 = 0$	Subclass B5	Subclass B4
$\Phi_2 \neq 0, \Phi_3 = 0$	Subclass B6	-
$\Phi_2 \neq 0, \Phi_3 \neq 0$	-	Subclass B1, B2, B3

All the obtained metrices can be presented in the form

$$dS^2 = \Omega(2dx^0dx^1 + G(x^2, x^3)dx^{1^2} + Tdx^{2^2} + (1-T)dx^{3^2}).$$

In what follows, we also use the notation

$$cos_\omega x = \begin{matrix} cosx, \ if \ \omega < 0, \\ chx, \ if \ \omega > 0. \end{matrix}$$

Class A.

In all of the solutions for the metrices of class A, there hold relations (3).

Solution A1.

$$\Omega = e^{-\beta\gamma\rho x^0} cos_\omega{}^2(\lambda\Phi)e^{\Phi}, \qquad \Phi = \alpha x^0 + \beta x^1 + \frac{1}{\gamma t_2} - \frac{1}{\gamma t_3}, \qquad \omega = \pm\lambda^2.$$

$$\nu = -16\,\alpha\beta\gamma^2, \qquad \kappa = -6\beta^2\gamma^2\rho.$$

Solution A2.

$$\Omega = f_1{}^2\, e^{\Phi_1}, \qquad \frac{f_1{}''}{f_1} = \omega\Phi_1'{}^2,$$

$\Phi = \Phi_1(x^1)$ is an arbitrary function, ω is an arbitrary constant.

Solution A3.

$$\Omega = f_0{}^2\, e^{\Phi_0}, \qquad \frac{\kappa\rho}{24} + \frac{f_0{}''}{f_0} = \omega\Phi_0'{}^2,$$

$\Phi = \Phi_0(x^0)$ is an arbitrary function, ω is an arbitrary constant.

Class B.

Everywhere in class B, unless otherwise is specified, there hold relations (4).

Solution B1.

$$\Omega = e^{\Phi}\, cos_\omega{}^2(\lambda\Phi), \qquad T = \text{const}, \qquad \Phi = \alpha x^0 + \beta x^1 + \gamma x^2 + \delta x^3, \qquad \omega = \pm\lambda^2.$$

$$\alpha = \frac{1}{2}(\gamma^2(T-1) - \delta^2 T).$$

Solution B2.

$$\Omega = \frac{e^{\Phi}\, cos_\omega{}^2(\lambda\Phi)}{1-t_0}, \qquad G = (\alpha - 4\,\frac{\gamma^2}{\beta^2}\,x^{2\,2})\,t_0, \qquad t_0{}' = \frac{4}{\beta}\,\gamma\, t_0{}^2(1-t_0),$$

$$\Phi = \Phi_0 + \beta\, x^1 + \gamma\, x^{2\,2} + \delta\, x^3, \qquad \Phi_0' = -\frac{\delta^2}{2\beta} + \frac{\delta^2 - \alpha\,\beta^2}{2\beta}\, t_0, \qquad \omega = \pm\lambda^2.$$

Solution B3.

$$\Omega = f_0^2\, cos_\omega{}^2(\lambda\Phi)\, e^{\Phi}, \qquad \frac{f_0{}'}{f_0} = -\frac{1}{2}\frac{t_0'}{t_0} + \frac{2\,\gamma}{\beta}\, t_0, \qquad G = (\alpha - 4\,\frac{\gamma^2}{\beta^2}\, x^{2\,2})\, t_0 + 4\,\frac{\delta^2}{\beta^2}\, x^{3\,2}(t_0 - 1),$$

$$t_0' = \frac{4}{\beta}\, t_0(1-t_0)(\gamma t_0 + \delta(t_0 - 1)),$$

$$\Phi = \Phi_0 + \beta\, x^1 + \gamma\, x^{2\,2} + \delta\, x^{3\,2}, \qquad \Phi_0' = -\frac{\alpha\,\beta}{2}\, t_0, \qquad \omega = \pm\lambda^2.$$

Solution B4.

$$\Omega = f_1{}^2\, e^{\Phi_1} \qquad T = \text{const}, \qquad G = 0, \qquad \frac{f_1{}''}{f_1} = \omega\, \Phi_1'{}^2,$$

$\Phi = \Phi_1(x^1)$, is an arbitrary function, ω is an arbitrary constant.

Solution B5.

$$\Omega = f_0{}^2\, e^{\Phi_0}, \qquad \frac{f_0{}''}{f_0} = \frac{1}{4}\, t_0{}^2 \quad 3(2\mu - 2\nu - \kappa) + 2(-2\mu + 2\nu + 3\kappa)t_0 - 3\kappa\, t_0{}^2 \quad + \frac{\nu}{2} + \omega\, \Phi_0'{}^2,$$

$\Phi = \Phi_0(x^0)$ is an arbitrary function, ω is an arbitrary constant.

Solution B6.

$$\Phi = \ln \frac{x^2}{\sqrt[3]{t_0 f_0{}^2}}, \qquad \Omega = \frac{f_0{}^2}{x^2}\, e^{\Phi},$$

$$f_0'' = -\frac{f_0}{12}(-8\nu + (\kappa + 6\nu - 2\mu)t_0 + (6\kappa + 12\nu - 17\mu)t_0{}^2 + 5(-3\kappa - 2\nu + 2\mu)t_0{}^3 + 8\kappa\, t_0{}^4) + \frac{f_0'}{3}(\ln f_0 t_0)',$$

ω is an arbitrary constant.

5 Conclusion.

In the present article, we have obtained conformally-flat Stackel spaces in the scalar-tensor gravity theory of Brans-Dicke which admit a complete separation of variables in the Hamilton-Jacobi equations according to type (1.1). A complete form of the metric tensor and scalar field is presented. This work was supported by RFBR Grant N 06-01-00609-a and by President Grant SS-2553.2008.2.

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Accelerating Cosmologies and Black Holes in Dilatonic Einstein-Gauss-Bonnet Theory

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Abstract

It is known that there appear higher order quantum corrections in the low-energy effective theories of superstrings, which are the leading candidates of fundamental theories of elementary particles. We review our recent studies of accelerating cosmologies and black holes in such theories, and summarize useful formulae in these studies.

1 Introduction

One of the most important problems in theoretical physics is the formulation of the quantum theory of gravity and its application to physical system to understand physics at strong gravity. The leading candidates for the fundamental theory including all the forces of elementary particles are the ten-dimensional superstring theories and eleven-dimensional M-theory, which are hoped to provide models of accelerated expansion and inflation of the universe upon compactification to four dimensions and also give proper description of black holes. There has been interest in the application of string theories to these subjects. Since it is still difficult to study geometrical settings in superstring theories, most analyses have been performed by using low-energy effective theories inspired by string theories. These effective theories are the supergravities which typically involve not only the metric but also the dilaton field (as well as several form fields).

Recent cosmological observations have confirmed the existence of an early inflationary epoch and the accelerated expansion of the present universe [28]. There are many attempts to derive such models in the context of string effective theories, but most of them assume some additional matter or need special settings. >From the viewpoint of the fundamental theories, however, it is desirable if such models are obtained without making special assumptions.

It has been shown that a model with a certain period of accelerated expansion can be obtained from the higher-dimensional vacuum Einstein equation if one assumes a time-dependent hyperbolic internal space [2] and that this class of models is obtained [3] from what are known as S-branes [4] in the limit of a vanishing flux of three-form fields. In the latter case, this desirable property is obtained also for spherical and flat internal spaces. (For other attempts to realize inflation in the context of string theories, see, for instance, Refs. [5, 6].) Unfortunately, this class of models does not give a sufficiently large inflation to resolve the cosmological problems.

It is also known that higher-order corrections can give rise to inflationary solutions [7]. This is a very desirable setting, since there are terms of higher orders in the curvature that arise as corrections to the lowest effective supergravity action in superstrings and M-theory [8]. The simplest such

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correction is the Gauss-Bonnet (GB) term in the low-energy effective heterotic string. (We ignore other gauge fields and forms for simplicity.) When the dilaton is dropped or is set to a constant, this correction is known as the first term except for the cosmological constant and Einstein-Hilbert action in the Lovelock gravity [9] which is the most general metric theory of gravity yielding conserved second order equations of motion in arbitrary number of dimensions D. It is a natural generalization of Einstein's general relativity (GR) to higher dimensions without ghost, and for this reason it was conjectured [10] and indeed found to be the low-energy effective theory of strings.

There are many works that investigate cosmology with the GB correction in systems of four and more dimensions (see, for instance, Refs. [11, 12]). However, most investigations consider a pure GB term without a dilaton, or assume a constant dilaton, which is not a solution of the heterotic string, and they do not discuss cosmological solutions with a dynamical dilaton in higher dimensions. It is thus important to analyze a system including dynamical dilatons. There has also been an attempt to obtain inflationary solutions in M-theory with higher-order quantum corrections [13].

An interesting approach to accelerating cosmologies and inflation has been considered for Einstein theory with some additional scalars [14, 6]. In this dynamical system method, one considers the solution space restricted by the constraint equation resulting from a component of the Einstein equation. If the field equations are written as an autonomous system, we can find fixed points in this space. Then all possible solutions can be expressed as trajectories between these fixed points in the solution space. This is a very powerful method for examining possible solutions, and it is applicable even if exact solutions cannot be obtained. In particular, it is possible to find solutions with (transient) accelerating expansion which may be relevant to cosmology. In fact, the existence of an eternally accelerating solution, first found in Ref. [5], has been proven for hyperbolic internal and external spaces, without giving an explicit solution [14]. We have applied this method to the dilatonic Einstein-GB theory and found that there are interesting accelerating cosmological solutions [16]. In section 2, we summarize our results on this topic.

Another subject that quantum gravity may have significant effects is the physics of black holes. Attempts at understanding black holes in the Einstein-Maxwell-dilaton system were initiated in Refs. [17, 18], in which a static spherically symmetric black hole solution with dilaton hair was found in four dimensions. After this, many solutions were discussed in various models. It is also natural to ask how the black hole solutions are affected by the higher order terms in these effective theories.

There have been also many works on the black hole solutions in the Lovelock theories [19, 20]. In the four-dimensional spacetime, the GB term does not give any contribution because it becomes a surface term and gives a topological invariant. Boulware and Deser [10] discovered a static, spherically symmetric black hole solutions of such models in more than four dimensions. In the system with a negative cosmological constant, black holes can have horizons with nonspherical topology such as torus, hyperboloid, and other compactified submanifolds. These solutions were originally found in general relativity and are called topological black holes [22]. It is of interest to see how these are modified by the presence of a dilaton.

One more motivation for the study of black holes is the following. There has been recently a renewed interest in these solutions for the application to the calculation of shear viscosity in strongly coupled gauge theories using black hole solutions in five-dimensional Einstein-GB theory via AdS/CFT correspondence [23]. Almost all these studies consider a pure GB term without a dilaton, or assume a constant dilaton, which is not a solution of the heterotic string. It is, however, expected that AdS/CFT correspondence is valid within the effective theories of superstring. It is thus again important to investigate how the properties of black holes are modified when the dilaton is present. The inclusion of the dilaton was also considered by Boulware and Deser [10], but exact black hole solutions and their thermodynamic properties were not discussed. Callan *et al.* [24] considered black hole solutions in the theory with a higher-curvature term $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and the dilaton field, and Refs. [25] took both the GB term and the dilaton field into account in four-dimensional spacetime. We summarize the main results we obtained in our study of this topic in section 3, but we refer the readers to the original papers [16, 26, 27] for the details. The reason is that rather than restating known results already published, it is more useful to summarize formulae on the field equations in the Einstein and string frames in time-dependent and static solutions for further future study of these subjects. These are given as three Appendices.

2 Accelerating Cosmologies from Dilatonic Gauss-Bonnet theory

In this section, we summarize the main results in Ref. [16]. We consider cosmological solutions with a dilaton field and the GB correction from heterotic string theory by extending the dynamical

Table 1: Fixed points of the autonomous system and their properties. Here $a(\tau)$ is the four-dimensional scale factor and τ is the cosmic time.

Label	(u'_1, u'_2, ϕ')	Eigenvalues of small perturbations	Stability	$\frac{da}{d\tau}$	$\frac{d^2a}{d\tau^2}$
M	$(0,0,0)$	$(0,0)$	unstable	–	–
P_1	$(0.292373, -0.36066, -0.954846)$	$(1.52555, 1.52555)$	unstable	<0	<0
$\tilde{\mathrm{P}}_1$	$(-0.292373, 0.36066, 0.954846)$	$(-1.52555, -1.52555)$	stable	>0	<0
P_2	$(0.91822, -0.080285, 0.585906)$	$(-2.41943, -2.41943)$	stable	>0	>0
$\tilde{\mathrm{P}}_2$	$(-0.91822, 0.080285, -0.585906)$	$(2.41943, 2.41943)$	unstable	<0	>0
P_3	$(0.161307, 0.161307, -9.30437)$	$(0.874329, 0.874324)$	unstable	>0	<0
$\tilde{\mathrm{P}}_3$	$(-0.161307, -0.161307, 9.30437)$	$(-0.87433, -0.874324)$	stable	<0	<0

system method.

The action, metric and the field equations in the Einstein frame are given in Eqs. (A.1), (A.2) and (A.11) – (A.14) in Appendix A, respectively, with $\alpha = 0, \beta = -1/2, \gamma = \frac{1}{2}, p = 3, q = 6$ and $D = 10$. For simplicity, we consider both the external and internal spaces are flat. (The case of non-flat spaces are under study [28].) In this case, it is possible [16] to rewrite the field equations as an autonomous system if we introduce the new time variable T defined through the relation

$$\partial_t = e^{\phi/4}\partial_T, \quad \text{i.e.} \quad \frac{dT}{dt} = e^{\phi/4}. \tag{2.1}$$

We have derived all the fixed points and analyzed their stability in the system. We find that there are seven fixed points in this system:

$$\begin{aligned}&(u'_1, u'_2, \phi') = \mathrm{M}(0,0,0), \quad \mathrm{P}_1(\mp 0.292373, \pm 0.36066, \pm 0.954846),\\ &\mathrm{P}_2(\pm 0.91822, \mp 0.080285, \pm 0.585906), \quad \mathrm{P}_3(\pm 0.161307, \pm 0.161307, \mp 9.30437),\end{aligned} \tag{2.2}$$

where the prime denotes T derivative and the labels are indicated for upper signs and those lower signs are denoted with tildes. Their properties are summarized in Table 1.

We find that the solutions of $\tilde{\mathrm{P}}_1$, P_2 and P_3 give the expanding solutions in the Einstein frame. Among these, only P_2 gives accelerating expansion. In this accelerated solution, we have the cosmic time τ defined by

$$T = \frac{4}{\phi'} e^{\frac{\phi'}{4}t}, \quad \frac{d\tau}{dT} = e^{3u_2 - \phi/4} = e^{-0.387T}. \tag{2.3}$$

We then find that the four-dimensional scale factor $a(\tau)$ becomes

$$a(\tau) = e^{u_1 + 3u_2} = e^{0.677T} \sim |\tau|^{-1.75}, \tag{2.4}$$

which gives super-inflation and τ changes from $-\infty$ to 0 as T changes from $-\infty$ to ∞. We have also studied the stability of these solutions and the results are given in Table 1.

We draw the flow diagram for solutions around the fixed points, which is used to examine what kind of solutions are possible. Using this, we can find many interesting solutions. For example, there are solutions starting from a decelerated expanding region that approach the accelerated expanding solution (P_2), solutions starting from a decelerated contracting region that approach the accelerated contracting solution ($\tilde{\mathrm{P}}_3$), and solutions starting from an accelerated expanding region that approach the decelerated expanding solution ($\tilde{\mathrm{P}}_1$). We can see that there are several accelerating cosmological solutions in this theory, including solutions that flow into the non-accelerating fixed point $\tilde{\mathrm{P}}_1$ and solutions that flow into P_2, with a Big Rip singularity. It is possible that stringy effects remove this kind of singularity, and if this is the case, these solutions may represent viable cosmologies.

It is interesting to investigate whether or not these solutions represent a viable cosmological solution. A step towards this is to examine if they provide sufficient e-folding to solve cosmological problems. An investigation of the solution flowing into the fixed point P_2 indicates that it is difficult to obtain sufficient e-folding before arriving at the fixed point, but we can easily obtain sufficient e-folding if the solution arrives at the fixed point.

When we consider the accelerating expansion of the present universe, the fine-tuning problem of the initial conditions is always an annoying one. To answer this problem, we have examined solutions by changing the initial conditions near the fixed point P_2. By doing so, we find that there are several solutions that flow into P_2. This means that there is a certain range of initial conditions that leads to accelerating expansion. In this sense, these solutions may be capable of explaining the naturalness of the accelerating expansion. To examine how large an area of these initial conditions can give such behavior and whether the present model can yield realistic behavior needs further study.

It is an interesting question to examine the cases of nonvanishing curvatures of external and internal spaces, and also to study how much these results carry over if we consider this problem in the string frame. This is under study [28] using the formulae summarized in Appendix A.

3 Black Holes in Dilatonic Gauss-Bonnet theory

We now turn to the problem of black holes [26, 27, 29]. The metric and the field equations in the Einstein frame are given in Eq. (B.1) and (B.5) – (B.8) in Appendix B.

For asymptotically flat solutions in the theory without the cosmological constant, we parametrize the metric as

$$ds_D^2 = -\left(1 - \frac{2Gm}{r^{D-3}}\right) e^{-2\delta} dt^2 + \left(1 - \frac{2Gm}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Sigma_{D-2,k}^2, \tag{3.1}$$

where the last piece denotes $(D-2)$-dimensional space with constant curvature of signature k, and the mass function $m = m(r)$ and the lapse function $\delta = \delta(r)$ depend only on the radial coordinate r.

We obtain the solutions under the following asymptotic behaviors:

(1). Asymptotic flatness at spatial infinity ($r \to \infty$):

$$m(r) \to M, \quad (M < \infty), \quad \delta(r) \to 0, \quad \phi(r) \to 0. \tag{3.2}$$

(2). The existence of a regular horizon r_H (the subscript H means quantities at the horizon):

$$2m_H = r_H^{D-3}, \quad |\delta_H| < \infty, \quad |\phi_H| < \infty. \tag{3.3}$$

(3). No other horizon outside of the event horizon and the regularity of spacetime for $r > r_H$:

$$2m(r) < r^{D-3}, \quad |\delta(r)| < \infty, \quad |\phi(r)| < \infty. \tag{3.4}$$

Under these conditions, we have derived black hole solutions for dimensions $D = 4$ up to $D = 10$.

Given the boundary conditions at the horizon, ϕ'_H is determined by the quadratic equation

$$\begin{aligned} & 2C\gamma\Big[2(D-3) + (D-4)(3D-11)C + (D-4)C^2\Big\{(D-4)_5 + (D-2)(3D-11)\gamma^2\Big\} \\ & +2(D-2)_5\, C^3\gamma^2\Big] r_H^2 \phi_H'^2 + 2\Big[(D-1)_2\,(D-4)C^2\Big\{2 + 2C - (D-4)_5\, C^2\Big\}\gamma^2 \\ & -\{1 + (D-4)C\}^2 \{2(D-3) + (D-4)_5\, C\}\Big] r_H \phi'_H \\ & \qquad +(D-1)_2\, C\Big[2(D-2) - 4(D-4)C - (D-4)^2(D+1)C^2\Big]\gamma = 0, \end{aligned} \tag{3.5}$$

where we have defined

$$C = \frac{2(D-3)e^{-\gamma\phi_H}}{r_H^2}. \tag{3.6}$$

In all dimensions, we find only the smaller solution of Eq. (3.5) gives regular black holes for given r_H and ϕ_H.

For $D = 4$, we find that regular black hole solutions exist only for $r_H \geq 1.47126$ in the unit $\alpha_2 = 1$. It is found that regular black hole solutions exist for all $r_H > 0$ in $D = 5$ and beyond.

We have examined thermodynamic properties. In $D = 4$, the GB term has the tendency to raise the temperature compared to the non-dilatonic solution (Schwarzschild black hole).

In $D = 5$ and non-dilatonic case, the temperature increases as the mass of the black hole becomes small for large black holes, which means that the heat capacity is negative. Below the mass $M = 2.976072$, the temperature decreases as the mass becomes small. The sign of the heat capacity changes at this mass, which is the same as the Reissner-Nordström black hole solution, signalling the second order phase transition. As the black hole becomes small through Hawking radiation, the temperature becomes extremely law, and the solution cannot reach the singularity with zero horizon radius. This is favorable feature from the point of view of cosmic censorship hypothesis.

In the dilatonic case, we find that the thermodynamic properties change drastically. The heat capacity is negative in all the mass range, and the temperature blows up at the singular solution. This is due to the nontrivial coupling between the dilaton field and the GB term and the resultant divergence of the dilaton field at the horizon.

In $D \geq 6$, the behavior of the temperature is qualitatively the same as that in the non-dilatonic case. The dilaton field has tendency to lower the temperature for the large black hole, while it raises the temperature for small black hole. The temperature diverges for the zero mass “solution" and the black hole continues evaporating.

In GR, the horizon radius of the black hole is related to entropy by $S = \pi r_H^2$ in $D = 4$. In GB gravity, entropy is not obtained by a quarter of the area of the event horizon but should be calculated by Wald's formula [3, 31]. We find that there is no qualitative difference between the dilatonic and non-dilatonic cases in this respect. The solution disappears at the nonzero finite mass for $D = 4$ (the dilatonic solution) and $D = 5$. (There is black hole solution for $r_H > 0$ in $D = 5$ but there is a lower bound on the mass.) Entropy of the dilatonic black hole is always larger than that of the non-dilatonic black hole with the same mass.

Due to the space limit, we refer the readers to our original paper [27] for results on the asymptotically AdS solutions. One thing worth mentioning is that there is no such solution without cosmological constant.

It is interesting to extend this work to topological black holes with the hypersurfaces of zero and negative curvatures [29], and also to extremal solutions with cosmological terms [32] extending the work in four dimensions [33]. Also we intend to check how much the results here carry over if we consider solutions in the string frame [34]. We hope to report on these subjects in the near future.

4 Conclusion

In this paper, we have summarized our recent results on the accelerating cosmologies, inflation and black holes in dilatonic Einstein-GB theory, which arises as a low-energy effective theories in heterotic string theories. The higher order terms as well as the dilaton are found to have profound effects on the results. These studies are not only interesting for their own right but important because the effect of dilaton has not been studied much. In particular, we find that there is a significant deviation from non-dilatonic case for $D = 5$ in the properties of black holes. This case is particularly important in understanding $D = 4$ field theories via gauge/gravity correspondence. We hope that our study stimulates further developments in this direction. In this respect, it is our hope that the formulae we give here are useful for further study.

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A Field equations for time-dependent solutions

We consider the following low-energy effective action in a general frame:

$$S = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-g}\left[e^{\alpha\phi}\{R+\beta(\partial_\mu\phi)^2\}+\alpha_2 e^{-\gamma\phi}R^2_{\rm GB}\right]. \tag{A.1}$$

Here, κ_D^2 is a D-dimensional gravitational constant, ϕ is a dilaton field, α, β, γ and $\alpha_2 = \alpha'/8$ are numerical coefficients, and $R^2_{\rm GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the GB correction. We note that

$\alpha = -2, \beta = 4, \gamma = 2$ corresponds to the string frame whereas $\alpha = 0, \beta = -1/2, \gamma = 1/2$ to the Einstein frame.

Let us consider the time-dependent metric in D-dimensional space:

$$ds_D^2 = -e^{2u_0(t)}dt^2 + e^{2u_1(t)}ds_p^2 + e^{2u_2(t)}ds_q^2\,, \tag{A.2}$$

where $D = 1 + p + q$. The external p-dimensional and internal q-dimensional spaces (ds_p^2 and ds_q^2) are chosen to be maximally symmetric, with the signature of the curvature given by σ_p and σ_q, respectively.

We find that the Riemann tensors are given by

$$\begin{aligned} &R^t{}_{itj} = e^{-2u_0} X g_{ij}, \quad R^t{}_{atb} = e^{-2u_0} Y g_{ab}, \quad R^i{}_{jkl} = e^{-2u_0} A_p (g^i{}_k g_{jl} - g^i{}_l g_{jk}), \\ &R^i{}_{ajb} = e^{-2u_0} \dot u_1 \dot u_2 g^i{}_j g_{ab}, \quad R^a{}_{bcd} = e^{-2u_0} A_q (g^a{}_c g_{bd} - g^a{}_d g_{bc}), \end{aligned} \tag{A.3}$$

where i, j and a, b run over the p-dimensional and q-dimensional spaces, respectively, and

$$\begin{aligned} A_p &\equiv \dot u_1^2 + \sigma_p e^{2(u_0-u_1)}, \quad A_q \equiv \dot u_2^2 + \sigma_q e^{2(u_0-u_2)}, \\ X &\equiv \ddot u_1 - \dot u_0 \dot u_1 + \dot u_1^2, \qquad Y \equiv \ddot u_2 - \dot u_0 \dot u_2 + \dot u_2^2\,. \end{aligned} \tag{A.4}$$

Throughout this paper, the dot denotes derivative with respect to the time t. The Ricci tensors are

$$\begin{aligned} R_{tt} = -pX - qY, \quad R_{ij} &= e^{-2u_0}\{X + (p-1)A_p + q\dot u_1 \dot u_2\} g_{ij}, \\ R_{ab} &= e^{-2u_0}\{Y + (q-1)A_q + p\dot u_1 \dot u_2\} g_{ab}, \end{aligned} \tag{A.5}$$

and scalar curvature is given by

$$R = e^{-2u_0}\{2pX + 2qY + p_1 A_p + q_1 A_q + 2pq\dot u_1 \dot u_2\}. \tag{A.6}$$

The GB term is given by

$$\begin{aligned} R_{\rm GB}^2 \;=\; & e^{-4u_0}\Big\{ p_3 A_p^2 + 2p_1 q_1 A_p A_q + q_3 A_q^2 + 4\dot u_1 \dot u_2 (p_2 q A_p + p q_2 A_q) + 4 p_1 q_1 \dot u_1^2 \dot u_2^2 \\ & + 4pX\left[(p-1)_2 A_p + q_1 A_q + 2(p-1)q \dot u_1 \dot u_2\right] + 4qY\left[p_1 A_p + (q-1)_2 A_q + 2p(q-1)\dot u_1 \dot u_2\right] \Big\}, \end{aligned} \tag{A.7}$$

where we have defined

$$\begin{aligned} (p-m)_n &\equiv (p-m)(p-m-1)(p-m-2)\cdots(p-n)\,, \\ (q-m)_n &\equiv (q-m)(q-m-1)(q-m-2)\cdots(q-n)\,. \end{aligned} \tag{A.8}$$

Multiplying (A.7) by $\sqrt{-g}e^{-\gamma\phi} = e^{u_0 + pu_1 + qu_2 - \gamma\phi}$ and carrying out a partial integration, we find that the action reduces to the following (up to an overall factor):

(1) Einstein-Hilbert action

$$\mathcal{L}_1 = e^{-u_0 + pu_1 + qu_2 + \alpha\phi}\Big[p_1 A_p + q_1 A_q - 2(p_1 \dot u_1{}^2 + pq\dot u_1 \dot u_2 + q_1 \dot u_2{}^2) - 2\alpha(p\dot u_1 + q\dot u_2)\dot\phi - \beta\dot\phi^2\Big]\,. \tag{A.9}$$

(2) GB action

$$\begin{aligned} \mathcal{L}_2 \;=\; & \alpha_2 e^{-3u_0 + pu_1 + qu_2 - \gamma\phi}\Big\{ p_3 A_p{}^2 + 2p_1 q_1 A_q A_p + q_3 A_q{}^2 - 4A_p(p_3 \dot u_1{}^2 + p_2 q \dot u_1 \dot u_2 + p_1 q_1 \dot u_2{}^2) \\ & - 4A_q(p_1 q_1 \dot u_1{}^2 + p q_2 \dot u_1 \dot u_2 + q_3 \dot u_2{}^2) + \frac{4}{3}(2p_3 \dot u_1{}^4 + \\ & 2p_2 q \dot u_1{}^3 \dot u_2 + 3p_1 q_1 \dot u_1{}^2 \dot u_2{}^2 + 2pq_2 \dot u_1 \dot u_2{}^3 + 2q_3 \dot u_2{}^4) \\ & + 4\gamma\dot\phi\Big[(p_2 \dot u_1 + p_1 q \dot u_2)A_p + (pq_1 \dot u_1 + q_2 \dot u_2)A_q - \frac{2}{3}\;\; p_2 \dot u_1^3 + q_2 \dot u_2^3\;\;\Big]\Big\}\,. \end{aligned} \tag{A.10}$$

Now the field equations are

$$F \equiv F_1 + F_2 = 0\,, \tag{A.11}$$

$$F^{(p)} \equiv f_1^{(p)} + f_2^{(p)} + X\;\; g_1^{(p)} + g_2^{(p)}\;\; + Y\;\; h_1^{(p)} + h_2^{(p)}\;\; - Z\;\; i_1^{(p)} + i_2^{(p)}\;\; = 0\,, \tag{A.12}$$

$$F^{(q)} \equiv f_1^{(q)} + f_2^{(q)} + Y\;\; g_1^{(q)} + g_2^{(q)}\;\; + X\;\; h_1^{(q)} + h_2^{(q)}\;\; - Z\;\; i_1^{(q)} + i_2^{(q)}\;\; = 0\,, \tag{A.13}$$

$$F_\phi \equiv Z + \frac{\alpha}{2\beta}\left[p_1 A_p + q_1 A_q + 2pX + 2qY + 2pq\dot{u}_1\dot{u}_2\right] + \frac{\alpha}{2}\dot{\phi}^2 - \frac{\gamma\alpha_2}{2\beta}e^{2u_0-(\alpha+\gamma)\phi}R^2_{\rm GB} = 0, \quad (A.14)$$

where $R^2_{\rm GB}$ is given in Eq. (A.7), and we have

$$\begin{aligned}
Z &= \ddot{\phi} + (-\dot{u}_0 + p\dot{u}_1 + q\dot{u}_2)\dot{\phi}\,,\\
F_1 &= p_1 A_p + q_1 A_q + 2pq\dot{u}_1\dot{u}_2 + 2\alpha(p\dot{u}_1 + q\dot{u}_2)\dot{\phi} + \beta\dot{\phi}^2\,,\\
f_1^{(p)} &= (p-1)_2 A_p + q_1 A_q + 2(p-1)q\dot{u}_1\dot{u}_2 - 2\alpha\dot{\phi}\dot{u}_1 - (\beta - 2\alpha^2)\dot{\phi}^2\,,\\
f_1^{(q)} &= p_1 A_p + (q-1)_2 A_q + 2p(q-1)\dot{u}_1\dot{u}_2 - 2\alpha\dot{\phi}\dot{u}_2 - (\beta - 2\alpha^2)\dot{\phi}^2\,,\\
g_1^{(p)} &= 2(p-1)\,,\quad g_1^{(q)} = 2(q-1)\,,\quad h_1^{(p)} = 2q\,,\quad h_1^{(q)} = 2p\,,\quad i_1^{(p)} = i_1^{(q)} = -2\alpha\,, \qquad (A.15)
\end{aligned}$$

and

$$\begin{aligned}
F_2 = \alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big\{&p_3 A_p^2 + 2p_1 q_1 A_p A_q + q_3 A_q^2 + 4(p_2 q A_p + p q_2 A_q + p_1 q_1 \dot{u}_1\dot{u}_2)\dot{u}_1\dot{u}_2\\
&- 4\gamma\dot{\phi}\;(p_2\dot{u}_1 + p_1 q\dot{u}_2)A_p + (pq_1\dot{u}_1 + q_2\dot{u}_2)A_q + 2(p_1 q\dot{u}_1 + pq_1\dot{u}_2)\dot{u}_1\dot{u}_2\;\Big\},
\end{aligned}$$

$$\begin{aligned}
f_2^{(p)} &= \alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big\{(p-1)_4 A_p^2 + 2(p-1)_2 q_1 A_p A_q + q_3 A_q^2\\
&\quad + 4\left[(p-1)_3 q A_p + (p-1)q_2 A_q + (p-1)_2 q_1 \dot{u}_1\dot{u}_2\right]\dot{u}_1\dot{u}_2 + 4\gamma\dot{\phi}\;((p-1)_2 A_p + q_1 A_q\\
&\quad + 2(p-1)q\dot{u}_1\dot{u}_2)(\dot{u}_1 + \gamma\dot{\phi}) + 2((p-1)_2\dot{u}_1 A_p + q_1\dot{u}_2 A_q + (p-1)q\dot{u}_1\dot{u}_2(\dot{u}_1 + \dot{u}_2))\;\Big\},\\
f_2^{(q)} &= \alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big\{p_3 A_p^2 + 2p_1(q-1)_2 A_p A_q + (q-1)_4 A_q^2\\
&\quad + 4\left[p_2(q-1)A_p + p(q-1)_3 A_q + p_1(q-1)_2\dot{u}_1\dot{u}_2\right]\dot{u}_1\dot{u}_2 + 4\gamma\dot{\phi}\;(p_1 A_p + (q-1)_2 A_q\\
&\quad + 2p(q-1)\dot{u}_1\dot{u}_2)(\dot{u}_2 + \gamma\dot{\phi}) + 2(p_1\dot{u}_1 A_p + (q-1)_2\dot{u}_2 A_q + p(q-1)\dot{u}_1\dot{u}_2(\dot{u}_1 + \dot{u}_2))\;\Big\},\\
g_2^{(p)} &= 4(p-1)\alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big[(p-2)_3 A_p + q_1 A_q + 2(p-2)q\dot{u}_1\dot{u}_2 - 2\gamma((p-2)\dot{u}_1 + q\dot{u}_2)\dot{\phi}\Big],\\
g_2^{(q)} &= 4(q-1)\alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big[p_1 A_p + (q-2)_3 A_q + 2p(q-2)\dot{u}_1\dot{u}_2 - 2\gamma(p\dot{u}_1 + (q-2)\dot{u}_2)\dot{\phi}\Big],\\
h_2^{(p)} &= 4q\alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big[(p-1)_2 A_p + (q-1)_2 A_q + 2(p-1)(q-1)\dot{u}_1\dot{u}_2\\
&\quad -2\gamma((p-1)\dot{u}_1 + (q-1)\dot{u}_2)\dot{\phi}\Big],\\
h_2^{(q)} &= 4p\alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}\Big[(p-1)_2 A_p + (q-1)_2 A_q + 2(p-1)(q-1)\dot{u}_1\dot{u}_2\\
&\quad -2\gamma((p-1)\dot{u}_1 + (q-1)\dot{u}_2)\dot{\phi}\Big],\\
i_2^{(p)} &= \alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}4\gamma\Big[(p-1)_2 A_p + q_1 A_q + 2(p-1)q\dot{u}_1\dot{u}_2\Big],\\
i_2^{(q)} &= \alpha_2 e^{-2u_0-(\alpha+\gamma)\phi}4\gamma\Big[p_1 A_p + (q-1)_2 A_q + 2p(q-1)\dot{u}_1\dot{u}_2\Big]. \qquad (A.16)
\end{aligned}$$

The basic relations, Eqs. (A.11) – (A.14), are not all independent as they satisfy

$$\dot{F} + (p\dot{u}_1 + q\dot{u}_2 - 2\dot{u}_0 + \alpha\dot{\phi})F = p\dot{u}_1 F^{(p)} + q\dot{u}_2 F^{(q)} + 2\beta\,\dot{\phi}F_\phi\,. \qquad (A.17)$$

B Field equations for static solutions

We reduce the action for the metric ansatz

$$ds_D^2 = -e^{2u(r)}dt^2 + e^{2v(r)}dr^2 + e^{2w(r)}d\Sigma^2_{D-2,k}, \qquad (B.1)$$

where the last piece denotes $(D-2)$-dimensional space with constant curvature of signature k. The action can be obtained from Ref. [16] by the replacement

$$t \to -ir,\quad p=1,\quad ds_p^2 \to -dt^2,\quad \sigma_p = 0,\quad q = D-2,\quad \sigma_q = k,\quad u_0 \to v,\quad u_1 \to u,\quad u_2 \to w. \quad (B.2)$$

We find

$$R = e^{-2v}[(D-2)_3 A(r) - 2(u'' + (u'-v')u') - 2(D-2)(w'' + (w'-v')w') - 2(D-2)u'w'].$$

$$R^2_{\rm GB} = (D-2)_3 e^{-4v} \Big[(D-4)_5 A^2(r) + 8u'w'(w'' + w'(w'-v'))$$
$$-4A(r) \Big(U(r) + (D-4)(w'' + w'(w'+u'-v')) \Big) \Big]. \tag{B.3}$$

B.1 Einstein frame

In the Einstein frame with gauge field $A = f(r)dt$ and the cosmological constant

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \Big[R - \frac{1}{2}(\partial\phi)^2 + \alpha_2 e^{-\gamma\phi} R^2_{\rm GB} - e^{a\phi} F^2 - e^{b\phi}\Lambda \Big], \tag{B.4}$$

we have the field equations

$$\begin{aligned}
F &\equiv (D-2)_3 A(r) - 2(D-2)u'w' + \frac{1}{2}\phi'^2 + \alpha_2 (D-2)_3 e^{-2v-\gamma\phi}\Big[(D-4)_5 A^2(r) \\
&\quad -4(D-4)A(r)w'(u'-\gamma\phi') \\
&\quad +4\gamma\phi' u'(A(r) - 2w'^2)\Big] - 2e^{a\phi-2u} f'^2 - e^{2v+b\phi}\Lambda = 0, \qquad (B.5)\\
G &\equiv (D-2)_3 A(r) - 2(D-2)(w'' - w'v' + w'^2) - \frac{1}{2}\phi'^2 + \alpha_2 (D-2)_3 e^{-2v-\gamma\phi}\Big[(D-4)_5 A^2(r) \\
&\quad + 4\gamma\Big\{\phi'' - (v' - (D-4)w' + \gamma\phi')\phi'\Big\}A(r) - 4\Big\{(D-4)A(r) + 2\gamma w'\phi'\Big\}(w'' \\
&\quad -w'v' + w'^2)\Big] - 2e^{a\phi-2u} f'^2 - e^{2v+b\phi}\Lambda = 0, \qquad (B.6)\\
H &\equiv (D-3)_4 A(r) - 2(D-3)(w'' + w'(w'+u'-v')) - \frac{1}{2}\phi'^2 - 2U(r) \\
&\quad + \alpha_2 (D-3) e^{-2v-\gamma\phi}\Big[(D-4)_6 A^2(r) - 4(D-4)_5 A(r)(w'' + w'(w'+u'-v'-\gamma\phi')) \\
&\quad - 4(D-4)A(r)\Big\{\gamma^2\phi'^2 + U(r) - \gamma(\phi'' + (u'-v')\phi')\Big\} + 8\gamma\phi' u'\Big\{w'(2v' + \gamma\phi' \\
&\quad -(D-4)w') - (w'' + w'^2)\Big\} + 8(D-4)w'(u'-\gamma\phi')(w'' - w'v' + w'^2) \\
&\quad -8\gamma w'(u'\phi'' + \phi' U(r))\Big] + 2e^{a\phi-2u} f'^2 - e^{2v+b\phi}\Lambda = 0, \qquad (B.7)\\
F_\phi &\equiv \phi'' + (u' - v' + (D-2)w')\phi' - \alpha_2\gamma e^{2v-\gamma\phi} R^2_{\rm GB} + 2a e^{a\phi-2u} f'^2 - b e^{2v+b\phi}\Lambda = 0, \qquad (B.8)\\
&\quad \Big(e^{a\phi-u-v+(D-2)w} f'\Big)' = 0, \qquad (B.9)
\end{aligned}$$

where the prime denotes the derivative with respect to r, the GB term is given in (B.3) and we have defined

$$A(r) \equiv k e^{2(v-w)} - w'^2, \quad U(r) \equiv u'' + u'^2 - u'v', \tag{B.10}$$

Eqs. (B.5) – (B.8) are not all independent but satisfy

$$F' + [u' - 2v' + (D-2)w']F = u'G + (D-2)w'H + \phi' F_\phi. \tag{B.11}$$

This serves to check the consistency of the results.

If we are in the Einstein frame, we should choose

$$u = \frac{1}{2}\ln B(r) - \delta(r), \quad v = -\frac{1}{2}\ln B(r), \quad w = \ln r, \tag{B.12}$$

which is the choice in [26, 27]. Alternative choice would be

$$u = \frac{1}{2}\ln B(r), \quad v = -\frac{1}{2}\ln B(r), \quad w = \ln R(r), \tag{B.13}$$

which is the choice in [33].

B.2 String frame

In the string frame, our action without gauge filed and the cosmological constant is expressed as

$$
\begin{aligned}
&\sqrt{-g}\, e^{-2\phi}\Big[R+4(\partial_\mu\phi)^2+\alpha_2 R^2_{\rm GB}\Big]\\
&= e^{u-v+(D-2)w-2\phi}[(D-2)_3A(r)+2(D-2)u'w'+2(D-2)_3w'^2-4\{u'+(D-2)w'\}\phi'+4\phi'^2]\\
&+\alpha_2 e^{u-3v+(D-2)w-2\phi}\Big[(D-2)_5A^2(r)+4(D-2)_4\{u'w'+(D-5)w'^2\}A(r)\\
&-8(D-2)_3\{u'+(D-4)w'\}\phi'A(r)+\frac{8}{3}(D-2)_4\{u'w'^3+(D-5)w'^4\}\\
&-\frac{16}{3}(D-2)_4\phi'w'^3\Big],
\end{aligned}
\tag{B.14}
$$

up to total derivatives. The contribution to the field equation from the GB term can be read off from Ref. [16] by the replacement (B.2) with $\gamma=2$, or from Ref. [26, 27].

Variation of the action (B.14) yields

$$
\begin{aligned}
F &\equiv (D-2)_3A(r)-2(D-2)\{w''-w'(v'-w'+2\phi')+4\{\phi''-(v'+\phi')\phi'\}\\
&\quad+\alpha_2e^{-2v}(D-2)_3\Big[(D-4)_5A^2(r)-4(D-4)(w''+(w'-v')w')A(r)\\
&\quad+8\{\phi''-(v'-(D-4)w'+2\phi')\phi'\}A(r)-16\{w''+(w'-v')w'\}w'\phi'\Big]=0,
\end{aligned}
\tag{B.15}
$$

$$
\begin{aligned}
G &\equiv (D-2)_3A(r)-2(D-2)u'w'+4\{u'+(D-2)w'-\phi')\}\phi'\\
&\quad+\alpha_2e^{-2v}(D-2)_3\Big[(D-4)_5A^2(r)-4\{(D-4)w'(u'-2\phi')-2\phi'u'\}A(r)\\
&\quad-16u'w'^2\phi'\Big]=0,
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
H &\equiv (D-3)_4A(r)-2(D-3)\{w''+w'(u'-v'+w'-2\phi')\}+4\{\phi''+\phi'(u'-v'-\phi')\}-2U(r)\\
&\quad+\alpha_2e^{-2v}(D-3)\Big[(D-4)_6A^2(r)-4(D-4)_5A(r)\{w''+w'(u'-v'+w'-2\phi')\}\\
&\quad-4(D-4)A(r)\{4\phi'^2+U(r)-2(\phi''+(u'-v')\phi')\}+16\phi'u'w'\{v'+2\phi'-(D-4)w'\}\\
&\quad+8(w''-w'v'+w'^2)\{(D-4)u'w'-2(u'+(D-4)w')\phi'\}\\
&\quad-16w'\{u'\phi''+\phi'U(r)\}\Big]=0,
\end{aligned}
\tag{B.17}
$$

$$
\begin{aligned}
F_\phi &\equiv (D-2)_3A(r)-2\{U(r)-2u'\phi'\}-2(D-2)\{w''+w'(u'-v'+w'-2\phi')\}\\
&\quad+4\{\phi''-(v'+\phi')\phi'\}+\alpha_2e^{2v}R^2_{\rm GB}=0,
\end{aligned}
\tag{B.18}
$$

where F, G, H and F_ϕ are not all independent, so instead of using F and G it may be easier to use $F-G=0$ for our set of basic equations.

C Conformal Transformation

In order to go from the string frame to the Einstein frame, it is useful to give formulae for the conformal transformation. If we make the Weyl transformation

$$
g_{\mu\nu}=e^{-2\rho}\tilde g_{\mu\nu},
\tag{C.1}
$$

the Riemann tensor is transformed as

$$
\begin{aligned}
R^\mu{}_{\nu\alpha\beta} &= \tilde R^\mu{}_{\nu\alpha\beta}+\tilde g^\mu_\alpha\rho_{;\nu;\beta}-\tilde g^\mu_\beta\rho_{;\nu;\alpha}-\tilde g_{\nu\alpha}\rho^{;\mu}{}_{;\beta}+\tilde g_{\nu\beta}\rho^{;\mu}_{;\alpha}+\tilde g^\mu_\alpha\partial_\beta\rho\partial_\nu\rho-\tilde g^\mu_\beta\partial_\alpha\rho\partial_\nu\rho\\
&\quad+\tilde g_{\beta\nu}\partial_\alpha\rho\partial^\mu\rho-\tilde g_{\alpha\nu}\partial_\beta\rho\partial^\mu\rho-(\tilde g^\mu_\alpha\tilde g_{\nu\beta}-\tilde g^\mu_\beta\tilde g_{\nu\alpha})(\partial_\sigma\rho)^2,
\end{aligned}
\tag{C.2}
$$

where ; means covariant derivative, and the contraction on the rhs is made using $\tilde g_{\mu\nu}$. We also have

$$
\begin{aligned}
R_{\mu\nu} &= \tilde R_{\mu\nu}+(D-2)\rho_{;\mu;\nu}+\tilde g_{\mu\nu}\rho^{;\alpha}{}_{;\alpha}+(D-2)[\partial_\mu\rho\partial_\nu\rho-\tilde g_{\mu\nu}(\partial_\alpha\rho)^2],\\
R &= e^{2\rho}\Big[\tilde R+2(D-1)\rho_{;\mu}{}^{;\mu}-(D-1)_2(\partial_\mu\rho)^2\Big].
\end{aligned}
\tag{C.3}
$$

We find

$$
\begin{aligned}
R^{\mu}{}_{\nu\alpha\beta}{}^2 &= e^{4\rho}\Big[\tilde{R}^{\mu}{}_{\nu\alpha\beta}{}^2 + 8\tilde{R}_{\mu\nu}\rho^{;\mu;\nu} + 8\tilde{R}_{\mu\nu}\partial^\mu\rho\partial^\nu\rho - 4\tilde{R}(\partial_\mu\rho)^2 + 4(D-2)(\rho_{;\mu;\nu})^2 \\
&\quad +4(\rho_{;\mu}{}^{;\mu})^2 + 8(D-2)\rho_{;\mu;\nu}\partial^\mu\rho\partial^\nu\rho - 8(D-2)\rho_{;\mu}{}^{;\mu}(\partial_\nu\rho)^2 \\
&\quad +2(D-1)_2(\partial_\mu\rho)^2(\partial_\nu\rho)^2\Big],
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
R_{\mu\nu}{}^2 &= e^{4\rho}\Big[\tilde{R}_{\mu\nu}{}^2 + 2(D-2)\tilde{R}_{\mu\nu}\rho^{;\mu;\nu} + 2(D-2)\tilde{R}_{\mu\nu}\partial^\mu\rho\partial^\nu\rho + 2\tilde{R}\rho_{;\mu}{}^{;\mu} - 2(D-1)\tilde{R}(\partial_\mu\rho)^2 \\
&\quad + (D-2)^2(\rho_{;\mu;\nu})^2 + (3D-4)(\rho_{;\mu}{}^{;\mu})^2 + 2(D-2)^2\rho_{;\mu;\nu}\partial^\mu\rho\partial^\nu\rho \\
&\quad -2(D-2)(2D-3)\rho_{;\mu}{}^{;\mu}(\partial_\nu\rho)^2 \\
&\quad +(D-1)(D-2)^2(\partial_\mu\rho)^2(\partial_\nu\rho)^2\Big],
\end{aligned} \tag{C.5}
$$

$$
\begin{aligned}
R^2 &= e^{4\rho}\Big[\tilde{R}^2 + 4(D-1)\tilde{R}\rho_{;\mu}{}^{;\mu} - 2(D-1)_2\tilde{R}(\partial_\mu\rho)^2 + 4(D-1)^2(\rho_{;\mu}{}^{;\mu})^2 \\
&\quad - 4(D-1)^2(D-2)\rho_{;\mu}{}^{;\mu}(\partial_\nu\rho)^2 + (D-1)^2(D-2)^2(\partial_\mu\rho)^2(\partial_\nu\rho)^2\Big].
\end{aligned} \tag{C.6}
$$

Hence the Gauss-Bonnet (GB) combination is transformed as

$$
\begin{aligned}
R^2_{\rm GB} &\equiv R_{\mu\nu\alpha\beta}{}^2 - 4R_{\mu\nu}{}^2 + R^2 \\
&= e^{4\rho}\Big[\tilde{R}_{\mu\nu\alpha\beta}{}^2 - 4\tilde{R}_{\mu\nu}{}^2 + \tilde{R}^2 - 8(D-3)\tilde{R}_{\mu\nu}\rho^{;\mu;\nu} - 8(D-3)\tilde{R}_{\mu\nu}\partial^\mu\rho\partial^\nu\rho + 4(D-3)\tilde{R}\rho_{;\mu}{}^{;\mu} \\
&\quad - 2(D-3)_4\tilde{R}(\partial_\mu\rho)^2 - 4(D-2)_3(\rho_{;\mu;\nu})^2 + 4(D-2)_3(\rho_{;\mu}{}^{;\mu})^2 - 8(D-2)_3\rho_{;\mu;\nu}\partial^\mu\rho\partial^\nu\rho \\
&\quad -4(D-2)(D-3)^2\rho_{;\mu}{}^{;\mu}(\partial_\nu\rho)^2 + (D-1)_4(\partial_\mu\rho)^2(\partial_\nu\rho)^2\Big].
\end{aligned} \tag{C.7}
$$

In order to go into the Einstein frame and then to normalize the dilaton kinetic term properly, we should choose

$$
\rho = -\frac{2}{D-2}\phi = -\frac{1}{\sqrt{2(D-2)}}\varphi. \tag{C.8}
$$

The total action is then transformed as

$$
\begin{aligned}
&\sqrt{-g}\, e^{-2\phi}\Big[R + 4(\partial_\mu\phi)^2 + \alpha_2 R^2_{\rm GB}\Big] \\
&= \sqrt{-\tilde{g}}\Big[\tilde{R} - \frac{1}{2}(\partial_\mu\varphi)^2 + \alpha_2 e^{-\gamma\varphi}\Big\{\tilde{R}^2_{\rm GB} + 4(D-3)\tilde{R}_{\mu\nu}\gamma\varphi^{;\mu;\nu} - 2(D-3)\gamma^2\tilde{R}_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi \\
&\quad -2(D-3)\gamma\tilde{R}\varphi_{;\mu}{}^{;\mu} - \frac{1}{2}(D-3)_4\tilde{R}(\gamma\partial_\mu\varphi)^2 - (D-2)_3(\gamma\varphi_{;\mu;\nu})^2 + (D-2)_3(\gamma\varphi_{;\mu}{}^{;\mu})^2 \\
&\quad -(D-2)_3\gamma^3\varphi_{;\mu;\nu}\partial^\mu\varphi\partial^\nu\varphi \\
&\quad + \frac{1}{2}(D-2)(D-3)^2\gamma^3\varphi_{;\mu}{}^{;\mu}(\partial_\nu\varphi)^2 + \frac{1}{16}(D-1)_4\gamma^4(\partial_\mu\varphi)^2(\partial_\nu\varphi)^2\Big\}\Big],
\end{aligned} \tag{C.9}
$$

where

$$
\gamma \equiv \sqrt{\frac{2}{D-2}}. \tag{C.10}
$$

We thus see that the Einstein and string frames are different in terms higher order in derivatives in the presence of GB terms, which may cause difference in the results. This is why it is worth studying solutions in different frames.

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Six Puzzles for LCDM Cosmology

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Abstract

The ΛCDM cosmological model is a well defined, simple and predictive model which is consistent with the majority of current cosmological observations. Despite of these successes there are specific cosmological observations which differ from the predictions of ΛCDM at a level of 2σ or higher. These observations include the following: 1. Large Scale Velocity Flows (ΛCDM predicts significantly smaller amplitude and scale of flows than what observations indicate), 2. Brightness of Type Ia Supernovae (SnIa) at High Redshift z (ΛCDM predicts fainter SnIa at High z), 3. Emptiness of Voids (ΛCDM predicts more dwarf or irregular galaxies in voids than observed), 4. Profiles of Cluster Haloes (ΛCDM predicts shallow low concentration and density profiles in contrast to observations which indicate denser high concentration cluster haloes) 5. Profiles of Galaxy Haloes (ΛCDM predicts halo mass profiles with cuspy cores and low outer density while lensing and dynamical observations indicate a central core of constant density and a flattish high dark mass density outer profile), 6. Sizable Population of Disk Galaxies (ΛCDM predicts a smaller fraction of disk galaxies due to recent mergers expected to disrupt cold rotationally supported disks). Even though the origin of some of the above challenges may be astrophysical or related to dark matter properties, it should be stressed that even on galactic and cluster scales, the effects of dark energy on the equilibrium and stability of astrophysical systems are not negligible and they may play a key role in the resolution of the above puzzles. Here, I briefly review these six challenges of ΛCDM and discuss the possible dark energy properties required for their resolution.

1 Introduction

Accumulating diverse observational evidence have indicated that the universe has entered a phase of accelerating expansion. Such observations include direct geometrical probes (standard candles like SnIa [1, 2, 3, 4], gamma ray bursts [5] and standard rulers like the CMB sound horizon[6, 7]) and dynamical probes (growth rate of cosmological perturbations [8] probed by the redshift distortion factor or by weak lensing [9]).

All these observational probes are converging towards confirming the accelerating expansion of the universe assuming the homogeneity of the universe. They have ruled out at several σ a flat matter dominated universe and they have produced excellent fits for the simplest cosmological model predicting accelerating cosmic expansion. This model is based on the assumptions of flatness, validity of general relativity, the presence of the cosmological constant Λ and Cold Dark Matter (ΛCDM)[10].

From the theoretical viewpoint the main weak points of ΛCDM include [10]:

- The Fine Tuning Problem: What is the physical mechanism that sets the value of Λ to its observed value which is 120 orders of magnitude smaller than the physically anticipated value?

- The Coincidence Problem: Why is the energy density corresponding to the cosmological constant just starting to dominate the universe at the present cosmological time?

Despite of efforts to increase the complexity of ΛCDM (using eg quintessence[11] or modified gravity[8]) in order to address the above weak points there has been no successful alternative that addresses the above problems without replacing them with other similar ones involving fined tuned parameters. Since the theoretical weaknesses of the model have lead to no successful alternative it may be useful to identify the observational weak points of ΛCDM and use these as a guide to building alternative models.

In view of the fact that ΛCDM is a simple, well defined and predictive model, it is important and straightforward to test its validity using a wide range of observational probes. If some of these observational probes indicate inconsistency of ΛCDM with observations then it is interesting to consider the modifications of the model required to establish consistency with observations.

Most approaches in testing the consistency of ΛCDM with observations have focused on comparing ΛCDM with alternative models or parameterizations on the basis of a bayesian analysis using the geometrical and dynamical probes mentioned above[1, 2, 3, 4, 5, 6, 7, 8, 9]. Due to its simplicity and acceptable quality of χ^2 fit, ΛCDM usually comes out as a winner in such a comparison [13].

Despite of the simplicity and apparent consistency of ΛCDM with most cosmological observations there are specific observational challenges for the model which have developed and persisted during the past few years. Some of these challenges involve galactic scale phenomena and it has been common wisdom that they will be resolved once astrophysical effects on these scales are better understood. Other challenges however, involve phenomena on scales larger than $\sim 10h^{-1}Mpc$ and these may require more drastic modifications of the model in order to be resolved. Such large scale challenges of ΛCDM include the observed high amplitude of large scale velocity flows on scales $\gtrsim 100h^{-1}Mpc$ [14, 15], the unexpected brightness of high redshift Type Ia supernovae (SnIa)[16], the halos of massive clusters of galaxies which are more concentrated and denser than predicted by ΛCDM [17] and the emptiness of voids which is unexpected in the context of ΛCDM [18, 19]. On smaller (galactic) scales ΛCDM is challenged by observations of constant density galactic halo cores instead of the ΛCDM predicted cuspy central cores [20], the higher than expected density of outer galactic haloes [21] and the sizable population of cold rotationally supported disk galaxies [22].

Since the above effects are statistically significant at 2σ level or more it is unlikely that they are all statistical fluctuations. In fact, it is possible that the resolution of the above puzzles will require more than a better understanding of astrophysical effects present on galactic scales. It may require a significant modification of the cosmological scale properties of the standard ΛCDM model such as the properties of gravity, dark energy or dark matter.

The goal of the present paper is to review the above phenomena challenging the foundations of the standard ΛCDM cosmological model. I will also discuss possible features of the model that may require modification in order to improve consistency with the above observations.

It should be stressed that this is not a complete list of cosmological puzzles related to the standard ΛCDM cosmological model. There are other challenges related to the statistical isotropy of the CMB and the Axis of Evil [23] (anomalous alignment of CMB multipoles in the direction $l \simeq -100$, $b = 60$) which may be less related to the properties of dark matter or dark energy. Such challenges are not discussed in the present brief review even though they may be related to the high amplitude and coherence bulk flows discussed in the next section.

2 Challenging ΛCDM

2.1 Large Scale Velocity Flows

The bulk flow corresponding to the CMB dipole is closely related to the amplitude of fluctuations on large scales, and can be used to test cosmological models [24]. A number of large scale velocity surveys have been undertaken [25] in the past two decades and a significant amount of peculiar velocity data on a wide range of scales is currently available. The issue of comparing such sparse surveys with expectations from cosmological models has also been investigated by several studies [26].

A combined sample of peculiar velocity data has been recently used [15, 14] to investigate the amplitude and coherence scale of the dipole bulk flow. It was found that the dipole moment (bulk flow) of the combined sample extends [14] on scales up to $100h^{-1}Mpc$ ($z \leq 0.03$) and perhaps up to $600h^{-1}Mpc$ ($z < 0.2$ [15]) with amplitude larger than $400km/sec$ [14] (perhaps up to $1000km/sec$ [15]). The direction of the flow has been found consistently to be approximately in the direction $l \simeq 285°$, $b \simeq 10°$. The expected rms bulk flow in the context of ΛCDM normalized with WMAP5

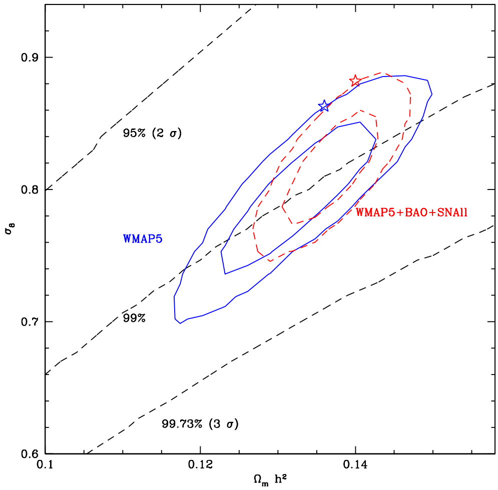

Figure 1: The $(\Omega_{0\mathrm{m}}, \sigma_8)$ χ^2 confidence contours obtained from the observed velocity flows (dashed lines) [14] are superposed with the corresponding contours obtained from WMAP5 data (blue solid lines) and from WMAP5+BAO+SN (red dashed line) (from Ref. [14]).

$(\Omega_{0\mathrm{m}}, \sigma_8) = (0.258, 0.796)$ on scales larger than $50h^{-1}Mpc$ is approximately $110km/sec$ while the probability that a flow magnitude larger than $400km/sec$ is realized in the context of the above ΛCDM normalization on scales larger than $50h^{-1}Mpc$ is less than 1%.

This is also demonstrated in Fig. 1 (from Ref. [14]) where the $(\Omega_{0\mathrm{m}}, \sigma_8)$ χ^2 confidence contours obtained from the observed velocity flows (dashed lines) are superposed with the corresponding contours obtained from WMAP5 data (blue solid lines) and from WMAP5+Baryon Acoustic Oscillations+SnIa (WMAP5+BAO+SN: red dashed line).The probability of consistency of bulk flow data with ΛCDM would be even lower if the data of Ref. [15] were considered where a flow of more than $600km/sec$ was observed on scales of $\sim 600h^{-1}Mpc$.

A potential resolution of the above described conflict between the high z WMAP5 normalization of ΛCDM and the low z normalization implied by the observed bulk flows could involve the existence of superhorizon sized non-Gaussian and non-inflationary inhomogeneities [27], a large void at distances of order gigaparsecs [28], or a redshift dependent σ_8 which changes by a factor of 2 between high z and low z due to an unknown physical reason. Other possibilities include a very large statistical fluctuation, a redshift dependence of Newton's constant or a redshift dependence of the dark energy equation of state parameter $w = w(z)$ leading to amplified gravity and dark energy clustering at early times ($w(z) > -1$ at $z > 0.2$).

2.2 Bright High z SnIa

As discussed in the introduction, geometrical tests of ΛCDM usually involve a bayesian comparison of ΛCDM with other dark energy parametrizations. This approach has not revealed so far any statistically significant weak points of the model with respect to the geometrical and dynamical probes considered.

Apart from the bayesian analysis approach, the ΛCDM model can be tested by comparing the real SnIa data with Monte Carlo simulations consisting of fictitious cosmological data that would have been obtained in the context of a ΛCDM cosmology. This comparison can be made on the basis of various statistics which attempt to pick up features of the data that can be reproduced with difficulty by a ΛCDM cosmology[16]. The existence of such features is hinted by the form of the likelihood contours in various parameter planes containing parameter values corresponding to flat ΛCDM. For example, most SnIa datasets producing likelihood contours in the $\Omega_\Lambda - \Omega_m$ parameter plane have the 1σ contour barely intersect the line of flatness $\Omega_\Lambda + \Omega_m = 1$ at the lower left side of the contour [3, 4]. Similarly, likelihood contours based on either SnIa standard candles or standard

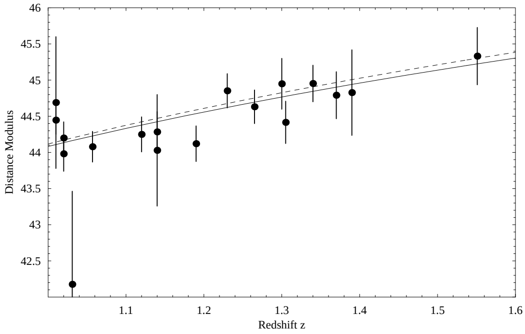

Figure 2: The Union08[4] distance moduli data superposed with the best fit ΛCDM model ($\Omega_{0\mathrm{m}} = 0.29$) dashed line and with the best fit $(w_0, w_1) = (-1.4, 2)$ model ($\Omega_{0\mathrm{m}} = 0.30$) continous line. Notice that at high redshifts z the distance moduli tend to be below the ΛCDM best fit while the trend is milder in the PDL crossing best fit model (from Ref. [16].

rulers (CMB sound horizon or Baryon Acoustic Oscillations) and constraining the parametrization [29]

$$w(z) = w_0 + w_1 \frac{z}{1+z} \tag{2.1}$$

systematically have the point corresponding to ΛCDM $(w_0, w_1) = (-1, 0)$ at the lower right edge of the 1σ contour while the best fit involves $w_0 < -1$, $w_1 > 0$ [3, 4, 30, 31]. This feature has persisted consistently over the last decade and over different accelerating expansion probes [30] (SnIa standard candles and CMB-BAO standard rulers). Even though the statistical significance of these features when viewed individually is relatively low, their persistent appearance makes it likely that there are systematic differences between the cosmological data and ΛCDM predictions.

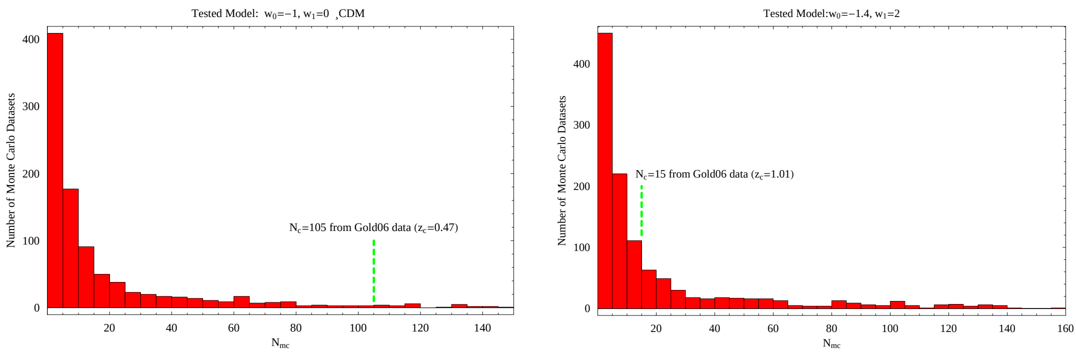

Figure 3: a: A histogram of the probability distribution of N_{mc} obtained using Monte Carlo ΛCDM data ($\Omega_{0\mathrm{m}} = 0.34$) in the context of the Gold06[3] dataset. The thick green dashed line corresponds to the crossing redshift z_c of the real Gold06 data. b: Similar histogram for the PDL crossing model $(w_0, w_1) = (-1.4, 2)$ (best fit $\Omega_{0\mathrm{m}} = 0.34$) instead of ΛCDM . Notice that the crossing redshift z_c corresponding to the real Gold06 data is a much more probable event in the context of this cosmological model (from Ref.[16]).

One such difference in the context of SnIa data has been recently pointed out by Kowalsky et. al. [4] where it was stated that there is ‘an unexpected brightness of SnIa data at $z > 1$’. This feature is even directly visible by observing the SnIa distance moduli superposed with the best fit ΛCDM model (dashed line in Fig. 1) where most high z moduli are below the best fit ΛCDM curve (obviously the reverse happens at low redshifts to achieve a good fit). Notice that this bias is smaller in the context of a parametrization that crosses the PDL $w = -1$ (continuous line in Fig. 1). [1]

This anomalous behavior of the data with respect to the ΛCDM best fit may be attributed to the systematic brightness trend of high redshift SnIa with respect to the best fit ΛCDM model. It is likely that this bias of the SnIa data with respect to ΛCDM best fit is also responsible for the systematic mild preference (at 1σ) of the SnIa data for a $w(z)$ crossing the $w = -1$ line.

In order to study quantitatively the likelihood of the existence of the above described bias in the context of a ΛCDM cosmology, we may use a statistic[16] (the Binned Normalized Differences (BND)) specially designed to pick up systematic brightness trends of the SnIa data with respect to a best fit cosmological model at high redshift. The BND statistic is based on binning the normalized differences between the SnIa distance moduli and the corresponding best fit values in the context of a specific cosmological model (eg ΛCDM). These differences are normalized by the standard errors of the observed distance moduli. We then focus on the highest redshift bin and extend its size towards lower redshifts until the Binned Normalized Difference (BND) changes sign (crosses 0) at a redshift z_c (bin size N_c). The bin size N_c of this crossing (the statistical variable) is then compared with the corresponding crossing bin size N_{mc} for Monte Carlo data realizations based on the best fit model. It may be shown[16] that the crossing bin size N_c obtained from the Union08 and Gold06 data with respect to the best fit ΛCDM model is anomalously large compared to N_{mc} of the corresponding Monte Carlo datasets obtained from the best fit ΛCDM in each case. In particular, only 2.2% of the Monte Carlo ΛCDM datasets are consistent with the Gold06 value of N_c (see Fig. 3a) while the corresponding probability for the Union08 value of N_c is 5.3%. Thus, according to this statistic, the probability that the high redshift brightness bias of the Union08 and Gold06 datasets is realized in the context of a $(w_0, w_1) = (-1, 0)$ model (ΛCDM cosmology) is less than 6%. The corresponding realization probability in the context of a $(w_0, w_1) = (-1.4, 2)$ model is more than 30% for both the Union08 and the Gold06 (see Fig. 3b) datasets indicating a much better consistency for this model with respect to the BND statistic.

This result reveals a potential challenge for ΛCDM cosmology and provides the motivation for obtaining additional SnIa data at high redshifts $z > 1$ which may confirm or disprove the anomalous high z SnIa brightness which is mainly responsible for the low probability of the high z SnIa data in the context of ΛCDM.

Clearly, the unexpected high z brightness of SnIa can be interpreted either as a trend towards more deceleration at high z than expected in the context of ΛCDM or as a statistical fluctuation or finally as a systematic effect perhaps due to a mild SnIa evolution at high z. However, in view of the fact that a similar mild trend for more deceleration than expected at high z is also observed in the context of standard rulers [30, 6, 31], the latter two interpretations are less likely than the first.

2.3 The Emptiness of Voids

Cosmological simulations performed in the context of ΛCDM predict [32] that many small dark matter haloes should reside in voids[19]. This is consistent with observations on large scales involving giant voids defined by $10^{12} M_\odot$ haloes [33]. Smaller voids however ($\sim 10 Mpc$) look very empty. Dwarf galaxies do not show a tendency to fill these voids even though ΛCDM predicts that many dwarf dark matter haloes should be in the voids. For example the ΛCDM model predicts that thousands of dwarf dark matter haloes should exist in the Local Group [34, 35, 36], while only ~ 50 are observed. Recent discoveries of very low luminosity dwarfs [37] and careful analysis of incompleteness effects in SDSS [38, 37] bring the theory and observations a bit closer, but the mismatch seems is still present.

Potential resolutions of the above tension between ΛCDM theory and observations involve incompleteness of observational sample, failure of many dwarf haloes to form stars or peculiar properties of dark matter and/or dark energy which accelerate growth of perturbations and allow gravity to clean up voids at early times.

2.4 Galaxy Halo Profiles

The ΛCDM theory predicts that dark matter halos have a specific density distribution that follows the well-known Navarro, Frenk, White (NFW) [39, 40] profile:

$$\rho_{NFW}(R) = \frac{\rho_s}{(R/r_s)(1 + R/r_s)^2} \tag{2.2}$$

[1]In the PDL crossing model we fix w_0, w_1 and vary Ω_{0m} only, in order to mimic the ΛCDM number of parameters.

where r_s and ρ_s are the characteristic radius and density of the distribution. A useful parameter characterizing the profile is the concentration parameter c defined as is $c = r_{vir}/r_s$ where r_{vir} is the virial radius of the system. r_s and ρ_s are related to each other (e.g. [41]), so eq. (2.2) is rather a one-parameter family of profiles.

A quite remarkable number of observations show that NFW profiles, displaying an inner "cusp", are inconsistent with data[42]. In fact, the latter indicate profiles with a different characteristic, a central density "core", i.e. a region where the dark matter density remains approximately constant.

In addition to the above well-known evidence for which in the inner regions of galaxies ($R < 2r_d$ where r_d is the stellar disk radius) the dark matter haloes show a flattish density profile, with amplitudes up to one order of magnitude lower than the ΛCDM predictions, at outer radii ($R > 4r_d$) the measured dark matter halo densities are found higher than the corresponding ΛCDM ones. The dark matter halo density, known to have a core in the internal regions, does not seem to converge to the NFW profile at $4 - 6r_d$ [21]. This implies an issue for ΛCDM that should be investigated in the future, when, due to improved observational techniques, the kinematic information will be extended to the $100kpc$ scale.

A possible resolution of the puzzle of higher than expected dark matter halo density in the galactic haloes is that massive halos themselves were assembled at high redshift[19]. If this is the case, modifying the properties of dark energy could play a role in shifting the epoch of galaxy formation towards earlier times. Alternatively, modified gravity theories or clustering of dark energy may also be considered as a potential resolution of this puzzle.

2.5 Cluster Halo Profiles

In the ΛCDM context, detailed N-body simulations have established a clear prediction that CDM-dominated cluster halos should have relatively shallow, low-concentration mass profiles, where the logarithmic gradient flattens continuously toward the center with a central slope tending towards r^{-1}, interior to a characteristic radius, $r_s \sim 100 - 200kpc \cdot h^{-1}$ [39, 40, 43, 44, 45, 46, 47].

Multiply-lensed images of various clusters [17] have been used to derive the inner mass profile [48], with the outer profile determined from weak lensing [49]. Together, the full profile has the predicted NFW form [40], but with a surprisingly high concentration $c = \frac{r_{vir}}{r_s}$ and high density when compared to the shallow profiles of the standard ΛCDM model [49, 50]. This result is verified by using not only the lensing based mass profile but also the X-ray and dynamical structure in model independent analyses [51].

A potential resolution of the above discrepancy between observed cluster profiles and ΛCDM predictions is that the central region of clusters collapsed, as in the case of galaxies, earlier than expected ie at $z > 1$, significantly earlier than in the standard ΛCDM , for which clusters form at $z < 0.5$.

The presence of massive clusters at high redshift ($z \sim 2$), and the old ages of their member galaxies [52, 53], may also imply clusters collapsed at relatively early times [54], for which accelerated growth factors have been proposed, adopting a generalized equation of state for dark energy [55]. Such an equation of state would allow for a non-negligible dark energy density at early times. Thus, as in the case of galaxy formation, the properties of dark matter and/or dark energy could also play a significant role in the resolution of this puzzle.

2.6 Overpopulation of Disk Galaxies

Roughly 70% of Milky-Way size dark matter halos are believed to host late-type, disk dominated galaxies [56]. Conventional wisdom dictates that disk galaxies result from fairly quiescent formation histories, and this has raised concerns about disk formation within the hierarchical ΛCDM cosmology [57, 58]. Recent evidence for the existence of a sizeable population of cold, rotationally supported disk galaxies at $z \sim 1.6$ [59] is particularly striking, given that the fraction of galaxies with recent mergers is expected to be significantly higher at that time [60]. High-resolution, dissipationless N-body simulations[61] studying the response of stellar Milky-Way type disks to such common mergers show that thin disks do not survive the bombardment. The remnant galaxies are roughly three times as thick and twice as kinematically hot as the observed thin disk of the Milky Way. However, despite of such indications a real evaluation of the severity of the problem is limited by both theoretical and observational concerns.

The role of dark energy in the resolution of this and other astrophysical scale puzzles should not be underestimated. For example, it has been demonstrated that the effects of dark energy on the equilibrium and stability of astrophysical structures is not negligible, and can be of relevance to describe features of astrophysical systems such as globular clusters, galaxy clusters or even galaxies [62, 63, 64, 65]. It has recently been demonstrated that the dark energy fluid changes certain aspects of astrophysical hydrostatic equilibrium. For example, the instability of previously viable astrophysical systems when dark energy is included has been demonstrated as due to the repulsive non local dark energy force acting on the matter distribution [66]. With the proper evolution of the dark energy equation of state, this repulsive force may also lead to a modification of the profile of the virialized structures thus addressing some of the above discussed puzzles on galactic and cluster scales.

3 Discussion - Conclusion

I have reviewed some of the potential challenges of the ΛCDM cosmological model pointing out that there are such challenges on both large and small cosmological scales. Even though some of the puzzles discussed here may be resolved by more complete observations or astrophysical effects, the possible requirement of more fundamental modifications of the ΛCDM model remains valid.

It is interesting to attempt to identify universal features which connect these puzzles and could therefore provide a guide for their simultaneous resolution. The large scale coherent velocity flows along with the high density dark matter haloes for both galaxies and clusters seem to hint towards a more effective mechanism for structure formation at early times ($z > 1$) than implied by ΛCDM . This improved effectiveness could possibly be provided by a mild evolution of Newton's constant G (higher G at $z > 0.5$) or by an evolution of the dark energy equation of state w such that $w(z) > -1$ at $z \gtrsim 0.5$ [55]. Both of these effects are expected to amplify structure formation at early times and it would be interesting to analyze quantitatively the predictions implied by the evolution of G or w with respect to the velocity flow and high dark matter density puzzles. The Bright High z SnIa puzzle would also benefit significantly by a mild evolution of w or G which would imply stronger deceleration at $z > 1$ than implied by ΛCDM .

The improved efficiency of gravity at early times could also help emptying the voids from dark matter haloes and their corresponding galaxies thus making theoretical predictions more consistent with observations. On the other hand, the increased gravitational acceleration would also produce higher peculiar velocities that could lead to more mass inside the voids. Therefore, the predicted emptiness of voids in models with an evolving G or w requires a detailed study.

In conclusion, the six puzzles for ΛCDM discussed in the present study provide a fertile ground for the development of both new theoretical model predictions on the corresponding observables and new observational data that would either establish or disprove these challenges for ΛCDM .

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Supersymmetric Cosmology and Dark Energy

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Abstract

Using the superfield approach we construct the $n = 2$ supersymmetric lagrangian for the FRW Universe with perfect fluid as matter fields. The obtained supersymmetric algebra allowed us to take the square root of the Wheeler-DeWitt equation and solve the corresponding quantum constraint. This model leads to the relation between the vacuum energy density and the energy density of the dust matter.

Introduction

This paper is for the anniversary volume on the occasion 50th birthday, Sergei Odintsov, our colleague and friend who made an extensive contribution to the cosmological and astrophysics fields.

Some time ago we have used the superfield formulation to investigate supersymmetric cosmological models [10]. The main idea is to extend the group of local time reparametrization of the cosmological models to the local time supersymmetry which is a subgroup of the four dimensional space-time supersymmetry. This local supersymmetry procedure has the advantage that, by defining the superfields on superspace, all the component fields in a supermultiplet can be manipulated simultaneously in a manner that automatically preserves supersymmetry. Besides, the fermionic fields are obtained in a clear manner as the supersymmetric partners of the cosmological bosonic variables.

More recently, using the superfield formulation the canonical procedure quantization for a closed FRW cosmological model filled with pressureless matter (dust) content and the corresponding superpartner was reported [4]. We have obtained the quantization for the energy-like parameter, and it was shown, that this energy is associated with the mass parameter quantization, and that such type of Universe has a quantized masses of the order of the Planck mass.

In the present work we are interested in the construction of the $n = 2$ supersymmetric lagrangian for the FRW Universe with barotropic perfect fluid as matter field including the cosmological constant. The simplest dark energy candidate is the cosmological constant stemming from energy density of the vacuum [11]. The obtained supersymmetric algebra allowed us to take the square root of the Wheeler-DeWitt equation and solve the corresponding quantum constraint.

Classical Action

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The classical action for a pure gravity system and the corresponding term of matter content, perfect fluid with a constant equation of state parameter γ; $p = \gamma\rho$, and the cosmological term is [4]

$$S = \int \Big[-\frac{c^2 R}{2N\tilde{G}} \left(\frac{dR}{dt}\right)^2 + \frac{Nkc^4}{2\tilde{G}} R + \frac{Nc^4\Lambda}{6\tilde{G}} R^3 + NM_\gamma c^2 R^{-3\gamma} \Big] dt. \qquad (0.1)$$

where c is the velocity of light in vacuum, $\tilde{G} = \frac{8\pi G}{6}$ where G is the Newtonian gravitational constant; $k = 1, 0, -1$ stands for spherical, plane or hyperspherical three space; $N(t), R(t)$ are the lapse function and the scale factor, respectively; M_γ is the mass by unit length$^{-\gamma}$.

The purpose of this work is the supersymmetrization of the full action (1.2) using the superfield approach. The action (1.2) is invariant under the time reparametrization

$$t' \to t + a(t), \qquad (0.2)$$

if the transformations of $R(t)$ and $N(t)$ are defined as

$$\delta R = a\dot{R}, \qquad \delta N = (aN)^{\cdot} \qquad (0.3)$$

The variation with respect to $R(t)$ and $N(t)$ lead to the classical equation for the scale factor $R(t)$ and the constraint, which generates the local reparametrization of $R(t)$ and $N(t)$. This constraint leads to the Wheeler-DeWitt equation in quantum cosmology.

In order to obtain the corresponding supersymmetric action for (1.2), we follow the superfield approach. For this, we extend the transformation of time reparametrization (0.2) to the $n = 2$ local supersymmetry of time $(t, \eta, \bar{\eta})$. Then, we have the following local supersymmetric transformation

$$\begin{aligned} \delta t &= a(t) + \frac{i}{2}[\eta\bar{\beta}'(t) + \bar{\eta}\beta'(t)], \\ \delta\eta &= \frac{1}{2}\bar{\beta}'(t) + \frac{1}{2}[\dot{a}(t) + ib(t)]\eta + \frac{i}{2}\dot{\bar{\beta}}'(t)\eta\bar{\eta}, \\ \delta\bar{\eta} &= \frac{1}{2}\beta'(t) + \frac{1}{2}[\dot{a}(t) - ib(t)]\bar{\eta} - \frac{i}{2}\dot{\beta}'(t)\eta\bar{\eta}, \end{aligned} \qquad (0.4)$$

where η is a complex odd parameter (η odd "time" coordinates), $\beta'(t) = N^{-1/2}\beta(t)$ is the Grassmann complex parameter of the local "small" $n = 2$ supersymmetry (SUSY) transformation, and $b(t)$ is the parameter of local $U(1)$ rotations of the complex η.

For the closed ($k = 1$) and plane ($k = 0$) FRW action we propose the following superfield generalization of the action (1.2), invariant under the $n = 2$ local supersymmetric transformation (0.4)

$$\begin{aligned} S_{susy} &= \int \Big[-\frac{c^2}{2\tilde{G}} I\!N^{-1} I\!R D_{\bar{\eta}} I\!R D_\eta I\!R + \frac{c^3\sqrt{k}}{2\tilde{G}} I\!R^2 + \frac{c^3\Lambda^{1/2}}{3\sqrt{3}\tilde{G}} I\!R^3 - \\ &- \frac{2\sqrt{2}M_\gamma^{1/2}}{(3-3\gamma)\tilde{G}^{1/2}} I\!R^{\frac{3-3\gamma}{2}} \Big] d\eta d\bar{\eta} dt, \end{aligned} \qquad (0.5)$$

where

$$D_\eta = \frac{\partial}{\partial\eta} + i\bar{\eta}\frac{\partial}{\partial t}, \qquad D_{\bar{\eta}} = -\frac{\partial}{\partial\bar{\eta}} - i\eta\frac{\partial}{\partial t}, \qquad (0.6)$$

are the supercovariant derivatives of the global "small" supersymmetry of the generalized parameter corresponding to t. The local supercovariant derivatives have the form $\tilde{D}_\eta = I\!N^{-1/2}D_\eta$, $\tilde{D}_{\bar{\eta}} = I\!N^{-1/2}D_{\bar{\eta}}$, and $I\!R(t,\eta,\bar{\eta}), I\!N(t,\eta,\bar{\eta})$ are superfields.

The Taylor series expansion for the superfields $I\!N(t,\eta,\bar{\eta})$ and $I\!R(t,\eta,\bar{\eta})$ are the following

$$I\!N(t,\eta,\bar{\eta}) = N(t) + i\eta\bar{\psi}'(t) + i\bar{\eta}\psi'(t) + V'(t)\eta\bar{\eta}, \qquad (0.7)$$

$$I\!R(t,\eta,\bar{\eta}) = R(t) + i\eta\bar{\lambda}'(t) + i\bar{\eta}\lambda'(t) + B'(t)\eta\bar{\eta}. \qquad (0.8)$$

In the expressions (0.7) and (0.8) we have introduced the redefinitions $\psi'(t) = N^{1/2}\psi(t)$, $V' = N(t)V(t) + \bar{\psi}(t)\psi(t)$, $\lambda' = \frac{\tilde{G}^{1/2}N^{1/2}}{cR^{1/2}}\lambda$ and $B' = \frac{\tilde{G}^{1/2}}{c}NB + \frac{\tilde{G}^{1/2}}{2cR^{1/2}}(\bar{\psi}\lambda - \psi\bar{\lambda})$. The components of the superfield $I\!\!N(t,\eta,\bar\eta)$ are gauge fields of the one-dimensional $n=2$ extended supergravity. $N(t)$ is the einbein, $\psi(t),\bar\psi(t)$ are the complex gravitino fields, and $V(t)$ is the $U(1)$ gauge field. The component $B(t)$ in (0.8) is an auxiliary degree of freedom (non-dynamical variable), and $\lambda,\bar\lambda$ are the fermion partners of the scale factor $R(t)$. After the integration over the Grassmann coordinates $\theta,\bar\theta$ we can rewrite the action (0.5) in its component form

$$\begin{aligned}
S_{susy} \quad = \quad \int \quad & -\frac{c^2R(DR)^2}{2N\tilde G}+\frac{i}{2}(\bar\lambda D\lambda-D\bar\lambda\lambda)-\frac{NR}{2}B^2-\frac{N\tilde G^{1/2}B}{2cR}\bar\lambda\lambda+\\
&+\frac{c^2\sqrt{k}RN}{\tilde G^{1/2}}B+\frac{c^2\sqrt{k}R^{1/2}}{2\tilde G^{1/2}}(\bar\psi\lambda-\psi\bar\lambda)+\frac{cN\sqrt{k}}{R}\bar\lambda\lambda+\\
&+\frac{c^2\Lambda^{1/2}}{\sqrt3\tilde G^{1/2}}NR^2B+\frac{c^2\Lambda^{1/2}R^{3/2}}{2\sqrt3\tilde G^{1/2}}(\bar\psi\lambda-\psi\bar\lambda)+\frac{2c\Lambda^{1/2}N}{\sqrt3}\bar\lambda\lambda-\\
&-\sqrt2cM_\gamma^{1/2}NR^{\frac{1-3\gamma}{2}}B-\frac{\sqrt2}{2}cM_\gamma^{1/2}R^{-\frac{3\gamma}{2}}(\bar\psi\lambda-\psi\bar\lambda)-\\
&-\sqrt2(1-3\gamma)\tilde G^{1/2}M_\gamma^{1/2}NR^{\frac{-3-3\gamma}{2}}\bar\lambda\lambda\Big\}\,dt.
\end{aligned}\tag{0.9}$$

So, the lagrangian for the auxiliary field has the form

$$\begin{aligned}
L_B \quad = \quad & -\frac{NR}{2}B^2-\frac{N\tilde G^{1/2}B}{2cR}\bar\lambda\lambda+\frac{c^2\sqrt{k}RN}{\tilde G^{1/2}}B+\frac{c^2\Lambda^{1/2}NR^2}{\sqrt3\tilde G^{1/2}}B-\\
&-\sqrt2cM_\gamma^{1/2}NR^{\frac{1-3\gamma}{2}}B.
\end{aligned}\tag{0.10}$$

From the expression (0.10) we can obtain the equation for the auxiliary field varying the Lagrangian with respect to B

$$B=\frac{c^2\sqrt{k}}{\tilde G^{1/2}}-\frac{\tilde G^{1/2}}{2cR^2}\bar\lambda\lambda+\frac{c^2\Lambda^{1/2}R}{\sqrt3\tilde G^{1/2}}-\sqrt2cM_\gamma^{1/2}R^{\frac{-3\gamma-1}{2}}.\tag{0.11}$$

Then, putting the expression (0.11) in (0.9) we have the following supersymmetric action

$$\begin{aligned}
S_{susy} \quad = \quad \int \quad & -\frac{c^2R(DR)^2}{2N\tilde G}+\frac{c^4NkR}{2\tilde G}+\frac{c^4N\Lambda R^3}{6\tilde G}+Nc^2M_\gamma R^{-3\gamma}+\\
&+\frac{c^4\sqrt{k}\Lambda^{1/2}R^2}{\sqrt3\tilde G}-\frac{\sqrt{2k}c^3}{\tilde G^{1/2}}M_\gamma^{1/2}R^{\frac{1-3\gamma}{2}}-\frac{\sqrt2c^3\Lambda^{1/2}M_\gamma^{1/2}}{\sqrt3\tilde G^{1/2}}R^{\frac{3-3\gamma}{2}}+\\
&+\frac{i}{2}(\bar\lambda D\lambda-D\bar\lambda\lambda)+\frac{cN\sqrt{k}}{2R}\bar\lambda\lambda+\frac{\sqrt3}{2}c\Lambda^{1/2}N\bar\lambda\lambda+\\
&+\frac{(-1+6\gamma)}{\sqrt2}N\tilde G^{1/2}M_\gamma^{1/2}R^{\frac{-3-3\gamma}{2}}\bar\lambda\lambda+\frac{c^2\sqrt{k}R^{1/2}}{2\tilde G^{1/2}}(\bar\psi\lambda-\psi\bar\lambda)\\
&+\frac{c^2\Lambda^{1/2}}{2\sqrt3\tilde G^{1/2}}R^{3/2}(\bar\psi\lambda-\psi\bar\lambda)-\frac{\sqrt2}{2}cM_\gamma^{1/2}R^{-\frac{3\gamma}{2}}(\bar\psi\lambda-\psi\bar\lambda)\quad dt,
\end{aligned}\tag{0.12}$$

where $DR=\dot R-\frac{i\tilde G^{1/2}}{2cR^{1/2}}(\psi\bar\lambda+\bar\psi\lambda)$ and $D\lambda=\dot\lambda-\frac12V\lambda$, $D\bar\lambda=\dot{\bar\lambda}+\frac12V\bar\lambda$.

Supersymmetric Quantum Model

In this section we will proceed with the quantization analysis of the system. The classical canonical Hamiltonian is calculated in the usual way for the systems with constraints. It has the form

$$H_c=NH+\frac12\bar\psi S-\frac12\psi\bar S+\frac12VF,\tag{0.13}$$

where H is the Hamiltonian of the system, S and $\bar S$ are the supercharges and F is the $U(1)$ rotation generator. The form of the canonical Hamiltonian (0.13) explains the fact that $N,\psi,\bar\psi$ and V are Lagrangian multipliers which only enforce the first-class constraints $H=0, S=0, \bar S=0$ and $F=0$, which express the invariance under the conformal $n=2$ supersymmetric transformations.

The first-class constraints may be obtained from the action (0.12) varying $N(t), \psi(t), \bar{\psi}(t)$ and $V(t)$, respectively. The first-class constraints are

$$
\begin{aligned}
H &= -\frac{\tilde{G}}{2c^2 R}\pi_R^2 - \frac{c^4 kR}{2\tilde{G}} - \frac{c^4 \Lambda R^3}{6\tilde{G}} - M_\gamma c^2 R^{-3\gamma} + \frac{\sqrt{2}c^3\Lambda^{1/2}M_\gamma^{1/2}}{\sqrt{3}\tilde{G}^{1/2}} R^{\frac{3-3\gamma}{2}} - \\
&- \frac{c^4\sqrt{k}\Lambda^{1/2}R^2}{\sqrt{3}\tilde{G}} + \frac{\sqrt{2k}c^3}{\tilde{G}^{1/2}} M_\gamma^{1/2} R^{\frac{1-3\gamma}{2}} - \frac{c\sqrt{k}}{2R}\bar{\lambda}\lambda - \frac{\sqrt{3}}{2}c\Lambda^{1/2}\bar{\lambda}\lambda - \\
&- \frac{(6\gamma-1)}{\sqrt{2}}\tilde{G}^{1/2}M_\gamma^{1/2}R^{\frac{-3-3\gamma}{2}}\bar{\lambda}\lambda, \qquad (0.14)\\
S &= \frac{i\tilde{G}^{1/2}}{cR^{1/2}}\pi_R - \frac{c^2\sqrt{k}R^{1/2}}{\tilde{G}^{1/2}} - \frac{c^2\Lambda^{1/2}R^{3/2}}{\sqrt{3}\tilde{G}^{1/2}} + \sqrt{2}cM_\gamma^{1/2}R^{-\frac{3\gamma}{2}} \quad \lambda, \qquad (0.15)\\
\bar{S} &= -\frac{i\tilde{G}^{1/2}}{cR^{1/2}}\pi_R - \frac{c^2\sqrt{k}R^{1/2}}{\tilde{G}^{1/2}} - \frac{c^2\Lambda^{1/2}R^{3/2}}{\sqrt{3}\tilde{G}^{1/2}} + \sqrt{2}cM_\gamma^{1/2}R^{-\frac{3\gamma}{2}} \quad \bar{\lambda}, \qquad (0.16)\\
F &= -\bar{\lambda}\lambda, \qquad (0.17)
\end{aligned}
$$

where $\pi_R = -\frac{c^2 R}{\tilde{G}N}\dot{R} + \frac{icR^{1/2}}{2N\tilde{G}^{1/2}}(\bar{\psi}\lambda + \psi\bar{\lambda})$ is the canonical momentum associated to R. The canonical Dirac brackets are defined as

$$\{R, \pi_R\} = 1, \quad \{\lambda, \bar{\lambda}\} = i. \qquad (0.18)$$

With respect to these brackets the super-algebra for the generators $H, S, \bar{S}$ and F becomes

$$\{S, \bar{S}\} = -2iH, \quad \{S, H\} = \{\bar{S}, H\} = 0, \quad \{F, S\} = iS, \quad \{F, \bar{S}\} = i\bar{S}. \qquad (0.19)$$

In a quantum theory the brackets (0.18) must be replaced by anticommutators and commutators, they can be considered as generators of the Clifford algebra. We have

$$
\begin{aligned}
\{\lambda, \bar{\lambda}\} &= -\hbar, \qquad [R, \pi_R] = i\hbar \qquad \text{with} \quad \pi_R = -i\hbar\frac{\partial}{\partial R} \qquad (0.20)\\
\bar{\lambda} &= \xi^{-1}\lambda^\dagger\xi = -\lambda^\dagger, \qquad \{\lambda, \lambda^\dagger\} = \hbar, \qquad \lambda^\dagger\xi = \xi\lambda^\dagger \quad \text{and} \quad \xi^\dagger = \xi.
\end{aligned}
$$

Then, for the operator $\bar{S}$ the following equation is satisfied

$$\bar{S} = \xi^{-1}S^\dagger\xi. \qquad (0.21)$$

Therefore, the anticommutator of supercharges S and their conjugated operator $\bar{S}$ under our defined conjugation has the form

$$\overline{\;S, \bar{S}\;} = \xi^{-1} \;\; S, \bar{S} \;\; \xi = \;\; S, \bar{S} \;\; , \qquad (0.22)$$

and the Hamiltonian operator is self-conjugated under the operation $\bar{H} = \xi^{-1}H^\dagger\xi$. We can choose the matrix representation for the fermionic parameters $\lambda, \bar{\lambda}$ and ξ as

$$\lambda = \sqrt{\hbar}\sigma_-, \qquad \bar{\lambda} = -\sqrt{\hbar}\sigma_+, \qquad \xi = \sigma_3, \qquad (0.23)$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, where σ_1, σ_2, σ_3 are the Pauli matrices.

In the quantum level we must consider the nature of the Grassmann variables λ and $\bar{\lambda}$, with respect to these we perform the antisymmetrization, then we can write the bilinear combination in the form of the commutators, $\bar{\lambda}, \lambda \to \frac{1}{2}[\bar{\lambda}, \lambda]$, and this leads to the following quantum Hamiltonian H.

$$
\begin{aligned}
H_{quantum} &= -\frac{\tilde{G}}{2c^2}R^{-1/2}\pi_R R^{-1/2}\pi_R - \frac{c^4 kR}{2\tilde{G}} - \frac{c^4\Lambda R^3}{6\tilde{G}} - M_\gamma c^2 R^{-3\gamma}\\
&+ \frac{\sqrt{2}c^3\Lambda^{1/2}M_\gamma^{1/2}}{\sqrt{3}\tilde{G}^{1/2}}R^{\frac{3-3\gamma}{2}} - \frac{c^4\sqrt{k}\Lambda^{1/2}R^2}{\sqrt{3}\tilde{G}} + \\
&+ \frac{\sqrt{2k}c^3}{\tilde{G}^{1/2}}M_\gamma^{1/2}R^{\frac{1-3\gamma}{2}} - \frac{c\sqrt{k}}{4R}[\bar{\lambda}, \lambda] - \frac{\sqrt{3}}{4}c\Lambda^{1/2}[\bar{\lambda}, \lambda] - \\
&- \frac{(6\gamma-1)}{2\sqrt{2}}\tilde{G}^{1/2}M_\gamma^{1/2}R^{\frac{-3-3\gamma}{2}}[\bar{\lambda}, \lambda]. \qquad (0.24)
\end{aligned}
$$

The supercharges S, $\bar{S}$ and the fermion number F have the following structures:

$$S = A\lambda, \qquad S^{\dagger} = A^{\dagger}\lambda^{\dagger} \tag{0.25}$$

where

$$A = \frac{i\tilde{G}^{1/2}}{c}R^{-1/2}\pi_R - \frac{c^2\sqrt{k}}{\tilde{G}^{1/2}}R^{1/2} - \frac{c^2\Lambda^{1/2}R^{3/2}}{\sqrt{3}\tilde{G}^{1/2}} + \sqrt{2}cM_{\gamma}^{1/2}R^{-\frac{3\gamma}{2}}, \tag{0.26}$$

and

$$F = -\frac{1}{2}[\bar{\lambda}, \lambda]. \tag{0.27}$$

An ambiguity exist in the factor ordering of these operators, such ambiguities always arise, when the operator expression contains the product of non-commuting operator R and π_R, as in our case. It is then necessary to find some criteria to know which factor ordering should be selected. The inner product is calculated performing the integration with the measure $R^{1/2}dR$. With this measure the conjugate momentum π_R is non-Hermitian with $\pi_R^{\dagger} = R^{-1/2}\pi_R R^{1/2}$. However, the combination $(R^{-1/2}\pi_R)^{\dagger} = \pi_R^{\dagger}R^{-1/2} = R^{-1/2}\pi_R$ is a Hermitian one, and $(R^{-1/2}\pi_R R^{1/2}\pi_R)^{\dagger} = R^{-1/2}\pi_R R^{1/2}\pi_R$ is Hermitian too. This choice in our supersymmetric quantum approach $n = 2$ eliminates the factor ordering ambiguity by fixing the ordering parameter $p = \frac{1}{2}$.

Superquantum Solutions

In the quantum theory, the first-class constraints $H = 0, S = 0, \bar{S} = 0$ and $F = 0$ become conditions on the wave function $\Psi(R)$. Furthermore, any physical state must be satisfied the quantum constraints

$$H\Psi(R) = 0, \quad S\Psi(R) = 0, \quad \bar{S}\Psi(R) = 0, \quad F\Psi(R) = 0, \tag{0.28}$$

where the first equation is the Wheeler-DeWitt equation for the minisuperspace model. The eigenstates of the Hamiltonian (0.24) have two components in the matrix representation (0.23)

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \tag{0.29}$$

However, the supersymmetric physical states are obtained applying the supercharges operators $S\Psi = 0, \bar{S}\Psi = 0$. With the conformal algebra given by (0.19), these are rewritten in the following form

$$(\lambda\bar{S} - \bar{\lambda}S)\Psi = 0. \tag{0.30}$$

Using the matrix representation for λ and $\bar{\lambda}$ we obtain the following differential equations for $\Psi_1(R)$ and $\Psi_2(R)$ components

$$\left(\frac{\hbar\tilde{G}^{1/2}}{c}R^{-1/2}\frac{\partial}{\partial R} - \frac{c^2\sqrt{k}R^{1/2}}{\tilde{G}^{1/2}} - \frac{c^2\Lambda^{1/2}R^{3/2}}{\sqrt{3}\tilde{G}^{1/2}} + \sqrt{2}cM_{\gamma}^{1/2}R^{-\frac{3\gamma}{2}}\right)\Psi_1(R) = 0. \tag{0.31}$$

$$\left(\frac{\hbar\tilde{G}^{1/2}}{c}R^{-1/2}\frac{\partial}{\partial R} + \frac{c^2\sqrt{k}R^{1/2}}{\tilde{G}^{1/2}} + \frac{c^2\Lambda^{1/2}R^{3/2}}{\sqrt{3}\tilde{G}^{1/2}} - \sqrt{2}cM_{\gamma}^{1/2}R^{-\frac{3\gamma}{2}}\right)\Psi_2(R) = 0. \tag{0.32}$$

Solving these equation, we have the following wave functions solutions

$$\Psi_1(R) = C\exp\Big[\frac{\sqrt{k}c^3R^2}{2\hbar\tilde{G}} + \frac{c^3\Lambda^{1/2}}{3\sqrt{3}\hbar\tilde{G}}R^3 - \frac{2\sqrt{2}c^2M_{\gamma}^{1/2}}{(3-3\gamma)\hbar\tilde{G}^{1/2}}R^{\frac{3-3\gamma}{2}}\Big], \tag{0.33}$$

$$\Psi_2(R) = \tilde{C}\exp\Big[-\frac{\sqrt{k}c^3R^2}{2\hbar\tilde{G}} - \frac{c^3\Lambda^{1/2}}{3\sqrt{3}\hbar\tilde{G}}R^3 + \frac{2\sqrt{2}c^2M_{\gamma}^{1/2}}{(3-3\gamma)\hbar\tilde{G}^{1/2}}R^{\frac{3-3\gamma}{2}}\Big]. \tag{0.34}$$

In the case of the flat universe ($k = 0$) and for the dust-like matter ($\gamma = 0$) we have the following solutions (using the relation $M_{\gamma=0} = \frac{1}{2}R^3\rho_{\gamma=0}$)

$$\Psi_1(R) = C_1\exp\Big[\frac{1}{\sqrt{6\pi}}\left(\frac{\rho_\Lambda}{\rho_{pl}}\right)^{1/2}\left(\frac{R}{l_{pl}}\right)^3 - \frac{\sqrt{2}}{\sqrt{6\pi}}\left(\frac{\rho_{\gamma=0}}{\rho_{pl}}\right)^{1/2}\left(\frac{R}{l_{pl}}\right)^3\Big], \tag{0.35}$$

$$\Psi_2(R) = C_2 \exp\Big[-\frac{1}{\sqrt{6\pi}} \left(\frac{\rho_\Lambda}{\rho_{pl}}\right)^{1/2} \left(\frac{R}{l_{pl}}\right)^3 + \frac{\sqrt{2}}{\sqrt{6\pi}} \left(\frac{\rho_{\gamma=0}}{\rho_{pl}}\right)^{1/2} \left(\frac{R}{l_{pl}}\right)^3 \Big], \tag{0.36}$$

where $\rho_{pl} = \frac{c^5}{\hbar G^2}$ is the Planck density and $l_{pl} = \left(\frac{\hbar G}{c^3}\right)^{1/2}$ is the Planck length.

We can see, that the function Ψ_1 in (0.3) has good behavior when $R \to \infty$ under the condition $\rho_\Lambda < 2\rho_{\gamma=0}$, while Ψ_2 does not. On the other hand, the wave function Ψ_2 in (0.36) has good behavior when $R \to \infty$ under the condition $\rho_\Lambda > 2\rho_{\gamma=0}$, because the principal contribution comes from the first term of the exponent, while Ψ_1 does not have good behavior. However, only the scalar product for the second wave function Ψ_2 is normalizable in the measure $R^{1/2}dR$ under the condition $\rho_\Lambda > 2\rho_{\gamma=0}$. This condition does not contradict the astrophysical observation at $\rho_\Lambda \approx (2-3)\rho_M$, due to the fact that the dust matter introduces the main contribution to the total energy density of matter ρ_M.

On the other hand, according to recent astrophysical data, our universe is dominated by a mysterious form of the dark energy [4], which counts to about 70 per cent of the total energy density. As a result, the universe expansion is accelerating [5, 6]. Vacuum energy density $\rho_\Lambda = \frac{c^2\Lambda}{8\pi G}$ is a concrete example of the dark energy.

Conclusion

The recent cosmological data give us the following range for the dark energy state parameter $\gamma = -0.96^{+0.08}_{-0.09}$. However, in the literature we can find different theoretical models for the dark energy with state parameter $\gamma > -1$ and $\gamma < -1$, see reviews [7, 8] and the articles [9, 10]. In the present work we have discussed the case for $\gamma = 0$ corresponding to the FRW universe with barotropic perfect fluid as matter field. In the case of the flat universe ($k = 0$) and the dust-like matter $\gamma = 0$ we have obtained two wave functions. However, only the second wave function is normalizable under the condition $\rho_\Lambda > 2\rho_{\gamma=0}$, which leads to the cosmological value $\Lambda > \frac{16\pi G}{c^2}\rho_{\gamma=0}$.

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Constraining Post-Newtonian $f(R)$ Gravity in the Solar System

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Abstract

We consider some models of $f(R)$ gravity that can be used to describe, in a suitable weak-field limit, the gravitational field of the Sun. Using a perturbative approach, we focus on the impact that the modifications of the gravitational field, due to the non-linearity of the gravity Lagrangian, have on the Solar System dynamics. We compare the theoretical predictions for the precession of the longitude of the pericentre ϖ of a test particle with the corrections to the standard Newtonian-Einsteinian precessions of the longitudes of perihelia of some planets of the Solar System recently estimated by E.V. Pitjeva by fitting large data sets with various versions of the EPM ephemerides.

1 Introduction

General Relativity (GR) has passed with excellent results many observational tests: a satisfactory agreement comes both from Solar System tests and from binary pulsars observations. As a matter of fact (see e.g. [49]), the current values of the PPN parameters are in agreement with GR predictions and, consequently, Einstein's theory is the classical theory of gravitational interactions accepted nowadays.

However, observations seem to question the general relativistic model of gravitational interactions on large scales. On the one hand, the data coming from the rotation curves of spiral galaxies [4] cannot be explained on the basis of Newtonian gravity or GR: the existence of a peculiar form of matter is postulated to reconcile the theoretical model with observations, i.e. dark matter, which is supposed to be a cold and pressureless medium, whose distribution is that of a spherical halo around the galaxies. Furthermore, dark matter can explain the mass discrepancy in galactic clusters [9]. On the other hand, a lot of observations, such as the light curves of the type Ia supernovæ and the cosmic microwave background (CMB) experiments [42, 36, 3], firmly state that our Universe is now undergoing a phase of accelerated expansion. Actually, the present acceleration of the Universe cannot be explained, within GR, unless the existence of a cosmic fluid having exotic properties is postulate, i.e. dark energy or introducing a cosmological constant which, in turn, brings about other problems, concerning its nature and origin [35].

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The main problem one has to face with dark matter and dark energy (or the cosmological constant) is understanding their nature, since they are introduced as *ad hoc* gravity sources in GR or its weak-field limit, Newtonian gravity.

In order to explain the observations another possibility exists: the query for dark matter and dark energy points out the failure of GR (and its approximation, Newtonian gravity) to deal with gravitational interaction at galactic, intergalactic and cosmological scales. The latter viewpoint led to the introduction of various modified gravity models.

In this paper, we are concerned with the so called $f(R)$ theories of gravity, where the gravitational Lagrangian depends on a function f of the scalar curvature R (see [5, 47] and references therein). These theories are also referred to as "extended theories of gravity", since they naturally generalize GR: in fact, when $f(R) = R$ the action reduces to the usual Einstein-Hilbert action, and Einstein's theory is obtained. These theories can be studied in the metric formalism, where the action is varied with respect to metric tensor, and in the Palatini formalism, where the action is varied with respect to the metric and the affine connection, which are supposed to be independent from one another (actually, there is also the possibility that the matter part of the action depends on the affine connection, and is then varied with respect to it: this is the so-called metric-affine formalism, but we are not concerned with this approach in this paper). In general, the two approaches are not equivalent: the solutions of the Palatini field equations are a subset on solutions of the metric field equations [27].

Actually, $f(R)$ theories provide cosmologically viable models, where both the inflation phase and the accelerated expansion are reproduced (see [31, 32, 33] and references therein). Furthermore, they have been used to explain the rotation curves of galaxies without need for dark matter [6, 15].

However, because of the excellent agreement of GR with Solar System and binary pulsar observations, every theory that aims at explaining galaxies dynamics and the accelerated expansion of the Universe, should reproduce GR at the Solar System scale, i.e. in a suitable weak-field limit. In other words, also for $f(R)$ theories the constraint holds to have correct Newtonian and post-Newtonian limits. This issue has been lively debated in the recent literature, where different approaches to the problem have been taken into account, both in the Palatini and metric formalism. A thorough discussion can be found in the recent review by Sotiriou and Faraoni [47]. In summary, with respect to the weak-field tests and, more in general, the non cosmological solutions (see e.g. [14]), it seems that there are difficulties in considering Palatini $f(R)$ gravity as a viable theory because the Cauchy problem is ill-posed and, furthermore, curvature singularities arise when dialing with simple stellar models; as for metric $f(R)$ gravity, there are models that are in agreement with the weak-field tests, but it seems that curvature singularities exist, in this case, for compact relativistic stars.

Without going into the details of this interesting debate, in this paper we want to test some models of $f(R)$ gravity that can be used to describe the gravitational field of the Sun. In particular, we are going to examine the impact that the modifications of the gravitational field of GR have on the Solar System dynamics. We apply a perturbative approach to compare the $f(R)$-induced secular precession of the longitude of the pericentre ϖ of a test particle with the latest determinations of the corrections to the usual perihelion precessions coming from fits of huge planetary data sets with various versions of the EPM ephemerides [37, 38, 39, 40, 41].

The paper is organized as follows: in Section 2 we briefly review the theoretical formalism of $f(R)$ gravity, both in the metric and Palatini approach, then, in Section 3 we outline a general approach to the perturbations of the gravitational field of GR, due to the non-linearity of the gravity Lagrangian. In Section 4 we compare the theoretical predictions with the observations. Finally, discussion and conclusions are in Section 5.

2 The field equations of $f(R)$ gravity

In this Section, we introduce the field equations of $f(R)$ gravity. We shall consider both the metric and the Palatini approach (see, e.g., [5] and [47]).

The equations of motion of $f(R)$ extended theories of gravity can be obtained by a variational principle, starting from the action:

$$A = A_{\mathrm{grav}} + A_{\mathrm{mat}} = \int [\sqrt{g} f(R) + 2\chi L_{\mathrm{mat}}(\psi, \nabla\psi)]\, d^4x. \tag{2.1}$$

The gravitational part of the Lagrangian is represented by a function $f(R)$ of the scalar curvature R. The total Lagrangian contains also a first order matter part L_{mat}, functionally depending on matter fields Ψ, together with their first derivatives, equipped with a gravitational coupling constant $\chi = \frac{8\pi G}{c^4}$. In the metric formalism, Γ is supposed to be the Levi-Civita connection of g and, consequently, the scalar curvature R has to be intended as $R \equiv R(g) = g^{\alpha\beta}R_{\alpha\beta}(g)$. On the contrary, in the Palatini formalism the metric g and the affine connection Γ are supposed to be independent, so that the scalar curvature R has to be intended as $R \equiv R(g,\Gamma) = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$, where $R_{\mu\nu}(\Gamma)$ is the Ricci-like tensor of the connection Γ.

In the metric formalism the action (2.1) is varied with respect to the metric g, and one obtains the following field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\quad)\, f'(R) = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.2}$$

where $f'(R) = df(R)/dR$, and $T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta L_{\mathrm{mat}}}{\delta g_{\mu\nu}}$ is the standard minimally coupled matter energy-momentum tensor. The contraction of the field equations (2.2) with the metric tensor leads to the scalar equation

$$3 \quad f'(R) + f'(R)R - 2f(R) = \frac{8\pi G}{c^4}T, \tag{2.3}$$

where T is the trace of the energy-momentum tensor. Eq. (2.3) is a differential equation for the scalar curvature R.

In the Palatini formalism, by independent variations with respect to the metric g and the connection Γ, we obtain the following equations of motion:

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.4}$$

$$\nabla^\Gamma_\alpha[\sqrt{g}f'(R)g^{\mu\nu}) = 0, \tag{2.5}$$

where ∇^Γ means covariant derivative with respect to the connection Γ. Actually, it is possible to show [12, 13] that the manifold M, which is the model of the space-time, can be a posteriori endowed with a bi-metric structure (M, g, h) equivalent to the original metric-affine structure (M, g, Γ), where Γ is assumed to be the Levi-Civita connection of h. The two metrics are conformally related by

$$h_{\mu\nu} = f'(R)\; g_{\mu\nu}. \tag{2.6}$$

The equation of motion (2.4) can be supplemented by the scalar-valued equation obtained by taking the contraction of (2.4) with the metric tensor:

$$f'(R)R - 2f(R) = \frac{8\pi G}{c^4}T. \tag{2.7}$$

Equation (2.7) is an algebraic equation for the scalar curvature R.

In order to compare the predictions of $f(R)$ gravity with Solar System data, we have to consider the solutions of the field equations (2.2),(2.4),(2.5) - supplemented by the constraints (2.3),(2.7) - in vacuum, since tests are based on the observations of the dynamics of the planets in the gravitational field of the Sun.

In particular, the vacuum field equations in the metric approach read

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\quad)\, f'(R) = 0, \tag{2.8}$$

supplemented with the scalar equation

$$3 \quad f'(R) + f'(R)R - 2f(R) = 0. \tag{2.9}$$

In the Palatini approach, the field equations become

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = 0, \tag{2.10}$$

$$\nabla^\Gamma_\alpha[\sqrt{g}f'(R)g^{\mu\nu}) = 0, \tag{2.11}$$

and they are supplemented by the scalar equation

$$f'(R)R - 2f(R) = 0 \tag{2.12}$$

We want to point out some general features of the scalar equations (2.9) and (2.12), which can help to understand the differences between the vacuum solutions in the two formalisms.

In Palatini $f(R)$ gravity, the trace equation (2.12) is an algebraic equation for R, which admits constant solutions $R = c_i$ [12], and it is identically satisfied if $f(R)$ is proportional to R^2. As a consequence, it is easy to verify that (if $f'(R) \neq 0$) the field equations become

$$R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu} \tag{2.13}$$

which are the same as GR field equations with a cosmological constant. In other words, in the Palatini formalism, in vacuum, we can have only solutions that describe space-times with constant scalar curvature R. Summarizing, eq. (2.13) suggests that all GR solutions with cosmological constant are solutions of vacuum Palatini field equations: the function $f(R)$ determines the solutions of algebraic equation (2.12).

In metric $f(R)$ gravity the trace equation (2.9) is a differential equation for R: this means that, in general, it admits more solutions than the corresponding Palatini equation. In particular, we notice that if $R =$ constant we obtain the Palatini case: so for a given $f(R)$ function, in vacuum, the solutions of the field equations of Palatini $f(R)$ gravity are a subset of the solutions of the field equations of metric $f(R)$ gravity [27]; however, in metric $f(R)$ gravity, vacuum solutions with variable R are allowed too (see, e.g., [30]).

3 Corrections to the gravitational potential

We have seen in the previous Section that, when $f(R) \neq R$, the field equations of $f(R)$ gravity are different from those of GR. Thus, it is evident that the solutions of such modified field equations describing the gravitational field of a point-like mass (e.g. the Sun) contain corrections to the GR solutions, both at Newtonian and post-Newtonian level. However, these corrections have to be small enough not to contradict the known tests of GR. Thus, it is possible to treat them perturbatively to evaluate their impact on the dynamics of the Solar System planets.

In this Section we want to outline the general procedure that we are going to apply to some solutions of $f(R)$ gravity that can be used to describe the gravitational field of the Sun, in order to compare the predictions of these gravity models with the existing data.

In general, we are going to deal with spherically symmetrical metrics, describing the space-time around a point-like mass M, which can be endowed with proper angular momentum $\boldsymbol{J}$. The weak-field and slow-motion approximations of these metrics will be sufficient for our purposes. Generally speaking, this means that the deviations from GR will be linear in some parameters deriving from the specific $f(R)$ gravity model.

On using spherical isotropic coordinates, these metric have the general form[3]

$$ds^2 = A(r)dt^2 + B(r)\left(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2\right) + 2C(r)sin^2\theta dt d\varphi, \tag{3.1}$$

where the angular momentum $\boldsymbol{J}$ is assumed to be perpendicular to the $\theta = \pi/2$ plane.

The gravitational (scalar) potential $\Phi(r)$ is read from the $A(r)$ function

$$A(r) = 1 + 2\Phi(r). \tag{3.2}$$

According to what stated before, we expect a gravitational potential in the form

$$\Phi(r) = \Phi^{\mathrm{N}}(r) + \Delta\Phi(r), \tag{3.3}$$

where $\Phi^{\mathrm{N}}(r) = -\frac{M}{r}$ is the Newtonian potential of a point-like mass M, and $\Delta\Phi(r) \ll \Phi_N(r)$ is a correction vanishing for $f(R) \to R$.

The $C(r)$ function accounts for the presence of the so-called gravito-magnetic effects [44, 28] induced by the rotation of the source of the gravitational field. In GR $C(r)$ is given by the suitable component of the gravito-magnetic vector potential of a gravito-magnetic dipole, i.e. $A_\varphi^{\mathrm{GR}}(r) = -\frac{2J}{r}$ (see e.g. [28]). As a consequence, we expect that the $C(r)$ function has the form

$$C(r) = A_\varphi^{\mathrm{GR}}(r) + \Delta A_\varphi(r) \tag{3.4}$$

[3]If not otherwise stated, here and henceforth we use units such that $G = c = 1$.

where, again, $\Delta A_\varphi(r) \ll A_\varphi(r)^{\rm GR}$ is a correction vanishing for $f(R) \to R$.

We can use the gravito-electromagnetic [44, 28] formalism to describe the total perturbing acceleration felt by a test particle in the metric (3.1)

$$\boldsymbol{W} = -\boldsymbol{\mathcal{E}}^{\rm G} - 2\boldsymbol{v} \quad \boldsymbol{\mathcal{B}}^{\rm G} \tag{3.5}$$

where

$$\boldsymbol{\mathcal{E}}^{\rm G} = -\frac{d\Delta\Phi(r)}{dr}\hat{\boldsymbol{r}} \tag{3.6}$$

and

$$\boldsymbol{\mathcal{B}}^{\rm G} = \boldsymbol{\nabla} \quad \boldsymbol{A}, \quad \boldsymbol{A} = \frac{\Delta A_\varphi}{r\sin\theta} \tag{3.7}$$

Hence, given the perturbing acceleration (3.5), we can calculate its effects on planetary motions within standard perturbative schemes (see, e.g., [29]). We may use the Gauss equations for the variations of the elements, which enable us to study the perturbations of the Keplerian orbital elements due to a generic perturbing acceleration, whatever its physical origin is. The Gauss equations for the variations of the semi-major axis a, the eccentricity e, the inclination i, the longitude of the ascending node Ω, the argument of pericentre ω and the mean anomaly $\mathcal{M}$ of a test particle in the gravitational field of a body M are [29]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{\overline{n}\sqrt{1-e^2}}\left[eW_r\sin v + W_\tau \quad \frac{p}{r} \quad \right], & (3.8)\\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{\overline{n}a} \quad W_r\sin v + W_\tau \quad \cos v + \frac{1}{e} \quad 1 - \frac{r}{a} \quad\quad , & (3.9)\\
\frac{di}{dt} &= \frac{1}{\overline{n}a\sqrt{1-e^2}}\, W_\nu \quad \frac{r}{a} \quad \cos(\omega+v), & (3.10)\\
\frac{d\Omega}{dt} &= \frac{1}{\overline{n}a\sin i\sqrt{1-e^2}}\, W_\nu \quad \frac{r}{a} \quad \sin(\omega+v), & (3.11)\\
\frac{d\omega}{dt} &= -\cos i\frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{\overline{n}ae} \quad -W_r\cos v + W_\tau \quad 1+\frac{r}{p} \quad \sin v \quad , & (3.12)\\
\frac{d\mathcal{M}}{dt} &= \overline{n} - \frac{2}{\overline{n}a}\, W_r \quad \frac{r}{a} \quad - \sqrt{1-e^2} \quad \frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt} \quad , & (3.13)
\end{aligned}$$

in which $\overline{n} = 2\pi/P$ is the mean motion[4], P is the test particle's orbital period, v is the true anomaly counted from the pericentre, $p = a(1-e^2)$ is the semilatus rectum of the Keplerian ellipse, W_r, W_τ, W_ν are the radial, transverse (in-plane components) and normal (out-of-plane component) projections of the perturbing acceleration $\boldsymbol{W}$, respectively, on the orthonormal frame $\{\hat{\boldsymbol{r}}, \hat{}, \hat{}\}$ comoving with the particle.

For our purposes it is useful to consider the longitude of the pericenter $\varpi = \omega + \cos i\ \Omega$. The Gauss equation for its variation under the action of an entirely radial perturbing acceleration W_r is

$$\frac{d\varpi}{dt} = -\frac{\sqrt{1-e^2}}{\overline{n}ae}W_r\cos v. \tag{3.14}$$

After being evaluated onto the unperturbed Keplerian ellipse, the acceleration (3.5) must be inserted into eq. (3.14); then, the average over one orbital period P must be performed. To this end it is useful also to recall the following relations where also the eccentric anomaly E is used

$$\begin{cases}
r = a(1-e\cos E), \\
dt = \frac{(1-e\cos E)}{\overline{n}}dE, \\
\cos v = \frac{\cos E - e}{1-e\cos E}, \\
\sin v = \frac{\sin E\sqrt{1-e^2}}{1-e\cos E}.
\end{cases} \tag{3.15}$$

[4]For an unperturbed Keplerian ellipse it is $\overline{n} = \sqrt{GM/a^3}$.

Table 1: Inner planets. First row: estimated perihelion extra-precessions, from Table 3 of [38]. The quoted errors are not the formal ones but are realistic. The units are arc-seconds per century ($''$ cy^{-1}). Second row: semi-major axes, in Astronomical Units (AU). Their formal errors are in Table IV of [37], in m. Third row: eccentricities. Fourth row: orbital periods in years.

	Mercury	Earth	Mars
$\langle\dot{\varpi}\rangle$ ($''$ cy^{-1})	-0.0036 ± 0.0050	-0.0002 ± 0.0004	0.0001 ± 0.0005
a (AU)	0.387	1.000	1.523
e	0.2056	0.0167	0.0934
P (yr)	0.24	1.00	1.88

Table 2: Outer planets. First row: estimated perihelion extra-precessions [39]. The quoted uncertainties are the formal, statistical errors re-scaled by a factor 10 in order to get the realistic ones. The units are arc-seconds per century ($''$ cy^{-1}). Second row: semi-major axes, in Astronomical Units (AU). Their formal errors are in Table IV of [37], in m. Third row: eccentricities. Fourth row: orbital periods in years.

	Jupiter	Saturn	Uranus
$\langle\dot{\varpi}\rangle$ ($''$ cy^{-1})	0.0062 ± 0.036	-0.92 ± 2.9	0.57 ± 13.0
a (AU)	5.203	9.537	19.191
e	0.0483	0.0541	0.0471
P (yr)	11.86	29.45	84.07

In fact, what we aim at is evaluating the perturbations induced on the longitudes of the perihelia by the corrections to the gravitational field due to $f(R)$ gravity, in order to compare them with the latest observational determinations. The astronomer E.V. Pitjeva (Institute of Applied Astronomy, Russian Academy of Sciences, St. Petersburg) processed almost one century of data of different types for the major bodies of the Solar System to improve the EPM planetary ephemerides [37, 40, 41]. Among other things, she simultaneously estimated corrections to the secular rates of the longitudes of perihelia ϖ of the inner [38] and of some of the outer [40, 41] planets of the Solar System as fit-for parameters of global solutions in which she contrasted, in a least-square way, the observations to their predicted values computed with a complete set of dynamical force models including all the known Newtonian (solar quadrupole mass moment J_2, $N-$body interactions with the major planets, 301 biggest asteroids, massive ring of the small asteroids, 20 largest trans-Neptunian objects and massive ring for the other ones) and Einsteinian[5] features of motion. As a consequence, any force that is not present in Newtonian gravity or GR is, in principle, accounted for by the estimated corrections to the usual apsidal precessions. For the sake of completeness, we reproduce in tables 1 and 2 the estimated perihelia extra-precessions for inner and outer planets, respectively.

What we want to do is to see whether the estimated perihelia extra-precessions are compatible with the perturbations of the gravitational field deriving from the non-linearity of $f(R)$.

Now, let us briefly outline how we are going to put $f(R)$ gravity on the test. In general a correction to the gravitational field of GR due to the non linearity of the gravity Lagrangian, in the weak-field and slow motion approximation, can be parameterized in terms of a parameter κ, where $\kappa \to 0$ as far as $f(R) \to R$. In other words, κ is a measure of the non-linearity of the Lagrangian. Let $\mathcal{P}(f(R))$ be the prediction of a certain effect induced by these modified gravity models, e.g. the secular precession of the perihelion of a planet: for all the $f(R)$ models that we are going to consider below, it turns out that

$$\mathcal{P}(f(R)) = \kappa g(a,e), \tag{3.16}$$

where g is a function of the system's orbital parameters a (semi-major axis) and e (eccentricity); such g is a peculiar consequence of the $f(R)$ gravity model. Now, let us take the ratio of $\mathcal{P}(f(R))$ for two different systems A and B, e.g. two Solar System's planets: $\mathcal{P}_{\mathrm{A}}(f(R))/\mathcal{P}_{\mathrm{B}}(f(R)) = g_{\mathrm{A}}/g_{\mathrm{B}}$. The

[5]The general relativistic gravito-magnetic Lense-Thirring force has not yet been modeled.

model's parameter κ has now been canceled, but we still have a prediction that retains a peculiar signature of that model, i.e. $g_{\rm A}/g_{\rm B}$. Of course, such a prediction is valid if we assume κ is not zero, which is just the case both theoretically (only if $f(R) = R$ then $\kappa = 0$) and observationally because κ is usually determined by other independent long-range astrophysical/cosmological observations. Otherwise, one would have the meaningless prediction 0/0. The case $\kappa = 0$ (or $\kappa \leq \overline{\kappa}$, i.e. when κ is negligibly small) can be, instead, usually tested by taking one perihelion precession at a time. If we have observational determinations $\mathcal{O}$ for A and B of the effect considered above such that they are affected also[6] by the $f(R)$ gravity model (it is just the case for the purely phenomenologically estimated corrections to the standard Newton-Einstein perihelion precessions, since any $f(R)$ gravity model has not been included in the dynamical force models of the ephemerides adjusted to the planetary data in the least-square parameters' estimation process by Pitjeva [37, 38]), we can construct $\mathcal{O}_{\rm A}/\mathcal{O}_{\rm B}$ and compare it with the prediction for it by $f(R)$, i.e. with $g_{\rm A}/g_{\rm B}$. Note that $\delta\mathcal{O}/\mathcal{O} > 1$ only means that $\mathcal{O}$ is compatible with zero, being possible a nonzero value smaller than $\delta\mathcal{O}$. Thus, it is perfectly meaningful to construct $\mathcal{O}_{\rm A}/\mathcal{O}_{\rm B}$. Its uncertainty will be conservatively evaluated as $|1/\mathcal{O}_{\rm B}|\delta\mathcal{O}_{\rm A} + |\mathcal{O}_{\rm A}/\mathcal{O}_{\rm B}^2|\delta\mathcal{O}_{\rm B}$. As a result, $\mathcal{O}_{\rm A}/\mathcal{O}_{\rm B}$ will be compatible with zero. Now, the question is: Is it the same for $g_{\rm A}/g_{\rm B}$ as well? If yes, i.e. if

$$\frac{\mathcal{O}_{\rm A}}{\mathcal{O}_{\rm B}} = \frac{\mathcal{P}_{\rm A}(f(R))}{\mathcal{P}_{\rm B}(f(R))} \tag{3.17}$$

within the errors, or, equivalently, if

$$\frac{\mathcal{O}_{\rm A}}{\mathcal{O}_{\rm B}} - \frac{\mathcal{P}_{\rm A}(f(R))}{\mathcal{P}_{\rm B}(f(R))} = 0 \tag{3.18}$$

within the errors, the $f(R)$ gravity model examined can still be considered compatible with the data, otherwise it is seriously challenged.

In next Section some solutions of $f(R)$ gravity that can be used to describe the gravitational field of the Sun will be tested according to the procedure that we have just described.

4 $f(R)$ weak-field solutions

In this Section we introduce some solutions of $f(R)$ gravity that have been used in the literature to the describe the weak gravitational field, and that can be considered as suitable models of the gravitational field of the Sun. We consider these modified gravitational fields and, within the perturbative scheme outlined above, compare the theoretical predictions with the estimated extra-precessions of the planetary perihelia.

4.1 Power law corrections

Starting from a Lagrangian of the form $f(R) = f_0 R^n$, Capozziello and collaborators [6], in the metric approach, look for solutions describing the gravitational field of a point-like source, in order to reproduce the galaxies rotation curves without need for dark matter. As a result, they obtain the following power-law form for the gravitational potential:

$$\Phi(r) = -\frac{M}{r}\left[1 + \quad \frac{r}{r_c} \quad ^{\beta}\right]. \tag{4.1}$$

The deviation from the Newtonian potential is parameterized by a power law, with two free parameters β and r_c. In particular, β is related to n, i.e. the exponential of the Ricci scalar in $f(R) = f_0 R^n$.

In this case, the correction to the gravitational potential is

$$\Delta\Phi(r) = -\frac{M}{r} \quad \frac{r}{r_c} \quad ^{\beta}, \tag{4.2}$$

[6]If they are differential quantities constructed by contrasting observations to predictions obtained by analytical force models of canonical Newtonian/Einsteinian effects, $\mathcal{O}$ are, in principle, affected also by the mis-modeling in them.

which clearly leads to the radial acceleration

$$W_r = \frac{(\beta - 1)M}{r_c^\beta} r^{\beta-2} \tag{4.3}$$

It yields the following perihelion precession [21]

$$\langle \dot{\varpi} \rangle = \frac{(\beta - 1)\sqrt{M}}{2\pi r_c^\beta} a^{\beta - \frac{3}{2}} G(e;\beta), \tag{4.4}$$

with $G(e_{\mathrm{A}};\beta)/G(e_{\mathrm{B}};\beta) \approx 1$ for all the planets of the Solar System.

Capozziello and collaborators [6] find $\beta = 0.817$ from a successful fit of several galactic rotation curves with no dark matter; it is ruled out by comparing for several pairs of planets $\Delta\dot{\varpi}_{\mathrm{A}}/\Delta\dot{\varpi}_{\mathrm{B}}$ to $\mathcal{P}_{\mathrm{A}}/\mathcal{P}_{\mathrm{B}}$ obtained from eq. (4.4) [21].

4.2 Schwarzschild-de Sitter-like corrections

The field equations (2.4-2.5) and the structural equation (2.7), in the Palatini formalism have the spherically symmetrical solution (see [1] and [45]):

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{k}{3}r^2\right) dt^2 - \left(1 + \frac{2M}{r} - \frac{k}{6}r^2\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \tag{4.5}$$

$k = c_i/4$, where $R = c_i$ is any of the solutions of the structural equation (2.7). In this case, we may write

$$\Phi(r) = -\frac{M}{r} + \frac{kr^2}{6}. \tag{4.6}$$

In this case, the perturbing potential is

$$\Delta\Phi(r) = \frac{kr^2}{6}, \tag{4.7}$$

and the induced the perturbing acceleration is

$$W_r = -\frac{1}{3}kr. \tag{4.8}$$

As any other Hooke-type extra-acceleration, eq. (4.8) induces a secular perihelion precession [23, 19]

$$\langle \dot{\varpi} \rangle \propto \frac{k}{n} = k\sqrt{\frac{a^3}{M}}. \tag{4.9}$$

By using eq. (4.9) to construct $\mathcal{P}_{\mathrm{A}}/\mathcal{P}_{\mathrm{B}}$ for different pairs of Solar System's planets and comparing them to $\Delta\dot{\varpi}_{\mathrm{A}}/\Delta\dot{\varpi}_{\mathrm{B}}$ yield a negative answer [19].

4.3 Logarithmic corrections

Sobouti [46] aims at determining a $f(R)$ able to explain the rotation curves of the galaxies obtained. In particular, working in the metric approach, solutions with R variable with the radial coordinate r are obtained. In this context, the gravitational potential reads:

$$\Phi(R) = -\frac{M}{r} + \frac{\alpha}{2} + \frac{\alpha}{2}\ln(r/2M). \tag{4.10}$$

The parameter α can be related to Modified Newtonian Dynamics (MOND, see e.g. [29]) characteristic acceleration A_0.

The Logarithmic correction to the Newtonian gravitational potential assumes the form

$$\Delta\Phi(r) = -\gamma M \ln\left(\frac{r}{r_0}\right), \tag{4.11}$$

and leads to a perturbing radial acceleration

$$W_r = \frac{\gamma M}{r}\,. \tag{4.12}$$

In particular, in order to agree with the potential (4.10), we must set $\gamma = -\alpha/2$, $r_0 = 2M$.

This kind of acceleration has been treated in [21] with the approach outlined here getting negative answers.

4.4 Yukawa-like corrections

In different works, both in the Palatini [16, 2] and metric approach (e.g. see [34, 48, 7]) Yukawa-like corrections are obtained. They lead to a gravitational potential in the form

$$\Phi(r) = -\frac{M}{r}\left[1 + \alpha \exp \quad -\frac{r}{\lambda} \quad \right] \tag{4.13}$$

The parameter α is related to the strength of the correction, while λ is related to the range of the modified potential.

The Yukawa correction to the Newtonian potential

$$\Delta\Phi(r) = -\frac{M\alpha}{r} \exp \quad -\frac{r}{\lambda} \tag{4.14}$$

yields an entirely radial extra-acceleration

$$W_r = -\frac{M\alpha}{r^2} \quad 1 + \frac{r}{\lambda} \quad \exp \quad -\frac{r}{\lambda} \tag{4.15}$$

By only assuming $\lambda \gg ae$, i.e. Yukawa-type long-range modifications of gravity, it is possible to obtain useful approximated expressions for the induced perihelion precession which, in turn, allow to obtain [17]

$$\lambda = \frac{a_{\mathrm{B}} - a_{\mathrm{A}}}{\ln \sqrt{\frac{a_{\mathrm{B}}}{a_{\mathrm{A}}} \frac{\Delta\dot{\varpi}_{\mathrm{A}}}{\Delta\dot{\varpi}_{\mathrm{B}}}}} \tag{4.16}$$

for the range and

$$\alpha = \frac{2\lambda^2 \Delta\dot{\varpi}}{\sqrt{Ma}} \exp \quad \frac{a}{\lambda} \quad . \tag{4.17}$$

for the strength. By using A = Earth, B = Mercury in eq. (4.16) one gets $\lambda = 0.182 \pm 0.183$ AU; such a value for λ, the data of Venus and eq. (4.17) yield $\alpha = (-1 \pm 4) \times 10^{-11}$ [20].

4.5 Gravito-magnetic effects

We have shown that the vacuum solutions of General Relativity with a cosmological constant can be used in Palatini $f(R)$ gravity. In particular, the Kerr-de Sitter solution, which describes a rotating black-hole in a space-time with a cosmological constant [10, 8, 23, 24, 25, 26], can be used to investigate Gravito-magnetic effects in extended theories of gravity.

In particular, the weak-field and slow-motion approximation of the Kerr-de Sitter is [22]

$$\begin{aligned} ds^2 = & \quad 1 - \frac{2M}{r} - \frac{k}{3}r^2 \quad dt^2 - \quad 1 + \frac{2M}{r} - \frac{k}{6}r^2 \quad dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad + \\ & + 2\frac{J}{M} \quad \frac{2M}{r} + \frac{k}{3}r^2 + \frac{5}{6}Mkr \quad \sin^2\theta d\phi dt. \end{aligned} \tag{4.18}$$

We obtain the following expression for the perturbing gravito-magnetic potential

$$\Delta A_{\varphi} = \frac{J}{M} \quad \frac{k}{3}r^2 + \frac{5}{6}Mkr \quad \sin^2\theta. \tag{4.19}$$

Furthermore, the perturbing acceleration is

$$\boldsymbol{W} = -2\boldsymbol{v} \quad \boldsymbol{\mathcal{B}}^{\mathrm{G}}, \tag{4.20}$$

where the gravito-magnetic field $\boldsymbol{\mathcal{B}}^{\mathrm{G}}$ is

$$\boldsymbol{\mathcal{B}}^{\mathrm{G}} = \frac{Jk}{3M}\hat{\boldsymbol{J}} + \frac{5Jk}{12}\frac{\left[\hat{\boldsymbol{J}} + \quad \hat{\boldsymbol{J}} \quad \hat{\boldsymbol{r}} \quad \hat{\boldsymbol{r}}\right]}{r}. \tag{4.21}$$

The resulting orbital effects are [22]

$$\langle \dot{a} \rangle = 0, \tag{4.22}$$

$$\langle \dot{e} \rangle = 0, \tag{4.23}$$

$$\dot{i} = 0, \tag{4.24}$$

$$\left\langle \dot{\Omega} \right\rangle = \frac{Jk}{3M} \quad 1 + \frac{5M}{2a} \quad , \tag{4.25}$$

$$\langle \dot{\omega} \rangle = -\frac{2Jk\cos i}{3M} \quad 1 + \frac{5M}{4a} \quad , \tag{4.26}$$

$$\left\langle \dot{\mathcal{M}} \right\rangle = \overline{n} + \frac{5Jk\cos i}{3M} \quad 1 + \frac{M}{a} \quad . \tag{4.27}$$

In the calculation we have neglected terms of order $\mathcal{O}(e^2)$.

By using the corrections $\Delta\dot{\varpi}$ separately for each Solar System's planet one gets $k \leq 10^{-29}$ m^{-2}. For all the Solar System's planets the perihelion rate can be satisfactorily approximated by

$$\langle \dot{\omega} \rangle \approx -\frac{2Jk\cos i}{3M}. \tag{4.28}$$

Since $\cos i_{\rm A}/\cos i_{\rm B} \approx 1$ for every pair of planets A and B, in this case $\mathcal{P}_{\rm A}/\mathcal{P}_{\rm B} \approx 1$; this possibility is ruled out by $\mathcal{O}_{\rm A}/\mathcal{O}_{\rm B} = \Delta\dot{\varpi}_{\rm A}/\Delta\dot{\varpi}_{\rm B}$, as in the case of the DGP [8] braneworld scenario [18].

5 Discussion and Conclusions

In this paper we have considered some solutions of $f(R)$ gravity, both in the Palatini and metric formalism, that can be used to describe the weak gravitational field around the Sun. In particular, we have focused on the impact that the modifications of the GR gravitational field, due to the non-linearity of $f(R)$, have on the Solar System dynamics. We have considered that these modifications have to be small in order not to contradict the known tests of GR and, as a consequence, we have treated them as perturbations. Thus, we have applied a perturbative approach to compare the $f(R)$-induced secular effects with the latest observationally determinations coming from various versions of the EPM planetary ephemerides. In particular, we have considered the ratios of the corrections to the standard secular precessions of the longitudes of perihelia estimated by E.V. Pitjeva for several pairs of planets in the Solar System. For all the models that we have considered (power law, Hooke-like force, logarithmic corrections, Yukawa-like force, gravito-magnetic effects) our results show that the perturbations deriving from the non-linearity of $f(R)$ are not compatible with the currently available apsidal extra-precessions of the Solar System planets. Moreover, the hypothesis that the examined $f(R)$-induced perturbations are zero, which cannot be tested by definition with our approach, is compatible with each perihelion extra-rate separately.

This might suggest that, on the one hand, the $f(R)$-induced secular effects cannot explain the observed extra-precessions and that, on the other hand, the effects of the non-linearity of the gravity Lagrangian are important on length scales much larger than the Solar System (e.g. on the cosmological scale) and their effects on local physics are probably negligible. It will be important to repeat such tests if and when other teams of astronomers will independently estimate their own corrections to the standard secular precessions of the perihelia.

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Scalar-Tensor theories and current Cosmology

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Abstract

Scalar-tensor theories are studied in the context of cosmological evolution, where the expansion history of the Universe is reconstructed. It is considered quintessence/phantom models, where inflation and cosmic acceleration are reproduced. Also, the non-minimally coupling regime between the scalar field and the Ricci scalar is studied and cosmological solutions are obtained. The Chamaleon mechanism is shown as a solution of the local gravity tests problems presented in this kind of theories.

1 Introduction

Since the supernovae observations were analyzed in 1998 (see [1]), the majority of the scientific community has accepted that the Universe is in an accelerated expansion phase. As it was unknown the possible mechanism to make this kind of repulsion gravity, this was called dark energy (for a review of possible candidates see [2]), which it is supposed to have an equation of state parameter (EoS) less than $-1/3$, and an energy density close to the critical density $\rho_{DE} \sim 10^{-3}eV$. The main candidate to dark energy has been the cosmological constant, which may represent the vacuum energy but there is unknown explanation as to why the energy density is of the order of the critical density, much smaller than vacuum energy density predicted by quantum field theory (for a review on the comological constant problem see [3]). Then, some others candidates have been proposed, one of them is the quintessence/phantom scalar field models ([4],[5],[6], [7] and [8]), where a scalar field minimally coupled is included, and the accelerated expansion is reproduces by this single field, whose EoS parameter is around -1. Phantom models where the EoS parameter is less than -1, have not been excluding by observational data, and together with quintessence scalar field models, it constitutes a good and simple candidate to dark energy. Currently, the main purpose of scalar-tensor theories is not just to explain the cosmic acceleration but to reproduce all the expansion history from the early accelerated expansion called inflation to the cosmic acceleration. Also, scalar-tensor theories with scalar fields non minimal coupled to the Ricci scalar are currently considered, which may present problems with local gravity tests, although these are avoided by using the so-called chamaleon mechanism (see [9]), such that this kind of Brans-Dicke theory are not excluded.
At the present article, we review some solutions of quintessence/phantom models, where some examples are given and the evolution of the Universe is reproduced. The possible future singularities in the phantom epochs are studied. Also Brans-Dicke-like theory is considered where cosmic solutions are reconstructed, and the chamaleon mechanism is shown in such a way that the local gravity constraints may be avoided by this kind of theories.

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2 Quintessence and phantom scalar-tensor cosmology

The Quintessence/phantom scalar field models are well studied (for a review see [2]), they are presented as an explanation of the current cosmological acceleration. Also the majority of inflation models are constructed by a single scalar field called inflaton. Recently, the main task in scalar-tensor theory is the possibility to unify both inflation and late-time acceleration by using a single scalar-field (see for example [6], [7], [8]), this real possibility is shown above by several examples. Let us start consider the action that define these kind of models, we consider a Universe filled with some kind of matter whose equation of state (EoS) is given by $p_m = w_m\rho_m$ (here w_m is a constant) and a scalar field minimally coupled to gravity, the action is given by:

$$S = \int dx^4\sqrt{-g}\left[\frac{1}{2\kappa^2}R \pm \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + L_m\right] , \tag{2.1}$$

here the sing ($\pm$) in front of the kinetic term define the nature of the scalar field, where the phantom-like field is given by (+) and the quintessence-like field by (-). The lagrangian L_m is the matter lagrangian density. The field equations are obtained by varying the action (2.1) with respect $g^{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{\kappa^2}{2}\left[2T^{(m)}_{\mu\nu} + T^{(\varphi)}_{\mu\nu}\right] , \tag{2.2}$$

where $T^{(m)\mu}_{\nu} = \left(\begin{array}{cccc} -\rho_m & p_m & p_m & p_m \end{array}\right)$ and $T^{(\varphi)\mu}_{\nu} = \partial^\mu\varphi\partial_\nu\varphi - g^\mu_\nu\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + V(\varphi)\right]$. We assume a flat FRW spacetime, the metric is given by:

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i^2 . \tag{2.3}$$

Then, the corresponding Friedmann equations are written as:

$$H^2 = \frac{\kappa^2}{3}\left(\rho_m + \rho_\varphi\right) , \qquad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_m + p_m + \rho_\varphi + p_\varphi\right) , \tag{2.4}$$

where ρ_ϕ and p_ϕ are given by:

$$\rho_\varphi = \mp\frac{1}{2}\dot{\varphi}^2 + V(\varphi) , \qquad p_\varphi = \mp\frac{1}{2}\dot{\varphi}^2 - V(\varphi) , \tag{2.5}$$

here the phantom case is given by the case of negative value of the kinetic term, just as it is the definition for a phantom field, where the weak energy condition ($\rho + p \geq 0$) is violated. In other hand, by varing the action (2.1) with respect ϕ, the scalar field equation is obtained:

$$\ddot{\varphi} + 3H\dot{\varphi} \mp V'(\varphi) = 0 , \tag{2.6}$$

here the prime on the potential term means a derivative respect ϕ. Then, by the equations (2.4) and (2.6) the solution for the scale parameter may be obtained. In addtion, the continuity equation for the matter term may be useful to find out the solution:

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0 . \tag{2.7}$$

Then, the matter component, which is suposed to have an EoS given by $p_m = w_m\rho_m$, behaves as $\rho_m \propto a^{-3(1+w_m)t}$. A simple way to resolve the above equations may be done by redefining the scalar field as:

$$\varphi = \int^\phi d\phi\sqrt{\pm\omega(\phi)} , \tag{2.8}$$

where the sign depends on the sign of $\omega(\phi)$, i.e. on the phantom behaviour of the scalar field. Hence, the action (2.1) takes the form:

$$S = \int dx^4\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_m\right] , \tag{2.9}$$

and the equations (2.5) are written as:

$$\rho_\phi = \frac{1}{2}\omega(\phi)\,\dot{\phi}^2 + V(\phi)\ , \qquad p_\phi = \frac{1}{2}\omega(\phi)\,\dot{\phi}^2 - V(\phi)\ . \tag{2.10}$$

Then, combining the Friedmann equations (2.4) with (2.10), one obtains:

$$\omega(\phi)\,\dot{\phi}^2 = -\frac{2}{\kappa^2}\dot{H} - (\rho_m + p_m)\ , \qquad V(\phi) = \frac{1}{\kappa^2}\left(3H^2 + \dot{H}\right) - \frac{\rho_m - p_m}{2}\ . \tag{2.11}$$

Now we can use the continuity equation for the matter contribution (2.7), and $V(\phi)$ and $\omega(\phi)$ can be expressed as:

$$\omega(\phi) = -\frac{2}{\kappa^2}f'(\phi) - (w_m+1)F_0 \mathrm{e}^{-3(1+w_m)F(\phi)}\ , \quad V(\phi) = \frac{1}{\kappa^2}\left(3f(\phi)^2 + f'(\phi)\right) + \frac{w_m - 1}{2}F_0\,\mathrm{e}^{-3(1+w_m)F(\phi)}\ , \tag{2.12}$$

where $f(\phi) \equiv F'(\phi)$, F is an arbitrary differentiable function of ϕ, and F_0 is an integration constant. Then the following solution is found (see [6]):

$$\phi = t\ , \qquad H(t) = f(t)\ , \tag{2.13}$$

which leads to

$$a(t) = a_0 \mathrm{e}^{F(t)}, \qquad a_0 = \left(\frac{\rho_{m0}}{F_0}\right)^{\frac{1}{3(1+w_m)}}\ . \tag{2.14}$$

Then, by the formulation shown above, one may construct models of the Universe in the mark of scalar-tensor theories where the cosmic acceleration and also inflation are well reproduced. A useful parameter to study the evolution of these models is defined by:

$$w_{eff} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2}\ , \tag{2.15}$$

where,

$$\rho = \rho_m + \rho_\phi\ , \qquad p = p_m + p_\phi \tag{2.16}$$

In order to study the accelerated epochs of the Universe, it is convenient to write also the expression for the acceleration of the scale parameter:

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = -\frac{\kappa^2}{6}(\rho + 3p)\ . \tag{2.17}$$

Then, the Universe will be in an accelerated phase when the effective fluid is given by $p < -1/3\rho$. Some examples are presented below, where the above formulation is used to reconstruct the evolution of the Universe. Note that in spite of the simplicity of the formulation derived, in general the expressions for the scalar potential and the kinetic term will be very complex, even for simple solutions of the Hubble parameter, as it will be seen in the following examples.

2.1 Inflation

As a first example, we show a simple model of inflation. The majority of inflation models are given by a single scalar field, where the lagrangian for the scalar field is $L = -(1/2)(\partial\phi)^2 - V(\phi)$. Some conditions on the kinetic term and on the scalar potential are impossed in order that inflation could take place (see [10]). These conditions called slow-roll conditions establishes that $\ddot{\phi} << 3H\dot{\phi}$ and $\dot{\phi}^2 << V(\phi)$, and they may be cast in a more usseful forms, the so called slow-roll parameters, which are given by:

$$\varepsilon = \frac{1}{3\kappa^2}\left(\frac{V'}{V}\right)^2 \ll 1\ , \quad \eta = \frac{1}{3\kappa^2}\frac{V''}{V} \ll 1\ . \tag{2.18}$$

Then, the Friedmann Equations and the equation of motion for the scalar field (2.6) are written as:

$$H^2 \approx \frac{\kappa^2}{3}V(\phi)\ , \qquad 3H\dot{\phi} + V'(\phi) \approx 0\ , \tag{2.19}$$

hence, by an specific potential, the solution for the equations (2.19) is found. The inflation period is characterized by an accelerated expansion in order to explain the horizon problem or the flatness problem (see [10]), then we may choose a potential that holds the conditions (2.18) and produces an accelerated expansion. As a simple and very well known inflation model, we choose the following scalar potential:

$$V(\phi) = V_0 \exp \sqrt{\frac{2\kappa^2}{\alpha}}\phi \ , \tag{2.20}$$

which gives a solution of the type $a(t) \sim t^{\alpha}$, where for $\alpha > 1$ the accelerated expansion takes place. Then, by this very simple example, it is shown how an scalar field reproduces the inflationary period of the Universe. In the following examples, some models are reconstructed where the cosmic acceleration takes place, and even some models where a unified scenario of the inflation and cosmic acceleration is presented.

2.2 Cosmic Acceleration

Let us now analize an example of how an scalar field may reproduce the behaviour of the dark energy in a natural way. By the action given in (2.9), the reconstruction of the kinetic and potential term for the scalar field given in equations (2.12) and (2.13) is used. In this case, we consider the function $f(\phi)$ studied in Ref.[7], and which is given by:

$$f(\phi) = H_0 + \frac{H_1}{\phi} \ , \tag{2.21}$$

where H_0 and H_1 are positive constants. Then, by equation (2.12), the expressions for the kinetic term and the scalar potential are given by:

$$\begin{aligned} \omega(\phi) &= \frac{1}{\kappa^2}\frac{2H_1}{\phi^2} - (w_m+1)F_0\phi^{-3H_1(1+w_m)}\mathrm{e}^{-3H_0(1+w_m)\phi} \ , \\ V(\phi) &= \frac{1}{\kappa^2}\ \frac{H_1(3H_1-1)}{\phi^2} + \frac{2H_0H_1}{\phi} + H_0 \ + \frac{w_m-1}{2}F_0\phi^{-3H_1(1+w_m)}\mathrm{e}^{-3H_0(1+w_m)\phi} \ . \end{aligned} \tag{2.22}$$

Then, by (2.13) the solution for the Hubble parameter is found:

$$H(t) = H_0 + \frac{H_1}{t} \ , \qquad a(t) = a_0 t^{H_1}\mathrm{e}^{H_0 t} \ . \tag{2.23}$$

Note that the solution (2.23) behaves as an effective cosmological constant at late times. For small times and a choice given by $w_m = 0$ (dust matter) and $H_1 = 2/3$, the Universe is dominated by matter at early times. This may be seen clearer by studying the effective parameter (2.15):

$$w_{eff} = -1 + \frac{2H_1}{H_1^2 + H_0^2\phi^2 + 2H_0H_1\phi} \ . \tag{2.24}$$

Then, the scalar field characterized by the kinetic and potential terms (2.22) reproduces a Universe where the dust matter dominates at the begining when $t \to 0$ and the effective parameter of the EoS(2.24) $w_{eff} \to 0$. At late times, the scalar field begins to dominate, and to produce an accelerated expansion $w_{eff} \to -1$ ($t \to \infty$), similar to an effective comological constant. Note that in this case, the energy density of the dark energy and its EoS parameter, is not constant and change with time. This fact, comoon in all quintessence/phantom models, opens the possibility to compare them with the cosmological constant model, and then to establish the evolution of the EoS by the observational data to constraint the models.

2.3 Inflation and late-time acceleration unified

Let us now consider some models where the inflation and late-time acceleration period are reproduced by the same scalar field. In this case, it is considered an scalar field that passes through quintessence and phantom epochs, and the possible future singularities (for a classification of future singularities, see [11]) driven by the phantom behaviour are studied. By the reconstruction (2.12) and (2.13), we consider as a first example the choice (see [8]):

$$f(\phi) = \frac{H_0}{t_s - \phi} + \frac{H_1}{\phi^2} \ . \tag{2.25}$$

We take H_0 and H_1 to be constants and t_s as the Rip time, as specified below. Using (2.12), we find that the kinetic function and the scalar potential are

$$\begin{aligned}\omega(\phi) &= -\frac{2}{\kappa^2}\left(\frac{H_0}{(t_s-\phi)^2}-\frac{2H_1}{\phi^2}\right)-(w_m+1)F_0\,(t_s-\phi)^{3(1+w_m)H_0}\exp\left(\frac{3(1+w_m)H_1}{\phi}\right),\\ V(\phi) &= \frac{1}{\kappa^2}\left(\frac{H_0(3H_0+1)}{(t_s-\phi)^2}+\frac{H_1}{\phi^3}\left(\frac{H_1}{\phi}-2\right)\right)+\frac{w_m-1}{2}F_0\,(t_s-\phi)^{3(1+w_m)H_0}\,\mathrm{e}^{\frac{3(1+w_m)H_1}{\phi}}\end{aligned}\tag{2.26}$$

respectively. Then, through the solution (2.13), we obtain the Hubble parameter and the scale factor

$$H(t)=\frac{H_0}{t_s-t}+\frac{H_1}{t^2}\,,\quad a(t)=a_0\,(t_s-t)^{-H_0}\,\mathrm{e}^{-\frac{H_1}{t}}\,.\tag{2.27}$$

Since $a(t)\to 0^+$ for $t\to 0$, we can fix $t=0$ as the beginning of the universe. On the other hand, at $t=t_s$ the universe reaches a Big Rip singularity, thus we keep $t<t_s$. In order to study the different stages that our model will pass through, we calculate the acceleration parameter and the first derivative of the Hubble parameter. They are

$$\dot H=\frac{H_0}{(t_s-t)^2}-\frac{2H_1}{t^3}\,,\quad \frac{\ddot a}{a}=H^2+\dot H=\frac{H_0}{(t_s-t)^2}(H_0+1)+\frac{H_1}{t^2}\left(\frac{H_1}{t^2}-\frac{2H_1}{t}+\frac{2H_0}{t_s-t}\right).\tag{2.28}$$

As we can observe, for t close to zero, $\ddot a/a>0$, so that the universe is accelerated during some time. Although this is not a phantom epoch, since $\dot H<0$, such stage can be interpreted as corresponding to the beginning of inflation. For $t>1/2$ but $t\ll t_s$, the universe is in a decelerated epoch ($\ddot a/a<0$). Finally, for t close to t_s, it turns out that $\dot H>0$, and then the universe is superaccelerated, such acceleration being of phantom nature and ending in a Big Rip singularity at $t=t_s$, where the scale parameter $a(t)\to\infty$, and the Ricci scalar $R=6(2H^2+\dot H)\to\infty$.

Our second example also exhibits unified inflation and late time acceleration, but in this case we avoid phantom phases and, therefore, Big Rip singularities. We consider the following model given in Ref.[8]:

$$f(\phi)=H_0+\frac{H_1}{\phi^n}\,,\tag{2.29}$$

where H_0 and $H_1>0$ are constants and n is a positive integer (also constant). The case $n=1$ yields an initially decelerated universe and a late time acceleration phase. We concentrate on cases corresponding to $n>1$ which gives, in general, three epochs: one of early acceleration (interpreted as inflation), a second decelerated phase and, finally, accelerated expansion at late times. In this model, the scalar potential and the kinetic parameter are given, upon use of Eqs. (2.12) and (2.29), by

$$\omega(\phi) = \frac{2}{\kappa^2}\frac{nH_1}{\phi^{n+1}}-(w_m+1)F_0\,\mathrm{e}^{-3(w_m+1)\left(H_0\phi-\frac{H_1}{(n-1)\phi^{n-1}}\right)}\,,\tag{2.30}$$

$$V(\phi) = \frac{1}{\kappa^2}\frac{3}{\phi^{n+1}}\left[\left(\frac{H_0\phi^{n/2}+H_1}{\phi^{n-1}}\right)^2-\frac{nH_1}{3}\right]+\frac{w_m-1}{2}F_0\,\mathrm{e}^{-3(w_m+1)\left(H_0\phi-\frac{H_1}{(n-1)\phi^{n-1}}\right)}\tag{2.31}$$

Then, the Hubble parameter given by the solution (2.13) can be written as

$$H(t)=H_0+\frac{H_1}{t^n}\,,\quad a(t)=a_0\exp\left(H_0t-\frac{H_1}{(n-1)t^{n-1}}\right).\tag{2.32}$$

We can fix $t=0$ as the beginning of the universe because at this point $a\to 0$, so $t>0$. The effective EoS parameter (2.15) is

$$w_{\text{eff}}=-1+\frac{2nH_1t^{n-1}}{(H_0t^n+H_1)^2}\,.\tag{2.33}$$

Thus, when $t\to 0$ then $w_{\text{eff}}\to -1$ and we have an acceleration epoch, while for $t\to\infty$, $w_{\text{eff}}\to -1$ which can be interpreted as late time acceleration. To find the phases of acceleration and deceleration for $t>0$, we study $\ddot a/a$, given by:

$$\frac{\ddot a}{a}=\dot H+H^2=-\frac{nH_1}{t^{n+1}}+\left(H_0+\frac{H_1}{t^n}\right)^2.\tag{2.34}$$

For sufficiently large values of n we can find two positive zeros of this function, which means two corresponding phase transitions. They happen, approximately, at

$$t_{\pm} \approx \left[\sqrt{nH_1} \; \frac{1 \pm \sqrt{1 - \frac{4H_0}{n}}}{2H_0} \right]^{2/n} , \tag{2.35}$$

so that, for $0 < t < t_-$, the universe is in an accelerated phase interpreted as an inflationary epoch; for $t_- < t < t_+$ it is in a decelerated phase (matter/radiation dominated); and, finally, for $t > t_+$ one obtains late time acceleration, which is in agreement with the current cosmic expansion.

3 Brans-Dicke-like Cosmology

Let us now consider an scalar-tensor theory where the scalar field ϕ does not couple minimally to Ricci scalar in the action for the gravity field (2.1). This kind of theories were suggested in 1961 by Brans and Dicke (see [12]) in the context to include Mach's principle in a theory of gravity that in some limit General Relativity(GR) was recovered, such that GR was constrasted. Recently, this kind of theories have become important in the context of Cosmology in order to explain the current acceleration of the Universe and even the inflation too. Also, the mathematical equivalence to some modified gravity theories as F(R)-theories, make them to play an important role (see [13]). In other hand, the main problems of these theories, as the coupling appeared with the normal matter when a conformal transformation is performed, it seems to be resolved by the so-called chamaleon mechanism, which avoid possible violations of the Equivalence principle at small scales.

3.1 Reconstruction of comological solutions

In the preceding section we have considered an action, (2.1), in which the scalar field is minimally coupled to gravity. In the present section, the scalar field couples to gravity through the Ricci scalar (see [14] for a review on cosmological applications). We begin from the action

$$S = \int d^4x\sqrt{-g} \; \left((1 + f(\phi))\frac{R}{\kappa^2} - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) \right) , \tag{3.1}$$

where $f(\phi)$ is an arbitrary function of the scalar field ϕ. Then, the effective gravitational coupling depends on ϕ, as $\kappa_{eff} = \kappa[1 + f(\phi)]^{-1/2}$. One can work in the Einstein frame, by performing the scale transformation

$$g_{\mu\nu} = [1 + f(\phi)]^{-1}\tilde{g}_{\mu\nu} . \tag{3.2}$$

The tilde over g denotes an Einstein frame quantity. Thus, the action (3.1) in such a frame assumes the form [15]

$$S = \int d^4x\sqrt{-\tilde{g}} \left\{ \frac{\tilde{R}}{2\kappa^2} - \left[\frac{\omega(\phi)}{2(1+f(\phi))} + \frac{6}{\kappa^2(1+f(\phi))} \left(\frac{d(1+f(\phi)^{1/2})}{d\phi} \right)^2 \right] \partial_\mu\phi\partial^\mu\phi - \frac{V(\phi)}{[1+f(\phi)]^2} \right\} . \tag{3.3}$$

The kinetic function can be written as $W(\phi) = \frac{\omega(\phi)}{1+f(\phi)} + \frac{3}{\kappa^2(1+f(\phi))^2} \left(\frac{df(\phi)}{d\phi} \right)^2$, and the extra term in the scalar potential can be absorbed by defining the new potential $U(\phi) = \frac{V(\phi)}{[1+f(\phi)]^2}$, so that we recover the action (2.1) in the Einstein frame, namely

$$S = \int dx^4\sqrt{-\tilde{g}} \; \left(\frac{\tilde{R}}{\kappa^2} - \frac{1}{2}\,W(\phi)\,\partial_\mu\phi\partial^\mu\phi - U(\phi) \right) . \tag{3.4}$$

We assume that the metric is FRW and spatially flat in this frame

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\sum_i dx_i^2 , \tag{3.5}$$

then, the equations of motion in this frame are given by

$$\widetilde{H}^2 = \frac{\kappa^2}{6}\rho_\phi \,, \tag{3.6}$$

$$\dot{\widetilde{H}} = -\frac{\kappa^2}{4}\left(\rho_\phi + p_\phi\right) \,, \tag{3.7}$$

$$\frac{d^2\phi}{d\tilde{t}^2} + 3\tilde{H}\frac{d\phi}{d\tilde{t}} + \frac{1}{2W(\phi)}\left[W'(\phi)\left(\frac{d\phi}{d\tilde{t}}\right)^2 + 2U'(\phi)\right] = 0 \,, \tag{3.8}$$

where $\rho_\phi = \frac{1}{2}W(\phi)\dot{\phi}^2 + U(\phi)$, $p_\phi = \frac{1}{2}W(\phi)\dot{\phi}^2 - U(\phi)$, and the Hubble parameter is $\widetilde{H} \equiv \frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}}$. Then,

$$W(\phi)\dot{\phi}^2 = -4\dot{\widetilde{H}} \,, \qquad U(\phi) = 6\widetilde{H}^2 + 2\dot{\widetilde{H}}. \tag{3.9}$$

Note that $\dot{\widetilde{H}} > 0$ is equivalent to $W < 0$; superacceleration is due to the "wrong" (negative) sign of the kinetic energy, which is the distinctive feature of a phantom field. The scalar field could be redefined to eliminate the factor $W(\phi)$, but this would not correct the sign of the kinetic energy.

If we choose $W(\phi)$ and $U(\phi)$ as $\omega(\phi)$ and $V(\phi)$ in (2.12),

$$W(\phi) = -\frac{2}{\kappa^2}g'(\phi) \,, \quad U(\phi) = \frac{1}{\kappa^2}\left(3g(\phi)^2 + g'(\phi)\right) \,, \tag{3.10}$$

by using a function $g(\phi)$ instead of $f(\phi)$ in (2.12), we find a solution as in (2.13),

$$\phi = \tilde{t} \,, \quad \widetilde{H}(\tilde{t}) = g(\tilde{t}) \,. \tag{3.11}$$

In (3.10) and hereafter in this section, we have dropped the matter contribution for simplicity.

We consider the de Sitter solution in this frame,

$$\widetilde{H} = \widetilde{H}_0 = \text{const.} \;\rightarrow\; \tilde{a}(\tilde{t}) = \tilde{a}_0 e^{\widetilde{H}_0\tilde{t}} \,. \tag{3.12}$$

We will see below that accelerated expansion can be obtained in the original frame corresponding to the Einstein frame (3.5) with the solution (3.12), by choosing an appropriate function $f(\phi)$. From (3.12) and the definition of $W(\phi)$ and $U(\phi)$, we have

$$W(\phi) = 0 \;\rightarrow\; \omega(\phi) = -\frac{3}{[1+f(\phi)]\,\kappa^2}\left(\frac{df(\phi)}{d\phi}\right)^2 \,, \quad U(\phi) = \frac{6}{\kappa^2}\widetilde{H}_0^2 \;\rightarrow\; V(\phi) = \frac{6}{\kappa^2}\widetilde{H}_0^2[1+f(\phi)]^2 \,. \tag{3.13}$$

Thus, the scalar field has a non-canonical kinetic term in the original frame, while in the Einstein frame the latter can be positive, depending on $W(\phi)$. The correspondence between conformal frames can be made explicit through the conformal transformation (3.2). Assuming a spatially flat FRW metric in the original frame,

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i^2 \,, \tag{3.14}$$

then, the relation between the time coordinate and the scale parameter in these frames is given by

$$t = \int \frac{d\tilde{t}}{([1+f(\tilde{t})]^{1/2}} \,, \quad a(t) = [1+f(\tilde{t})]^{-1/2}\,\tilde{a}(\tilde{t}) \,. \tag{3.15}$$

Now let us discuss the late-time acceleration in the model under discussion.

As an example, we consider the coupling function between the scalar field and the Ricci scalar

$$f(\phi) = \frac{1-\alpha\phi}{\alpha\phi} \,, \tag{3.16}$$

where α is a constant. Then, from (3.13), the kinetic function $\omega(\phi)$ and the potential $V(\phi)$ are

$$\omega(\phi) = -\frac{3}{\kappa^2\alpha^2}\frac{1}{\phi^3} \,, \qquad V(\phi) = \frac{6\widetilde{H}_0}{\kappa^2\alpha^2}\frac{1}{\phi^2} \,, \tag{3.17}$$

respectively. The solution for the current example is found to be

$$\phi(t) = \tilde{t} = \frac{1}{\alpha}\left(\frac{3\alpha}{2}t\right)^{2/3}, \qquad a(t) = \tilde{a}_0\left(\frac{3\alpha}{2}t\right)^{1/3}\exp\left[\frac{\tilde{H}_0}{\alpha}\left(\frac{3\alpha}{2}t\right)^{2/3}\right]. \tag{3.18}$$

We now calculate the acceleration parameter to study the behavior of the scalar parameter in the original frame,

$$\frac{\ddot{a}}{a} = -\frac{2}{9}\frac{1}{t^2} + \tilde{H}_0\left(\frac{2}{3\alpha}\right)^{1/3}\left[\frac{1}{t^{4/3}} + \tilde{H}_0\left(\frac{2}{3\alpha}\right)^{1/3}\frac{1}{t^{2/3}}\right]. \tag{3.19}$$

We observe that for small values of t the acceleration is negative; after that we get accelerated expansion for large t; finally, the universe ends with zero acceleration as $t \to \infty$. Thus, late time accelerated expansion is reproduced by the action (3.1) with the function $f(\phi)$ given by Eq. (3.16).

3.2 Chamaleon mechanism

The kind of theories described by the above action (3.1) have problems when matter is included and one perform a conformal transformation and the Einstein frame is recovered, then the local gravity tests may be violated by a fith force that appeared on a test particle and the violation of the Equivalence Principle is presented. This kind of problems are well constrained by the experiments to a certain value of the coupling parameter as it is pointed below. Recently, a very interesting idea originally proposed in Ref. [9] avoids the constrains from local gravity tests in such a way that the effects of the scalar field are negleible at small scales but it adquires an important role for large scales, whose effects may produce the current acceleration of the Universe

Let us start by rewriting the action in the Einstein frame (3.3) in a similar form as the original Brans-Dicke action by redefining the scalar field ϕ and rewriting the kinetic term $\omega(\phi)$ in terms of the coupling $(1 + f(\phi))$, then the action is given by:

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - U(\sigma) + \mathrm{e}^{4\beta\sigma}\widetilde{L_m}\right], \tag{3.20}$$

where:

$$\mathrm{e}^{-2\beta\sigma} = 1 + f(\phi), \tag{3.21}$$

here β is a constant. As it is seen in the action (3.20), the matter density lagrangian couples to the scalar field σ, such that a massive test particle will be under a fith force, and the equation of motion will be:

$$\ddot{x}^\mu + \tilde{\Gamma}^\mu_{\lambda\nu}\dot{x}^\lambda\dot{x}^\nu = -\beta\partial^\mu\sigma, \tag{3.22}$$

where x^μ represents the four-vector describing the path of a test particle moving in the metric $\tilde{g}^(\mu\nu)$, and $\tilde{\Gamma}^\mu_{\nu\lambda}$ are the Christoffel symbols for the metric $\tilde{g}^{\mu\nu}$. From the equation (3.22), the scalar field σ can be seen as a potential from a force given by:

$$F_\sigma = -M\beta\partial^\mu\sigma, \tag{3.23}$$

where M is the mass of a test particle. Then, this kind of theories reproduces a fith force which it has been tested by the experiments to a limit $\beta < 1.6 \times 10^{-3}$ (Ref. [16]). The aim of the chamaleon mechanism is that it makes the fith force negleible for small scales passing the local tests. Such mechanism works in the following way, by varying the action (3.20) with respect the scalar field σ, the equation of motion for the scalar field is obtained:

$$\nabla^2\sigma = U_{,\sigma} - \beta\mathrm{e}^{4\beta\sigma}\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu}, \tag{3.24}$$

where the energy-momentum tensor is given by $\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\partial L_m}{\partial\tilde{g}^{\mu\nu}}$. For simlicity we restrict to dust matter $\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} = -\tilde{\rho}_m$, where the energy density may be written in terms of the conformal transformation (3.21) as $\rho_m = \tilde{\rho}_m\mathrm{e}^{3\beta\sigma}$. Then, The equation for the scalar field (3.24) is written in the following way:

$$\nabla^2\sigma = U_{,\sigma} + \beta\rho_m\mathrm{e}^{\beta\sigma}, \tag{3.25}$$

here it is showed that the dynamics of the scalar field depends on the matter energy density. We may write the right side of the equation (3.25) as an effective potential $U_{eff} = U(\sigma) + \rho_m e^{\beta\sigma}$. Then, the behaviour of the scalar field will depend on the effective potential, and the solutions for the equation (3.25) are given by studying U_{eff}. By imposing to the scalar potential $U(\sigma)$ to be a monotonic decreasing function, the effective potential will have a minimum that will govern the solution for the scalar field (for more details see [9]), this minimum is given by:

$$U_{,\sigma}(\sigma_{min}) + \beta\rho_m e^{\beta\sigma_{min}} = 0 , \tag{3.26}$$

which depends on the local matter density. At this minimum, the scalar field mass will be given by:

$$m_\sigma^2 = U_{,\sigma\sigma}(\sigma_{min}) + \beta^2\rho_m e^{\beta\sigma_{min}} . \tag{3.27}$$

Then, because of the characteristics of the scalar potential $U(\sigma)$, larger values of the local density ρ_m corresponds to small values of σ_{min} and large values of m_σ, so it is possible for sufficiently large values of the scalar field mass to avoid Equivalence Principle violations and fith forces on the Earth. As the energy density ρ_m becomes smaller, the scalar field mass m_σ decreases and σ_{min} increases, such that at large scales (when $\rho \sim H_0^2$), the effects of the scalar field become detected, where the accelerated expansion of the Universe may be a possible effect. Then, one may restrict the original scalar potential $V(\phi)$ and the coupling $(1 + f(\phi))$ trough the mechanism showed above. The effective potential U_{eff} is written in terms of ϕ by the equation (3.21) as:

$$U_{eff}(\phi) = U(\phi) + \frac{\rho_m}{(1 + f(\phi))^{1/2-}} , \tag{3.28}$$

where $U(\phi) = V(\phi)/(1 + f(\phi))$. Then, by giving a function $f(\phi)$ and a scalar potential $V(\phi)$, one may construct using the conditions described above on the mass m_σ, a cosmological model that reproduces the current accelerated expansion and at the same time, avoid the local test of gravity.

4 Conclusions

Scalar-tensor theories have been studied along the article, where quintessence/phantom models and Brans-Dicke-like theory have been presented and cosmological evolution have been reproduced. There are some problems in this kind of theories, as for example in the models that try to reproduce the whole expansion history, where the grateful exit from inflation such that the large scale structure should be well studied. In spite of the possible succesful of scalar-tensor theories, the cosmological constant problem still remain as one of the deepest problems in theoretical physics, so it may be researched depper. Also, the coincidence problem that may be resolved by scalar-tensor theory, have not yet a natural explanation. In other hand, the interesting chamaleon mechanism (see [9]) supposes one way to avoid local gravity tests for the non-minimally scalar-tensor theories, and then, it may be a good candidate to dark energy, although some more relativistic models should be constructed.

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Casimir densities in brane models with compact internal spaces

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Abstract

We investigate the Wightman function, the vacuum expectation values of the field squared and the energy-momentum tensor for a massive scalar field with general curvature coupling parameter subject to Robin boundary conditions on two codimension one parallel branes located on $(D+1)$-dimensional background spacetime $AdS_{D_1+1} \times \Sigma$ with a warped internal space Σ. The general case of different Robin coefficients on separate branes is considered. Unlike to the purely AdS bulk, the vacuum expectation values induced by a single brane, in addition to the distance from the brane, depends also on the position of the brane in the bulk. The brane induced parts in these expectation values vanish when the brane position tends to the AdS horizon or AdS boundary. For strong gravitational fields corresponding to large values of the AdS energy scale, the both single brane and interference parts of the expectation values integrated over the internal space are exponentially suppressed. An application to the higher dimensional generalization of the Randall-Sundrum brane model with arbitrary mass terms on the branes is discussed. For large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane.

1 Introduction

The braneworld scenario provides an interesting alternative to the standard Kaluza-Klein compactification of the extra dimensions. The simplest phenomenological models describing such a scenario are the five-dimensional Randall-Sundrum type braneworld models (for a review see [1]). From the point of view of embedding these models into a more fundamental theory, such as string/M-theory, one may expect that a more complete version of the scenario must admit the presence of additional extra dimensions compactified on an internal manifold. From a phenomenological point of view, the consideration of more general spacetimes offer a richer geometrical structure and may provide interesting extensions of the Randall-Sundrum mechanism for the geometric origin of the hierarchy. More extra dimensions also relax the fine-tunings of the fundamental parameters. These models can provide a framework in the context of which the stabilization of the radion field naturally takes place. In addition, a richer topological structure of the field configuration in transverse space provides the possibility of more realistic spectrum of chiral fermions localized on the brane. Several variants of the Randall-Sundrum scenario involving cosmic strings and other global defects of various codimensions have been investigated in higher dimensions (see, for instance, [2] and references therein).

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Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted great deal of attention [3]-[47]. A class of higher dimensional models with the topology $\mathrm{AdS}_{D_1+1} \times \Sigma$, where Σ is a one-parameter compact manifold, and with two branes of codimension one located at the orbifold fixed points, is considered in Refs. [25, 26]. In both cases of the warped and unwarped internal manifold, the quantum effective potential induced by bulk scalar fields is evaluated and it has been shown that this potential can stabilize the hierarchy between the Planck and electroweak scales without fine tuning. In addition to the effective potential, the investigation of local physical characteristics in these models is of considerable interest. Local quantities contain more information on the vacuum fluctuations than the global ones and play an important role in modelling a self-consistent dynamics involving the gravitational field. In papers [39, 40, 41] we have studied the bulk and surface Casimir densities for a scalar field with an arbitrary curvature coupling parameter obeying Robin boundary conditions on two codimension one parallel branes embedded in the background spacetime $\mathrm{AdS}_{D_1+1} \times \Sigma$ with a warped internal space Σ. For an arbitrary internal space Σ, the application of the generalized Abel-Plana formula [48] allowed us to extract form the vacuum expectation values the part due to the bulk without branes and to present the brane induced parts in terms of exponentially convergent integrals for the points away from the branes. In the present paper we review these results.

The paper is organized as follows. In the next section we evaluate the Wightman function in the region between the branes. By using the generalized Abel-Plana formula, we present this function in the form of a sum of the Wightman function for the bulk without boundaries and boundary induced parts. The vacuum expectation value of the bulk energy-momentum tensor for a general case of the internal space Σ is discussed in section 3. The interaction forces between the branes are discussed in section 4. The surface Casimir densities and the energy balance are considered in section 5. The last section contains a summary of the work.

2 Wightman function

For a free scalar field $\varphi(x)$ with curvature coupling parameter ζ the equation of motion has the form

$$g^{MN}\nabla_M\nabla_N + m^2 + \zeta R \quad \varphi(x) = 0, \tag{2.1}$$

where $M, N = 0, 1, \ldots, D$, and R is the scalar curvature. We will assume that the background spacetime has a topology $AdS_{D_1+1} \times \Sigma$, where Σ is a D_2-dimensional compact manifold. The corresponding line element has the form

$$ds^2 = g_{MN}dx^M dx^N = e^{-2k_D y}\eta_{\mu\nu}dx^\mu dx^\nu - e^{-2k_D y}\gamma_{ij}dX^i dX^j - dy^2, \tag{2.2}$$

with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$ being the metric for the D_1-dimensional Minkowski spacetime $R^{(D_1-1,1)}$ and the coordinates X^i cover the manifold Σ, $D = D_1 + D_2$. Here and below $\mu, \nu = 0, 1, \ldots, D_1 - 1$ and $i, j = 1, \ldots, D_2$. The scalar curvature for the metric tensor from (3.1) is given by the expression $R = -D(D+1)k_D^2 - e^{2k_D y}R_{(\gamma)}$, where $R_{(\gamma)}$ is the scalar curvature for the metric tensor γ_{ik}. In the discussion below, in addition to the coordinate y we will use the radial coordinate z defined by the relation $z = e^{k_D y}/k_D$. In terms of the coordinate z, the metric tensor is conformally related to the metric of the direct product space $R^{(D_1,1)} \times \Sigma$ by the conformal factor $(k_D z)^{-2}$.

Our main interest in this paper will be the Wightman function and the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor induced by two infinite parallel branes of codimension one with the coordinates $y = a$ and $y = b$, $a < b$. We will assume that on this branes the scalar field obeys the boundary conditions

$$(\tilde{A}_j + \tilde{B}_j\partial_y)\varphi(x) = 0, \quad y = j, \ j = a, b, \tag{2.3}$$

with constant coefficients $\tilde{A}_j$, $\tilde{B}_j$. In the orbifolded version of the model which corresponds to a higher dimensional Randall-Sundrum braneworld these coefficients are expressed in terms of the surface mass parameters and the curvature coupling of the scalar field. In quantum field theory the imposition of boundary conditions modifies the spectrum for the zero–point fluctuations and as a result the VEVs for physical observables are changed. These effects can either stabilize or destabilize the branewolds and have to be taken into account in the self-consistent formulation of the braneworld dynamics.

As a first stage in the investigations of local quantum effects, we will consider the positive frequency Wightman function defined as the expectation value $G^+(x, x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$. In the region between the branes, $a<y<b$, the Wightman function is presented as the mode-sum:

$$\begin{aligned} G^{+}(x,x') &= \frac{k_D^{D-1}(zz')^{D/2}}{2^{D_1+1}\pi^{D_1-3}z_a^2}\sum_{\beta}\psi_\beta(X)\psi_\beta^*(X')\int d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}} \\ &\times \sum_{n=1}^{\infty}\frac{h_{\beta\nu}(u)}{[A_b^2+B_b^2(\eta^2u^2-\nu^2)]\,\bar{J}_\nu^{(a)2}(u)/\bar{J}_\nu^{(b)2}(\eta u)-A_a^2+B_a^2(u^2-\nu^2)}|_{u=\gamma_{\nu,n}}, \quad (2.4)\end{aligned}$$

where $\mathbf{x}=(x^1,x^2,\ldots,x^{D_1-1})$ represents the spatial coordinates in $R^{(D_1-1,1)}$, $\Delta\mathbf{x}=\mathbf{x}-\mathbf{x}'$, $\eta=z_b/z_a$, and

$$h_{\beta\nu}(u) = ug_\nu(u,uz/z_a)g_\nu(u,uz'/z_a)\frac{e^{-i\Delta t\sqrt{u^2/z_a^2+k^2+\lambda_\beta^2}}}{\sqrt{u^2/z_a^2+k^2+\lambda_\beta^2}}, \quad (2.5)$$

$$g_\nu(u,v) = J_\nu(v)\bar{Y}_\nu^{(a)}(u)-\bar{J}_\nu^{(a)}(u)Y_\nu(v),\ \nu=\sqrt{(D/2)^2-D(D+1)\zeta+m^2/k_D^2}, \quad (2.6)$$

with $k=|\mathbf{k}|$, $\Delta t=t-t'$, $z_j=e^{k_Dj}/k_D$, $j=a,b$, $J_\nu(x)$, $Y_\nu(x)$ are the Bessel and Neumann functions. In formula (2.6), for a given function $F(x)$ we use the notation

$$\bar{F}^{(j)}(x)=A_jF(x)+B_jxF'(x),\ A_j=\tilde{A}_j+\tilde{B}_jk_DD/2,\quad B_j=\tilde{B}_jk_D,\ j=a,b. \quad (2.7)$$

In the discussion below we will assume values of the curvature coupling parameter for which ν is real. For imaginary ν the ground state becomes unstable [49]. In (2.4), the modes $\psi_\beta(X)$ are the eigenfunctions for the operator $\Delta_{(\gamma)}+\zeta R_{(\gamma)}$:

$$\Delta_{(\gamma)}+\zeta R_{(\gamma)}\ \ \psi_\beta(X)=-\lambda_\beta^2\psi_\beta(X),\ \int d^{D_2}X\,\sqrt{\gamma}\psi_\beta(X)\psi_{\beta'}^*(X)=\delta_{\beta\beta'}, \quad (2.8)$$

with eigenvalues λ_β^2, and $\Delta_{(\gamma)}$ is the Laplace-Beltrami operator for the metric γ_{ij}. From the boundary condition on the branes we receive that the eigenvalues $\gamma_{\nu,n}$ have to be solutions to the equation

$$g_\nu^{(ab)}(\gamma_{\nu,n},\eta\gamma_{\nu,n})\equiv\bar{J}_\nu^{(a)}(\gamma_{\nu,n})\bar{Y}_\nu^{(b)}(\eta\gamma_{\nu,n})-\bar{Y}_\nu^{(a)}(\gamma_{\nu,n})\bar{J}_\nu^{(b)}(\eta\gamma_{\nu,n})=0. \quad (2.9)$$

This equation determines the tower of radial Kaluza-Klein (KK) masses.

Applying to the sum over n in (2.4) a variant of the generalized Abel-Plana formula [48], the Wightman function is presented in two equivalent forms ($j=a,b$)

$$\begin{aligned} G^{+}(x,x') &= G_0^{+}(x,x')+\langle\varphi(x)\varphi(x')\rangle^{(j)}-\frac{k_D^{D-1}(zz')^{D/2}}{2^{D_1-1}\pi^{D_1}}\sum_\beta\psi_\beta(X)\psi_\beta^*(X') \\ &\times \int d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}}\int_{\sqrt{k^2+\lambda_\beta^2}}^{\infty}du u G_\nu^{(j)}(uz_a,uz)G_\nu^{(j)}(uz_a,uz') \\ &\times \frac{\Omega_{j\nu}(uz_a,uz_b)}{\sqrt{u^2-k^2-\lambda_\beta^2}}\cosh(\Delta t\sqrt{u^2-k^2-\lambda_\beta^2}), \quad (2.10)\end{aligned}$$

where $I_\nu(u)$ and $K_\nu(u)$ are the modified Bessel functions and

$$\begin{aligned} \Omega_{a\nu}(u,v) &= \frac{\bar{K}_\nu^{(b)}(v)/\bar{K}_\nu^{(a)}(u)}{\bar{K}_\nu^{(a)}(u)\bar{I}_\nu^{(b)}(v)-\bar{K}_\nu^{(b)}(v)\bar{I}_\nu^{(a)}(u)}, \\ \Omega_{b\nu}(u,v) &= \frac{\bar{I}_\nu^{(a)}(u)/\bar{I}_\nu^{(b)}(v)}{\bar{K}_\nu^{(a)}(u)\bar{I}_\nu^{(b)}(v)-\bar{K}_\nu^{(b)}(v)\bar{I}_\nu^{(a)}(u)}, \quad (2.11) \\ G_\nu^{(j)}(u,v) &= I_\nu(v)\bar{K}_\nu^{(j)}(u)-\bar{I}_\nu^{(j)}(u)K_\nu(v),\ j=a,b. \end{aligned}$$

In (2.10), the term

$$
\begin{aligned}
G_0^+(x,x') &= \frac{k_D^{D-1}(zz')^{\frac{D}{2}}}{2^{D_1}\pi^{D_1-1}} \sum_\beta \psi_\beta(X)\psi_\beta^*(X') \int d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}} \\
&\times \int_0^\infty du\, u \frac{e^{-i\Delta t\sqrt{u^2+k^2+\lambda_\beta^2}}}{\sqrt{u^2+k^2+\lambda_\beta^2}} J_\nu(uz)J_\nu(uz'),
\end{aligned}
\tag{2.12}
$$

does not depend on the boundary conditions and is the Wightman function for the $AdS_{D_1+1}\times\Sigma$ spacetime without branes. The second term on the right of Eq. (2.10) is given by the formula

$$
\begin{aligned}
\langle\varphi(x)\varphi(x')\rangle^{(a)} &= -\frac{k_D^{D-1}(zz')^{\frac{D}{2}}}{2^{D_1-1}\pi^{D_1}} \sum_\beta \psi_\beta(X)\psi_\beta^*(X') \int d\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{x}} \\
&\times \int_{\sqrt{k^2+\lambda_\beta^2}}^\infty du u \frac{\bar{I}_\nu^{(a)}(uz_a)}{\bar{K}_\nu^{(a)}(uz_a)} \frac{K_\nu(uz)K_\nu(uz')}{\sqrt{u^2-k^2-\lambda_\beta^2}} \cosh(\Delta t\sqrt{u^2-k^2-\lambda_\beta^2}),
\end{aligned}
\tag{2.13}
$$

for $j=a$, and the expression for $\langle\varphi(x)\varphi(x')\rangle^{(b)}$ is obtained from (2.13) by the replacements $a\to b$, I_ν K_ν. The term $\langle\varphi(x)\varphi(x')\rangle^{(j)}$ does not depend on the parameters of the brane at $z=z_{j'}$, $j'\neq j$, and is induced by a single brane at $z=z_j$ when the boundary $z=z_{j'}$ is absent. In the same way described above for the Wightman function, any other two-point function can be evaluated. Note that the expression for the Wightman function is not symmetric with respect to the interchange of the brane indices. The reason for this is that the boundaries have nonzero extrinsic curvature tensors and two sides of the boundaries are not equivalent. In particular, for the geometry of a single brane the VEVs are different for the regions on the left and on the right of the brane. In the region $y<a$ the Wightman has the form $G^+(x,x') = G_0^+(x,x') + \langle\varphi(x)\varphi(x')\rangle^{(a)}$, where the expression for the second term on the right hand-side is obtained from (2.13) by the replacement I_ν K_ν. Similarly, for the Wightman function in the region $y>b$ one has $G^+(x,x') = G_0^+(x,x') + \langle\varphi(x)\varphi(x')\rangle^{(b)}$, where the second term is given by formula (2.13) replacing $a\to b$.

In the higher dimensional generalization of the Randall-Sundrum braneworld based on the bulk $AdS_{D_1+1}\times\Sigma$ the Wightman function for untwisted scalar is given by formula (2.10) with an additional factor 1/2 and with Robin coefficients

$$
\tilde{A}_a/\tilde{B}_a = -c_a/2 - 2D\zeta k_D, \quad \tilde{A}_b/\tilde{B}_b = c_b/2 - 2D\zeta k_D 2. \tag{2.14}
$$

For twisted scalar field Dirichlet boundary conditions are obtained. The one-loop effective potential and the problem of moduli stabilization in this model with zero mass parameters c_j are discussed in Ref. [25].

3 Vacuum energy-momentum tensor

The VEV of the energy-momentum tensor can be evaluated by substituting the Wightman function and the VEV of the field squared into the formula

$$
\langle 0|T_{MN}|0\rangle = \lim_{x'\to x}\partial_M\partial_N' G^+(x,x') + \quad \zeta - \frac{1}{4} \quad g_{MN}\nabla_L\nabla^L - \zeta\nabla_M\nabla_N - \zeta R_{MN} \quad \langle 0|\varphi^2|0\rangle, \tag{3.1}
$$

where R_{MN} is the Ricci tensor. Substituting the expression for the Wightman function into this formula, for the components of the vacuum energy-momentum tensor in the region between the branes we obtain the formula

$$
\begin{aligned}
\langle 0|T_M^N|0\rangle &= \langle T_M^N\rangle^{(0)} + \langle T_M^N\rangle^{(j)} - \frac{2k_D^{D+1}z^D}{(4\pi)^{D_1/2}\Gamma(D_1/2)} \sum_\beta |\psi_\beta(X)|^2 \\
&\times \int_{\lambda_\beta}^\infty du\, u(u^2-\lambda_\beta^2)^{\frac{D_1}{2}-1}\Omega_{j\nu}(uz_a,uz_b)F_{\beta M}^{(+)N}[G_\nu^{(j)}(uz_j,uz)],
\end{aligned}
\tag{3.2}
$$

with the functions $F^{(+)N}_{\beta M}[g(v)]$, $g(v) = G^{(j)}_{\nu}(uz_j, v)$, defined by the relations

$$F^{(\pm)\sigma}_{\beta\mu}[g(v)] = \delta^{\sigma}_{\mu} \quad \frac{1}{4} - \zeta \quad \left\{ z^2 g^2(v)\eta_\beta(X) + 2v\frac{\partial}{\partial v}F[g(v)] + \frac{\pm v^2 - z^2\lambda^2_\beta}{D_1(\zeta - 1/4)}g^2(v) \right\}, \quad (3.3)$$

$$\begin{aligned} F^{(\pm)D}_{\beta D}[g(v)] &= \frac{1}{4} - \zeta \quad z^2 g^2(v)\eta_\beta(X) + \frac{1}{2}[-v^2 g'^2(v) \\ &\quad + D(4\zeta - 1)vg(v)g'(v) + \quad 2m^2/k^2_D - \nu^2 \pm v^2 \quad g^2(v)], \end{aligned} \quad (3.4)$$

for the components in the AdS part, and by the relations

$$F^{(\pm)i}_{\beta D}[g(v)] = \frac{k_D}{2}z^2(1 - 4\zeta)F[g(v)]\eta^i_\beta(X), \quad (3.5)$$

$$F^{(\pm)k}_{\beta i}[g(v)] = z^2 g^2(v)\frac{t^k_{\beta i}(X)}{|\psi_\beta(X)|^2} + \frac{1}{2}\delta^k_i(1 - 4\zeta)v\frac{\partial}{\partial v}F[g(v)], \quad (3.6)$$

with $t^k_{\beta i}(X) = -\gamma^{kl}t_{\beta il}(X)$, for the components having indices in the internal space. In these expressions we use the following notations

$$F[g(v)] = vg(v)g'(v) + \frac{1}{2} \quad D + \frac{4\zeta}{4\zeta - 1} \quad g^2(v), \quad (3.7)$$

$$\eta_\beta(X) = \frac{\triangle_{(\gamma)}|\psi_\beta(X)|^2}{|\psi_\beta(X)|^2}, \quad \eta^i_\beta(X) = -\gamma^{ik}\frac{\partial_k|\psi_\beta(X)|^2}{|\psi_\beta(X)|^2}, \quad (3.8)$$

$$\begin{aligned} t_{\beta ik}(X) &= \nabla_{(\gamma)i}\psi_\beta(X)\nabla_{(\gamma)k}\psi^*_\beta(X) + \\ &\quad \zeta - \frac{1}{4} \quad \gamma_{ik}\triangle_{(\gamma)} - \zeta\nabla_{(\gamma)i}\nabla_{(\gamma)k} - \zeta R_{(\gamma)ik} \quad |\psi_\beta(X)|^2, \end{aligned} \quad (3.9)$$

where $\nabla_{(\gamma)i}$ is the covariant derivative operator associated with the metric tensor γ_{ik}.

In formula (3.2),

$$\langle T^N_M\rangle^{(0)} = \frac{k^{D+1}_D z^D}{(4\pi)^{\frac{D_1}{2}}}\Gamma \quad 1 - \frac{D_1}{2} \quad \sum_\beta |\psi_\beta(X)|^2 \int_0^\infty du\, u(u^2 + \lambda^2_\beta)^{\frac{D_1}{2}-1}F^{(-)N}_{\beta M}[J_\nu(uz)], \quad (3.10)$$

is the VEV for the energy-momentum tensor in the background without branes, and the term $\langle T^N_M\rangle^{(j)}$ is induced by a single brane at $z = z_j$. For the left brane one has

$$\langle T^N_M\rangle^{(a)} = -\frac{2k^{D+1}_D z^D}{(4\pi)^{\frac{D_1}{2}}\Gamma \quad \frac{D_1}{2}} \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^\infty du\, u(u^2 - \lambda^2_\beta)^{\frac{D_1}{2}-1}\frac{\bar{I}^{(a)}_\nu(uz_a)}{\bar{K}^{(a)}_\nu(uz_a)}F^{(+)N}_{\beta M}[K_\nu(uz)], \quad (3.11)$$

and the corresponding expression for the right brane is obtained by the replacements $a \rightarrow b$, $I_\nu \quad K_\nu$. Unlike to the case of purely AdS bulk, here the VEVs for a single brane in addition to the distance from the brane depend also on the position of the brane in the bulk. In the limit when the AdS curvature radius tends to infinity we derive the formula for the vacuum energy-momentum tensor for parallel plates on the background spacetime with topology $R^{(D_1,1)} \times \Sigma$. In this limit for a homogeneous internal space ^D_D–component of the brane induced part in the VEV of the energy-momentum tensor vanishes.

The features of the single brane parts in the VEVs in the asymptotic regions of the parameters are as follows. For the points on the brane the vacuum energy-momentum tensor diverges. Near the brane the total vacuum energy-momentum tensor is dominated by the brane induced part and has opposite signs for Dirichlet and non-Dirichlet boundary conditions. Near the brane ^D_D– and ^i_D–components of this tensor have opposite signs in the regions $y < a$ and $y > a$. For large distances from the brane in the region $y > a$ the contribution of a given mode along Σ with nonzero KK mass is suppressed by the factor $e^{-2\lambda_\beta z}$. For the zero mode the brane induced VEV near the AdS horizon behaves as $z^{D_2 - 2\nu}$. In the purely AdS bulk ($D_2 = 0$) this VEV vanishes on the horizon for $\nu > 0$. For an internal spaces with $D_2 > 2\nu$ the VEV diverges on the horizon. The VEV integrated over the internal space vanishes on the AdS horizon for all values D_2 due to the additional warp factor coming from the volume element. For the points near the AdS boundary, the brane induced VEV vanishes as

$z^{D+2\nu}$ for diagonal components and as $z^{D+2\nu+2}$ for the ${}_D^i$–component. For small values of the length scale for the internal space, the contribution of nonzero KK masses is exponentially suppressed and the main contribution into the brane induced energy-momentum tensor comes from the zero mode. In the opposite limit, when the length scale of the internal space is large, to the leading order the vacuum energy-momentum tensor reduces to the corresponding result for a brane in the bulk AdS_{D+1} given in Ref. [36]. For strong gravitational fields corresponding to small values of the AdS curvature radius, the contribution from nonzero KK modes along Σ is suppressed by the factor $e^{-2\lambda_\beta|z-z_a|}$. For the zero KK mode the components of the brane induced vacuum energy-momentum tensor behave like $k_D^{D_1+1}e^{D_2k_Dy}\exp[(D_1+2\nu)k_D(y-a)]$ in the region $y<a$ and like $k_D^{D_1+1}e^{D_2k_Dy}\exp[2\nu k_D(a-y)]$ in the region $y>a$. The corresponding quantities integrated over the internal space contain additional factor $e^{-D_2k_Dy}$ coming from the volume element and are exponentially small in both regions. For fixed values of the other parameters, the brane induced VEV in the region $y>a$ vanishes as $z_a^{2\nu}$ when the brane position tends to the AdS boundary. When the brane position tends to the AdS horizon, $z_a\to\infty$, for massive KK modes along Σ the VEV of the energy-momentum tensor in the region $z<z_a$ is suppressed by the factor $e^{-2z_a\lambda_\beta}$. For the zero mode in the same limit the suppression is power-law with respect to z_a.

For the geometry of two branes, the VEV in the region between the branes is presented as

$$\langle 0|T_M^N|0\rangle = \langle T_M^N\rangle^{(0)} + \sum_{j=a,b}\langle T_M^N\rangle^{(j)} + \langle T_M^N\rangle^{(ab)}, \tag{3.12}$$

with separated boundary-free, single branes and interference parts. The latter is finite everywhere including the points on the branes. The surface divergences are contained in the single brane parts only. The both single brane and interference parts separately satisfy the continuity equation and are traceless for a conformally coupled massless scalar. The possible trace anomalies are contained in the boundary-free parts. In the limit $k_D\to 0$ we derive the corresponding results for two parallel Robin plates in the bulk $R^{(D_1,1)}\times\Sigma$. For small values of the length scale of the internal space corresponding to large KK masses, the interference part in the VEV of the energy-momentum tensor is suppressed by the factor $e^{-2\lambda_\beta(z_b-z_a)}$. The interference part vanishes as $z_a^{2\nu}$ when the left brane tends to the AdS boundary. Under the condition $z\ll z_b$ an additional suppression factor appears in the form $(z/z_b)^{D_1}$ for ${}_D^D$–component and in the form $(z/z_b)^{D_1+2\alpha_1}$ for the other components, where $\alpha_1=\min(1,\nu)$.

4 Interaction forces

Now we turn to the investigations of the vacuum forces acting on the branes. The corresponding effective pressure $p^{(j)}$ acting on the brane at $z=z_j$ is determined by ${}_D^D$–component of the vacuum energy-momentum tensor evaluated at the point of the brane location: $p^{(j)}=-\langle T_D^D\rangle_{z=z_j}$. For the region between two branes it can be presented as a sum of two terms: $p^{(j)}=p_1^{(j)}+p_{(\text{int})}^{(j)}$, $j=a,b$. The first term is the pressure for a single brane at $z=z_j$ when the second brane is absent. This term is divergent due to the surface divergences in the VEVs and needs additional renormalization. This can be done, for example, by applying the generalized zeta function technique to the corresponding mode-sum. Below we will be concentrated on the term $p_{(\text{int})}^{(j)}$. This term is the additional vacuum pressure induced by the presence of the second brane, and can be termed as an interaction force. It is determined by the last term on the right of formulae (3.2) evaluated at the brane location $z=z_j$. It is finite for all nonzero interbrane distances and is not changed by the renormalization procedure. Substituting $z=z_j$ into the second term on the right of formula (3.2), for the interaction part of the vacuum effective pressure one finds

$$p_{(\text{int})}^{(j)} = \frac{k_D^{D+1}z_j^D}{(4\pi)^{\frac{D_1}{2}}\Gamma\ \frac{D_1}{2}}\sum_\beta|\psi_\beta(X)|^2\int_{\lambda_\beta}^\infty duu(u^2-\lambda_\beta^2)^{\frac{D_1}{2}-1}\Omega_{j\nu}(uz_a,uz_b)F_\beta^{(j)}(uz_j), \tag{4.1}$$

where we have introduced the notation

$$F_\beta^{(j)}(u) = \ u^2-\nu^2+2m^2/k_D^2\ \ B_j^2 - D(4\zeta-1)A_jB_j - A_j^2 - 2\,(\zeta-1/4)\,z_j^2B_j^2\eta_\beta(X). \tag{4.2}$$

For small interbrane distances the interaction part dominates the single brane parts. For a Dirichlet scalar $\Omega_{j\nu}(uz_a,uz_b)>0$ and the vacuum interaction forces are attractive. For a given value of the

AdS energy scale k_D and one parameter manifold Σ with size L, the vacuum interaction forces (4.1) are functions on the ratios z_b/z_a and L/z_a. The first ratio is related to the proper distance between the branes and the second one is the ratio of the size of the internal space measured by an observer residing on the brane at $y=a$ to the AdS curvature radius k_D^{-1}. The quantity $p_{(\mathrm{int})}^{(j)}$ determines the force by which the scalar vacuum acts on the brane due to the modification of the spectrum for the zero-point fluctuations by the presence of the second brane. As the vacuum properties depend on the coordinate y, there is no a priori reason for the interaction terms to be equal for the branes $j=a$ and $j=b$, and the corresponding forces in general are different even in the case of the same Robin coefficients in the boundary conditions.

Taking the limit $k_D \to 0$ we obtain the result for the interaction forces between two Robin plates in the bulk $R^{(D_1-1,1)} \times \Sigma$. In this case, for a homogeneous internal space the interaction forces are the same even in the case of different Robin coefficients for separate branes. For the modes along Σ with large KK masses, the interaction forces are exponentially small. In particular, for sufficiently small length scales of the internal space this is the case for all nonzero KK modes and the main contribution to the interaction forces comes from the zero mode. For small interbrane distances, the interaction forces are repulsive for Dirichlet boundary condition on one brane and non-Dirichlet boundary condition on the another and are attractive for other cases. For small interbrane distances the contribution of the interaction term dominates the single brane parts, and the same is the case for the total vacuum forces acting on the branes. When the right brane tends to the AdS horizon, $z_b \to \infty$, the interaction force acting on the left brane vanishes as $e^{-2\lambda_\beta z_b}/z_b^{D_1/2}$ for the nonzero KK mode and like $z_b^{-D_1-2\nu}$ for the zero mode. In the same limit the corresponding force acting on the right brane behaves as $z_b^{D_2+D_1/2+1}e^{-2\lambda_\beta z_b}$ for the nonzero KK mode and like $z_b^{D_2-2\nu}$ for the zero mode. In the limit when the left brane tends to the AdS boundary the contribution of a given KK mode into the vacuum interaction force vanishes as $z_a^{D+2\nu}$ and as $z_a^{2\nu}$ for the left and right branes, respectively. For small values of the AdS curvature radius corresponding to strong gravitational fields, under the conditions $\lambda_\beta z_a \gg 1$ and $\lambda_\beta (z_b - z_a) \gg 1$, the contribution to the interaction forces is suppressed by the factor $e^{-2\lambda_\beta (z_b-z_a)}$. For the zero KK mode, the corresponding interaction forces integrated over the internal space behave as $k_D^{D_1+1}\exp[(D_1\delta_j^a + 2\nu)k_D(a-b)]$ for the brane at $y=j$ and are exponentially small. In the model without the internal space the suppression is relatively weaker.

5 Surface energy-momentum tensor

On manifolds with boundaries the energy-momentum tensor in addition to the bulk part contains a contribution located on the boundary. For an arbitrary smooth boundary ∂M with the inward-pointing unit normal vector n^L, the surface part of the energy-momentum tensor for a scalar field is given by the formula [50] $T_{MN}^{(\mathrm{s})} = \delta(x;\partial M)\tau_{MN}$, where the 'one-sided' delta-function $\delta(x;\partial M)$ locates this tensor on ∂M and

$$\tau_{MN} = \zeta\varphi^2 K_{MN} - (2\zeta - 1/2)h_{MN}\varphi n^L \nabla_L \varphi. \tag{5.1}$$

In this formula, h_{MN} is the induced metric on the boundary and K_{MN} is the corresponding extrinsic curvature tensor. From the point of view of physics on the brane at $y=j$, Eq. (5.1) corresponds to the gravitational source of the cosmological constant type with the surface energy density $\varepsilon_j^{(\mathrm{s})} = \langle 0|\tau_0^{(j)0}|0\rangle$ (surface energy per unit physical volume on the brane at $y=j$ or brane tension), stress $p_j^{(\mathrm{s})} = -\langle 0|\tau_1^{(j)1}|0\rangle$, and the equation of state $\varepsilon_j^{(\mathrm{s})} = -p_j^{(\mathrm{s})}$. It is noteworthy that this relation takes place for both subspaces on the brane.

For two-brane geometry the VEV of the surface energy density on the brane at $y=j$ is presented as the sum $\varepsilon_j^{(\mathrm{s})} = \varepsilon_{1j}^{(\mathrm{s})} + \Delta\varepsilon_j^{(\mathrm{s})}$. The first term on the right is the energy density induced on a single brane when the second brane is absent. This part is evaluated in [41] by using the generalized zeta function method. The second term is induced by the presence of the second brane and is given by the formula

$$\Delta\varepsilon_j^{(\mathrm{s})} = \frac{2C_j n^{(j)}(k_D z_j)^D B_j^2}{(4\pi)^{D_1/2}\Gamma(D_1/2)} \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^\infty du\, u(u^2-\lambda_\beta^2)^{\frac{D_1}{2}-1}\Omega_{j\nu}(uz_a, uz_b), \tag{5.2}$$

with the notation $C_j = \zeta - (2\zeta - 1/2)\tilde{A}_j/(k_D\tilde{B}_j)$. As we consider the region $a \leqslant y \leqslant b$, the energy density $\varepsilon_j^{(\mathrm{s})}$ is located on the surface $y=a+0$ for the left brane and on the surface $y=b-0$ for the right brane. The energy densities on the surfaces $y = a - 0$ and $y = b + 0$ are the same as for the corresponding single brane geometry. For an observer living on the brane at $y = j$ the corresponding effective D_1-dimensional cosmological constant is determined by the relation

$$\Lambda_{D_1 j} = 8\pi M_{D_1 j}^{2-D_1} e^{-D_2 k_D j} \int_{\Sigma} d^{D_2} X \sqrt{\gamma}\, \Delta\varepsilon_j^{(\mathrm{s})}, \tag{5.3}$$

where $M_{D_1 j}$ is the D_1-dimensional effective Planck mass scale for the same observer. In Ref. [41] it has been shown that for large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane. In the Randall-Sundrum braneworld model, for the interbrane distances solving the hierarchy problem between the gravitational and electroweak mass scales, the cosmological constant generated on the visible brane is of the right order of magnitude with the value suggested by the cosmological observations.

On background of manifolds with boundaries the total vacuum energy is splitted into bulk and boundary parts. In the region between two branes the bulk energy per unit coordinate volume in the D_1-dimensional subspace is obtained by the integration of the ${}_0^0$-component of the volume energy-momentum tensor over this region: $E^{(\mathrm{v})} = \int d^{D_2} X dy \sqrt{|g|} \langle 0|T_0^{(\mathrm{v})0}|0\rangle$. The surface energy per unit coordinate volume in the D_1-dimensional subspace, $E^{(\mathrm{s})}$, is related to the surface densities by the formula $E^{(\mathrm{s})} = \sum_{j=a,b} (k_D z_j)^{-D} \varepsilon_j^{(\mathrm{s})}$. Now it can be seen that the formal relation $E = E^{(\mathrm{v})} + E^{(\mathrm{s})}$ takes place for the unrenormalized VEVs, where

$$E = \frac{1}{2} \int \frac{d^{D_1 - 1}\mathbf{k}}{(2\pi)^{D_1 - 1}} \sum_{\beta} \sum_{n=1}^{\infty} (k^2 + m_n^2 + \lambda_\beta^2)^{1/2},\ m_n = \gamma_{\nu,n}/z_a, \tag{5.4}$$

is the total vacuum energy per unit coordinate volume of the D_1-dimensional subspace, evaluated as the sum of zero-point energies of elementary oscillators. The latter can be presented in the form $E = \sum_{j=a,b} E_j + \Delta E$, where E_a (E_b) is the vacuum energy for the geometry of a single brane at $y = a$ ($y = b$) in the region $y \geqslant a$ ($y \leqslant b$), and the interference term is given by the formula

$$\Delta E = \sum_{\beta} \int_{\lambda_\beta}^{\infty} du \frac{u(u^2 - \lambda_\beta^2)^{D_1/2 - 1}}{(4\pi)^{D_1/2} \Gamma(D_1/2)} \ln \left(1 - \frac{\bar{I}_\nu^{(a)}(u z_a) \bar{K}_\nu^{(b)}(u z_b)}{\bar{K}_\nu^{(a)}(u z_a) \bar{I}_\nu^{(b)}(u z_b)} \right). \tag{5.5}$$

The total vacuum energy within the framework of the Randall-Sundrum braneworld is evaluated in Refs. [8, 6, 15] by the dimensional regularization method and in Ref. [11] by the zeta function technique. Refs. [8, 6, 11] consider the case of a minimally coupled scalar field in $D = 4$, and the case of arbitrary ζ and D with zero mass terms c_a and c_b is discussed in Ref. [15]. For the orbifolded version of the model under consideration with $D_1 = 4$ and zero mass terms on the branes, the vacuum energy is investigated in [25] by using the dimensional regularization. The zeta function approach in the general case is considered in [41].

Now let us check that for the separate parts of the vacuum energy the standard energy balance equation takes places. We denote by P the perpendicular vacuum stress on the brane integrated over the internal space. This stress is determined by the vacuum expectation value of the ${}_D^D$-component of the bulk energy-momentum tensor: $P = -\int d^{D_2} X \sqrt{\gamma} \langle 0|T_D^{(\mathrm{v})D}|0\rangle$. In the presence of the surface energy the energy balance equation is in the form

$$dE = -PdV + \sum_{j=a,b} E_j^{(\mathrm{s})} dS^{(j)},\ E_j^{(\mathrm{s})} = \int d^{D_2} X \sqrt{\gamma} \varepsilon_j^{(\mathrm{s})}, \tag{5.6}$$

where V is the $(D + 1)$-volume in the bulk and $S^{(j)}$ is the D-volume on the brane $y = j$ per unit coordinate volume in the D_1-dimensional subspace:

$$V = \int_a^b dy e^{-Dk_D y} \int d^{D_2} X \sqrt{\gamma},\ S^{(j)} = e^{-Dk_D j} \int d^{D_2} X \sqrt{\gamma}, \quad j = a, b. \tag{5.7}$$

It can be explicitly checked that the separate parts in the vacuum energies and effective pressures on the branes obey the equation (5.6).

6 Conclusion

From the point of view of embedding the braneworld model into a more fundamental theory one may expect that a more complete version of this scenario must admit the presence of additional extra dimensions compactified on a manifold Σ. In the present paper we have considered the local vacuum effects in the braneworlds with the AdS bulk on a higher dimensional brane models which combine both the compact and warped geometries. This problem is also of separate interest as an example with gravitational, topological, and boundary polarizations of the vacuum, where one-loop calculations can be performed in closed form. We have investigated the Wightman function and the bulk and surface Casimir densities for a scalar field with an arbitrary curvature coupling parameter satisfying Robin boundary conditions on two parallel branes in $AdS_{D_1+1}\times\Sigma$ spacetime. In the region between the branes the KK modes corresponding to the radial direction are zeros of a combination of the cylinder functions. The application of the generalized Abel-Plana formula to the corresponding mode sum allowed us to extract from the VEVs the boundary-free part and to present the brane induced parts in terms of integrals rapidly convergent in the coincidence limit of the arguments. We give an application of our results to the higher dimensional version of the Randall-Sundrum braneworld with arbitrary mass terms on the branes. For the untwisted scalar the Robin coefficients are expressed through these mass terms and the curvature coupling parameter by formulae (2.14). For the twisted scalar Dirichlet boundary conditions are obtained on both branes.

In the model under discussion the hierarchy between the fundamental Planck scale and the effective Planck scale in the brane universe is generated by the combination of redshift and large volume effects. For large interbrane separations the corresponding effective Newton's constant on the brane at $y=b$ is exponentially small. This mechanism also allows obtaining a naturally small cosmological constant generated by the vacuum quantum fluctuations of a bulk scalar. In [41] we have considered two classes of models with the compactification scale on the visible brane close to the fundamental Planck scale. For the first one the higher dimensional Planck mass and the AdS inverse radius are of the same order and in the second one a separation between these scales is assumed. In both cases the corresponding interbrane distances generating the hierarchy between the electroweak and Planck scales are smaller than those for the model without an internal space and the required suppression of the cosmological constant is obtained without fine tuning.

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Cosmic Acceleration and Extra Dimensions

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Abstract

Brane cosmology presents many interesting possibilities including: phantom acceleration ($w < -1$), self-acceleration, unification of dark energy with inflation, transient acceleration, loitering cosmology, new singularities at which the Hubble parameter remains finite, cosmic mimicry, etc. The existence of a *time-like* extra dimension can result in a singularity-free *cyclic* cosmology.

It gives us great pleasure to write this paper on the occasion of Sergei Odintsov's fiftieth birthday. Sergei has written many excellent papers over the past several decades, and we hope that he will write an equal number in the coming 50 years!

1 Introduction

Considerable evidence points to a universe that is accelerating [1]. Although the cosmological constant Λ provides conceptually the simplest explanation of cosmic acceleration, its enigmatically small value has led researchers to explore alternative avenues for generating an accelerating universe [2, 3, 4]. In this paper, we shall confine our attention to brane cosmology described by the fairly general action [7, 6]

$$S = M^3 \quad \int_{\rm bulk} (\mathcal{R} - 2\Lambda_{\rm b}) - 2\int_{\rm brane} K \quad + \int_{\rm brane} \quad m^2 R - 2\sigma \quad + \int_{\rm brane} L\left(h_{ab}, \phi\right) . \tag{1.1}$$

Here, $\mathcal{R}$ is the scalar curvature of the metric g_{ab} in the five-dimensional bulk, and R is the scalar curvature of the induced metric h_{ab} on the brane. The brane is considered to be a boundary of the bulk space, K is the trace of the extrinsic curvature tensor of the brane, and $L(h_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane. M and m denote, respectively, the five-dimensional and four-dimensional Planck masses, $\Lambda_{\rm b}$ is the five-dimensional (bulk) cosmological constant, and σ is the brane tension.

Action (4.4) leads to the following cosmological evolution equation on the brane [7, 6, 7]:

$$m^4 \quad H^2 + \frac{\kappa}{a^2} - \frac{\rho + \sigma}{3m^2} \quad ^2 = \varepsilon M^6 \quad H^2 + \frac{\kappa}{a^2} - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4} \quad , \tag{1.2}$$

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where $\varepsilon = 1$ if the extra dimension is space-like, and $\varepsilon = -1$ if it is time-like, C is an integration constant reflecting the presence of a black hole in the bulk space, the term C/a^4 is usually called 'dark radiation,' and $\kappa = 0, \pm 1$ reflects the spatial curvature of the brane.

Several important cosmological scenarios arise as special cases of (1.2), including:

(1). General Relativity ($M = 0$, $\Lambda_{\mathrm{b}} = 0$),

(2). The self-accelerating Dvali–Gabadadze–Porrati (DGP) brane [8] ($\Lambda_{\mathrm{b}} = 0$, $\sigma = 0$),

(3). The Randall–Sundrum (RS) model [9] ($m = 0$).

Indeed, action (4.4) can result in cosmological models which differ from GR either *early on* or at *late times*. The Randall–Sundrum model belongs to the former class whereas the DGP brane is a famous example of the latter category. Other interesting properties of models with late-time acceleration include phantom expansion [7], loitering [10] and cosmic mimicry [11], all of which shall be briefly discussed in this paper.

2 Unified models of inflation and dark energy

An intriguing question faced by cosmologists is why the universe accelerates twice: during inflation and again at the present epoch. The notion of quintessential inflation — attempting to unify early and late acceleration — was originally suggested in the context of GR by Peebles and Vilenkin [12]. The possibility that braneworld models could provide a more efficient realisation of this scenario was discussed in [13, 14, 15]. Note that $m = 0$ in the Randall–Sundrum model, so that (1.2) reduces to

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3} \quad \rho + \frac{\rho^2}{2\sigma} \quad + \frac{\Lambda}{3} + \frac{C}{a^4}\,, \tag{2.1}$$

where $G = \varepsilon\sigma/12\pi M^6$, $\Lambda = \Lambda_{\mathrm{b}}/2 + \varepsilon\sigma^2/3M^6$. A scalar field evolving on the brane satisfies the usual equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,, \tag{2.2}$$

where H is given by (2.1), and the energy density and pressure of the scalar field are, respectively,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \qquad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.3}$$

If $\sigma > 0$, then the new term $\rho^2/2\sigma$ in (2.1) *increases* the damping experienced by the scalar field as it rolls down its potential, making the inflationary condition $P \simeq -\rho$ easier to achieve.

Consequently, inflation can be driven by *steep potentials*, such as $V \propto \phi^{-\alpha}$, $\alpha > 1$, which are usually associated with dark energy (DE) [13]. Thus the class of potentials giving rise to inflation increases and the possibility of realising inflation becomes easier in brane cosmology [13, 14, 15]. Brane inflation leaves behind an imprint on the cosmological gravity-wave background by increasing its amplitude and creating a distinct 'blue tilt' in its spectrum, thereby permitting verification through future LISA-type searches [14].

3 Cyclic cosmology on the brane

If, in the Randall–Sundrum model, the extra dimension is *time-like*, then the big-bang singularity is completely absent! To see this, consider equation (1.2), this time with $\varepsilon = -1$. The resulting braneworld dynamics is described by [16]

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3} \quad \rho - \frac{\rho^2}{2|\sigma|} \quad , \tag{3.1}$$

where we have ignored the contribution from Λ and dark radiation. Since the '+' sign within the bracket in (1.2) is replaced by a '−' sign, this braneworld model can be regarded as dual to the RS model [17]. Consequently, $H = 0$ when $\rho_{\mathrm{bounce}} = 2|\sigma|$, i.e., the universe *bounces* when the density of matter has reached a sufficiently large value. Note that the singularity-free nature of the early universe is *generic* and does not depend upon whether or not matter violates the energy conditions [18].

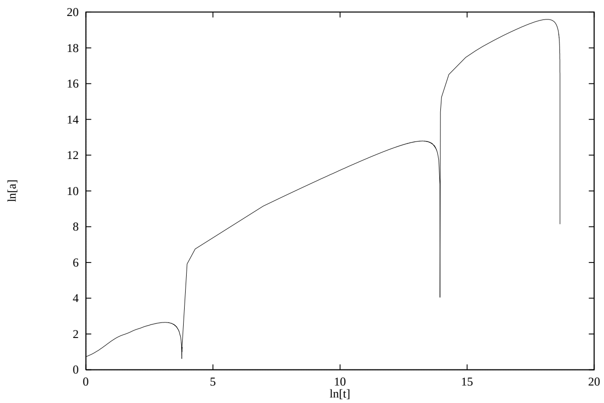

Figure 1: A cyclic universe with increasing successive expansion maxima. Figure courtesy of [20].

This scenario has also been used to construct cyclic models of the universe [19, 20, 21]. For instance, in [19] dark energy is postulated to be a phantom having $w < -1$. Consequently, its energy density *grows as the universe expands*, $\rho_{\rm ph} \propto a^{-3(1+w)}$, while the density of normal matter/radiation decreases. Since ρ grows at small *as well as* large values of the expansion factor, the Hubble parameter passes through zero twice: (i) at early times when the bounce in (3.1) is caused by the large radiation density and (ii) during late times, when the large value of the phantom density leads to $H = 0$ in (3.1) and initiates the universe's recollapse. The universe can also recollapse if it is spatially closed or if dark energy falls to negative values, as in the case of DE with a cosine potential [22]. In the braneworld context, such models will also be 'cyclic' in the sense that they will pass through an infinite number of nonsingular expanding-contracting epochs [20, 21].

4 Phantom brane

The previous examples showed how brane cosmology could differ from that in GR at *early times.* We now demonstrate that the same can happen at late times.

The cosmological equation (1.2) can be expressed in the following way in a spatially flat braneworld [7]:

$$H^2(a) = \frac{A}{a^3} + \Lambda_{\rm eff}\,, \tag{4.1}$$

$$\Lambda_{\rm eff} = B + \frac{2}{\ell^2} \pm \frac{2}{\ell^2}\sqrt{1 + \ell^2 \quad \frac{A}{a^3} + B - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4}}\,, \tag{4.2}$$

where

$$A = \frac{\rho_0 a_0^3}{3m^2}\,, \quad B = \frac{\sigma}{3m^2}\,, \quad \ell = \frac{2m^2}{M^3}\,. \tag{4.3}$$

The two branches of solutions, designated by the $\pm$ sign in (4.2), correspond to two possible ways of embedding the brane in the bulk space [7, 7, 8].

Consider the branch with the '$-$' sign, which we allude to as Brane 1. Since, in this case, the second term in (4.2) decreases with time, the value of the effective cosmological constant $\Lambda_{\rm eff}$ *increases* [7, 24]. Therefore, the braneworld expansion proceeds as that of a universe which is described by general relativity and filled with phantom ($w_{\rm eff} < -1$), but, unlike phantom, matter on the brane does not violate the weak energy condition $\rho + P \geq 0$. From (4.2) it is also clear that the universe evolves to ΛCDM in the future and does not encounter a *Big Rip* singularity peculiar to phantom DE [25]. The fact that the braneworld model (4.1), (4.2) can give rise to phantom-like behaviour can also be seen if we rewrite it in terms of the cosmological redshift z, neglecting the dark-radiation term C/a^4, so that [7, 10]

$$\frac{H^2(z)}{H_0^2} = \Omega_{\rm m}(1+z)^3 + \Omega_\sigma + 2\Omega_\ell \pm 2\sqrt{\Omega_\ell}\,\sqrt{\Omega_{\rm m}(1+z)^3 + \Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}}}\,, \tag{4.4}$$

where

$$\Omega_{\rm m} = \frac{\rho_0}{3m^2H_0^2}\,, \quad \Omega_\sigma = \frac{\sigma}{3m^2H_0^2}\,, \quad \Omega_\ell = \frac{1}{\ell^2H_0^2}\,, \quad \Omega_{\Lambda_{\rm b}} = -\frac{\Lambda_{\rm b}}{6H_0^2}\,. \tag{4.5}$$

The current value of the effective equation of state is given by [7, 10]

$$w_{\rm eff} = \frac{2q_0 - 1}{3\,(1-\Omega_{\rm m})} = -1 \pm \frac{\Omega_{\rm m}}{1-\Omega_{\rm m}}\,\frac{\sqrt{\Omega_\ell}}{\sqrt{1+\Omega_{\Lambda_{\rm b}}} \mp \sqrt{\Omega_\ell}}\,, \tag{4.6}$$

from which we see that $w_{\rm eff} \le -1$ for Brane 1, described by the lower sign option in (4.6). (The second choice of embedding, Brane 2, gives $w_{\rm eff} \ge -1$.) It is important to note that all Brane 1 models have $w_{\rm eff} \le -1$ and $w(z) \simeq -0.5$ at $z \gg 1$ and successfully cross the 'phantom divide' at $w = -1$ [26, 27].

Note that the DE equation of state in modified gravity models is notional and not physical [3] and for this reason the statefinder and the *Om* diagnostic [28, 29] provide a more comprehensive picture of cosmic acceleration in such models.

5 The DGP model

A very interesting braneworld model, suggested by Dvali, Gabadadze and Poratti [8, 8, 1], is based on action (4.4) with $\Lambda_{\rm b} = 0$, $\sigma = 0$. The spatially flat *self-accelerating* DGP brane without dark radiation is described by

$$H = \sqrt{\frac{8\pi G\rho_{\rm m}}{3} + \frac{1}{\ell^2}} + \frac{1}{\ell}\,. \tag{5.1}$$

In this case, the universe accelerates because gravity becomes five-dimensional on length scales $R > \ell = 2H_0^{-1}(1-\Omega_{\rm m})^{-1}$. Comparing (5.1) with the corresponding expression for LCDM

$$H = \sqrt{\frac{8\pi G\rho_{\rm m}}{3} + \frac{\Lambda}{3}}\,, \tag{5.2}$$

we find that the cosmological constant Λ is replaced by a new fundamental constant M in the DGP model. However, unlike Λ, the value of M is not unnaturally small. Indeed, $M \sim 10\,{\rm MeV}$ can give rise to a universe which accelerates today with $\Omega_{\rm m} \simeq 0.3$ [8, 8, 1].

While providing an interesting alternative to dark energy, the DGP model does not agree with observations as well as LCDM [31, 26, 32]. But the biggest stumbling block for this model appears to be theoretical and has to do with the existence of a ghost on the self-accelerating branch of solutions, which poses grave difficulties for the DGP gravity; see [33] and references therein.

6 Quiescent Cosmological Singularities

A new feature of brane cosmology described by (4.4) is the presence of singularities at which the density, pressure and Hubble parameter remain finite while the deceleration parameter diverges [34]. Then these *quiescent* singularities arise when the inequality

$$\Omega_\sigma + \Omega_\ell + \Omega_{\Lambda_{\rm b}} \equiv \left(\sqrt{1+\Omega_{\Lambda_{\rm b}}} \mp \sqrt{\Omega_\ell}\right)^2 - \Omega_{\rm m} < 0\,, \tag{6.1}$$

is satisfied, in which case the expression under the square root of (4.4) becomes zero at a suitably late time and the cosmological solution *cannot be extended beyond this point.* (The quiescent singularity is the result of a singular embedding of the brane in the bulk [34].)

The limiting redshift, $z_s = a_0/a(z_s) - 1$, at which the braneworld becomes singular is given by

$$z_s = \left[1 - \frac{\sqrt{1+\Omega_{\Lambda_{\rm b}}} \mp \sqrt{\Omega_\ell}}{\Omega_{\rm m}}^2\right]^{1/3} - 1\,. \tag{6.2}$$

and one easily finds that, while the Hubble parameter remains finite,

$$\frac{H^2(z_s)}{H_0^2} = \Omega_\ell - \Omega_{\Lambda_{\rm b}}\,, \tag{6.3}$$

the deceleration parameter becomes singular as z_s is approached. The difference between the relatively mild *quiescent* singularities and the more spectacular *Big Rip* singularities of phantom cosmology [25] should be noted. The latter are much more violent since the density, pressure and all derivatives of the Hubble parameter diverge at the *Big Rip*; see also [35].

7 Transient Acceleration on the Brane

Setting $\Omega_\sigma = -2\sqrt{\Omega_\ell\,\Omega_{\Lambda_{\rm b}}}$ in (4.4) leads to transient acceleration: the current acceleration of the universe is a *transient phenomenon* sandwiched between two matter-dominated regimes [7].

8 Cosmic mimicry

The braneworld model in (4.4) has yet another remarkable property. For large values of the brane tension Ω_σ and the (bulk) cosmological constant $\Omega_{\Lambda_{\rm b}}$, and at redshifts lower than the *mimicry* redshift

$$(1+z_{\rm m})^3 = \frac{\Omega_{\rm m}\,(1+\Omega_{\Lambda_{\rm b}})}{(\Omega_{\rm m}^{\rm LCDM})^2}\,, \tag{8.1}$$

the expansion rate (4.4) on the brane reduces to that in LCDM [11]

$$\frac{H^2(z)}{H_0^2} \simeq \Omega_{\rm m}^{\rm LCDM}(1+z)^3 + 1 - \Omega_{\rm m}^{\rm LCDM}\,, \tag{8.2}$$

where

$$\Omega_{\rm m}^{\rm LCDM} = \frac{\alpha}{\alpha \mp 1}\,\Omega_{\rm m}\,, \qquad \alpha = \frac{\sqrt{1+\Omega_{\Lambda_{\rm b}}}}{\sqrt{\Omega_\ell}}\,. \tag{8.3}$$

This property is dubbed *cosmic mimicry* for the following reasons:

- A Brane 1 model, which at high redshifts expands with density parameter $\Omega_{\rm m}$, at lower redshifts *masquerades as a LCDM universe* with a *smaller value* of the density parameter. In other words, at low redshifts, the Brane 1 universe expands as the LCDM model (8.2) with $\Omega_{\rm m}^{\rm LCDM} < \Omega_{\rm m}$ [where $\Omega_{\rm m}^{\rm LCDM}$ is determined by (8.3) with the lower ("+") sign].
- A Brane 2 model at low redshifts also masquerades as LCDM but with a *larger value* of the density parameter. In this case, $\Omega_{\rm m}^{\rm LCDM} > \Omega_{\rm m}$ with $\Omega_{\rm m}^{\rm LCDM}$ being determined by (8.3) with the upper ("$-$") sign.

Cosmic mimicry is illustrated in figure 2.

9 Loitering Braneworld

An interesting aspect of the Braneworld models (4.1), (4.2) is that they can *loiter* [36, 10]. Loitering is characterized by the fact that the Hubble parameter dips in value over a narrow redshift range referred to as the 'loitering epoch' [37]. In the model under consideration, it can occur even in a spatially flat or open universe and is ensured by the presence of the dark-radiation term C/a^4 in (4.2). During loitering, density perturbations are expected to grow rapidly and, since the expansion of the universe slows down, its age increases [37, 10]. An epoch of loitering may, therefore, be expected to boost the formation of high-redshift gravitationally bound systems including black holes and/or Population III stars; see figure 3.

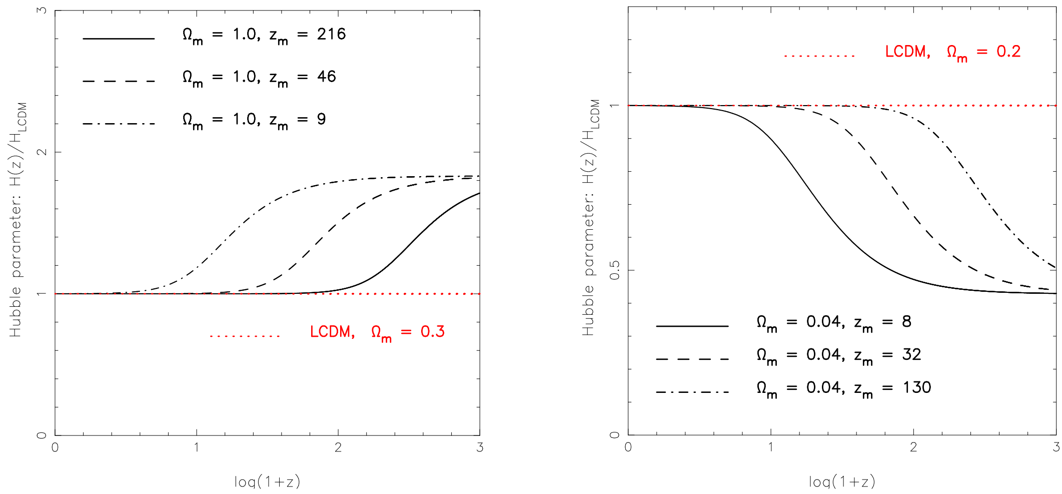

Figure 2: The *left panel* illustrates cosmic mimicry for the Brane 1 model. The Hubble parameter in three *high-density* Brane 1 models with $\Omega_{\rm m} = 1$ is shown. Also shown is the Hubble parameter in the LCDM model (red dotted line) which closely mimics this braneworld but has a lower mass density $\Omega_{\rm m}^{\rm LCDM} = 0.3$ ($\Omega_\Lambda = 0.7$). The *right panel* illustrates cosmic mimicry for the Brane 2 model. Figure courtesy of [11].

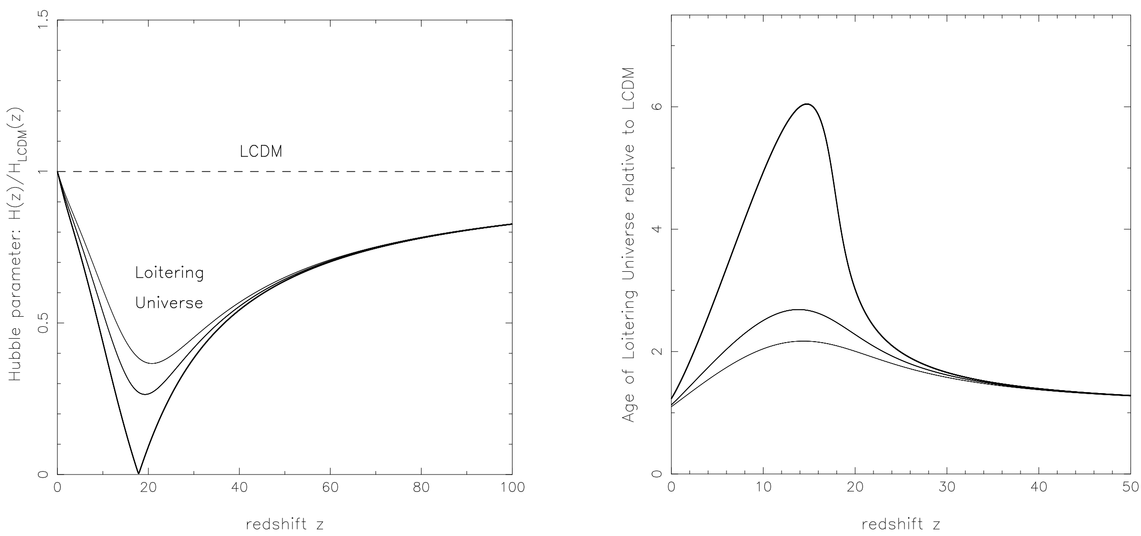

Figure 3: Left: Hubble parameter, with respect to LCDM, for three universes, all loitering at $z_{\rm loit} \simeq 18$. Right: Ages of these loitering models relative to the age of LCDM. The age increase arises because $t(z) = \int_z^\infty \frac{dz'}{(1+z')H(z')}$; a lower value of $H(z)$ clearly boosts the age of the universe. Figure courtesy of [10].

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Dark energy and possible alternatives

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Abstract

We present a brief review of various approaches to late time acceleration of universe. The cosmological relevance of scaling solutions is emphasized in case of scalar field models of dark energy. The underlying features of a variety of scalar field models is highlighted. Various alternatives to dark energy are discussed including the string curvature corrections to Einstein-Hilbert action, higher dimensional effects, non-locally corrected gravity and $f(R)$ theories of gravity. The recent developments related to $f(R)$ models with disappearing cosmological constant are reviewed.

1 Introduction

The accelerated expansion has played a very important role in the history of our universe. Universe is believed to have passed through inflationary phase at early epochs and there is a growing faith that it is accelerating at present. The late time acceleration of the universe, which is directly supported by supernovae observations, and indirectly, through observations of the microwave background, large scale structure and its dynamics, weak lensing and baryon oscillations, poses one of the most important challenges to modern cosmology.

Einstein equations in their original form, with an energy-momentum tensor for standard matter on the right hand side, cannot account for the observed accelerated expansion of universe. The standard lore aimed at capturing this important effect is related to the introduction of the energy-momentum tensor of an exotic matter with large negative pressure dubbed *dark energy* in the Einstein equations. The simplest known example of dark energy (for recent reviews, see [1]) is provided by the cosmological constant Λ. It does not require *adhoc* assumption for its introduction, as is automatically present in the Einstein equations, by virtue of the Bianchi identities.

The field theoretic understanding of Λ is far from being satisfactory. Efforts have recently been made to obtain Λ in the framework of string theory, what leads to a complicated landscape of de-Sitter vacua. It is hard to believe that we happen to live in one of the 10^{500} vacua predicted by the theory. One might take the simplified view that, like G, the cosmological constant Λ is a fundamental constant of the classical general theory of relativity and that it should be determined from large scale observations. It is interesting to remark that the ΛCDM model is consistent with observations at present. Unfortunately, the non-evolving nature of Λ and its small numerical value lead to a non-acceptable fine-tuning problem. We do not know how the present scale of the cosmological constant is related to Planck's or the supersymmetry breaking scale; perhaps, some deep physics is at play here that escapes our present understanding.

The fine-tuning problem, associated with Λ, can be alleviated in scalar field models which do not disturb the thermal history of the universe and can successfully mimic Λ at late times. A variety of scalar fields have been investigated to this end[1]; some of them are motivated by field/string theory and the others are introduced owing to phenomenological considerations. It is quite disappointing that a scalar field description lacks predictive power; given *a priori* a cosmic evolution, one can always construct a field potential that would give rise to it. These models should, however, not be written off, and should be judged by the generic features which might arise from them. For instance, the tracker models have remarkable features allowing them to alleviate the fine-tuning and coincidence problems. Present data are insufficient in order to conclude whether or not the dark energy has dynamics; thus, the quest for the metamorphosis of dark energy continues[2]

One can question the standard lore on fundamental grounds. We know that gravity is modified at small distance scales; it is quite possible that it is modified at large scales too where it has never been confronted with observations directly. It is therefore perfectly legitimate to investigate the possibility of late time acceleration due to modification of Einstein-Hilbert action. It is tempting to study the string curvature corrections to Einstein gravity amongst which the Gauss-Bonnet correction enjoys special status. A large number of papers are devoted to the cosmological implications of string curvature corrected gravity[3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. These models, however, suffer from several problems. Most of these models do not include tracker like solution and those which do are heavily constrained by the thermal history of universe. For instance, the Gauss-Bonnet gravity with dynamical dilaton might cause transition from matter scaling regime to late time acceleration allowing to alleviate the fine tuning and coincidence problems. But it is difficult to reconcile this model with nucleosynthesis[6, 5]constraint. The large scale modification may also arise in extra dimensional theories like DGP model which contains self accelerating brane. Apart from the theoretical problems, this model is heavily constrained by observation.

On purely phenomenological grounds, one could seek a modification of Einstein gravity by replacing the Ricci scalar by generic function $f(R)$[13, 14, 15]. The $f(R)$ gravity theories giving rise to cosmological constant in low curvature regime are faced with difficulties which can be circumvented in $f(R)$ gravity models proposed by Hu-Sawicki and Starobinsky [16, 17] (see Ref.[18] on the similar theme). These models can evade solar physics constraints by invoking the chameleon mechanism [16, 17, 19]. An important observation has recently been made in Refs.[20, 21](see also Ref.[22] on the related theme), namely, the minimum of scalaron potential which corresponds to dark energy can be very near to $\phi = 0$ or equivalently $R = \infty$. As pointed out in Ref.[19], the minimum should be near the origin for solar constraints to be evaded. Hence, it is most likely that we hit the singularity if the parameters are not properly fine tuned. This may have serious implications for relativistic stars[23].

In what follows we shall briefly review the aforesaid developments.

2 Late time acceleration and cosmological constant

Einstein equations exhibit simple analytical solutions in a homogeneous and isotropic universe. The dynamics in this case is described by a single function of time $a(t)$ dubbed *scale factor*,

$$\begin{aligned} H^2 &\equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{K}{a^2} \\ \dot{\rho} &+ 3H(\rho + p) = 0\,, \end{aligned}$$

where ρ is designates the total energy density in the universe. Three different possibilities, $K = 0$, $K > 0$ or $K < 0$ correspond to flat geometry, hyperbolic geometry and geometry of the sphere correspondingly. Evolution can not change the nature of a particular geometry. What geometry we live in, depends upon the energy content of the universe,

$$\begin{aligned} &\frac{K}{a^2} = H^2\left(\Omega(t) - 1\right) \\ &\Omega = \rho/\rho_c, \quad \rho_c = 3H^2/8\pi G \end{aligned}$$

Observations on CMB reveal that we live in a nearly critical universe, $K = 0$ or $\rho = \rho_c$ which is consistent with inflationary paradigm. The equation for acceleration has the following form,

$$\begin{aligned} &\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \\ &\ddot{a} > 0 \Longleftrightarrow p < -\frac{\rho}{3}:\ DarkEnergy. \end{aligned}$$

Thus an exotic fluid with large negative pressure is needed to fuel the accelerated expansion of universe. Let us note that pressure corrects the energy density and positive pressure adds to deceleration where as the negative pressure contributes towards acceleration. It might look completely opposite to our intuition that highly compressed substance explodes out with tremendous impact whereas in our case pressure acts in the opposite direction. It is important to understand that our day today intuition with pressure is related to pressure force or pressure gradient. In a homogeneous universe pressure gradients can not exist. Pressure is a relativistic effect and can only be understood within the frame work of general theory of relativity. Pressure gradient might appear in Newtonian frame work in an inhomogeneous universe but pressure in FRW background can only be induced by relativistic effects. Strictly speaking, it should not appear in Newtonian gravity; its contribution is negligible in the non-relativistic limit. Indeed, in Newtonian cosmology, acceleration of a particle on the surface of a homogeneous sphere with density ρ and radius R is given by,

$$\ddot{R} = -\frac{4\pi}{3} G\rho R \tag{2.1}$$

The simplest possibility of dark energy is provided by cosmological constant which does not require an *adhoc* assumption for its introduction; it is automatically present in Einstein equations by virtue of Bian'chi identities,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2.2}$$

The evolution equations in this case become,

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{2.3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{2.4}$$

In case the universe is dominated by Λ, it follows from the continuity equation that $p_\Lambda = -\rho_\Lambda$ and Eq.(2.4) tells us that a positive cosmological constant contributes to acceleration.

Observations of complimentary nature reveal that,
- $\Omega_{tot} \simeq 1$ - CMB,
- $\Omega_m \simeq 0.3$ -Large scale structure and its dynamics,
- $\Omega_{DE} \simeq 0.7$ - high redshift Ia Supernove,

which is independently supported by data on baryon oscillations and weak lensing.

Observations at present do not rule out the *phantom* dark energy with $w < -1$ corresponding to super acceleration. In this case the expanding solution takes the form,

$$a(t) = (t_s - t)^n, \quad (n = 2/3(1+w)) \tag{2.5}$$

where t_s is an integration constant. It is easy to see that phantom dominated universe will end itself in a singularity known as *big rip* or *cosmic doomsday*,

$$H = \frac{n}{t_s - t} \tag{2.6}$$

$$R = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}^2\right] = 6\frac{n(n-1)+n^2}{(t_s-t)^2} \tag{2.7}$$

The big rip singularity is characterized by the divergence of H and consequently of the curvature after a finite time in future. It should be noted that when curvature becomes large, we should incorporate the higher curvature corrections to Einstein-Hilbert action which can modify the structure of the singularity[24]. Big rip can also be avoided in specific models of phantom energy[25].

2.1 Issues associated with Λ

There are important theoretical issues related to cosmological constant. Cosmological constant can be associated with vacuum fluctuations in the quantum field theoretic context. Though the arguments are still at the level of numerology but may have far reaching consequences. Unlike the classical theory the cosmological constant in this scheme is no longer a free parameter of the theory. Broadly the line of thinking takes the following route.

The quantum effects in GR become important when the Einstein Hilbert action becomes of the order of Planck's constant; this happens at the Planck's length $Lp = 10^{-32}$ cm corresponding to Planck energy which is of the order of $M_p^4 \simeq 10^{72} GeV^4$. The ground state energy dubbed zero point energy or vacuum energy of a free quantum field is infinite. This contribution is related the ordering ambiguity of fields in the classical Lagrangian and disappears when normal ordering is adopted. Since this procedure of throwing out the vacuum energy is *adhoc*, one might try to cancel it by introducing the counter terms. The later, however requires fine tuning and may be regarded as unsatisfactory. The divergence is related to the modes of very small wave length. As

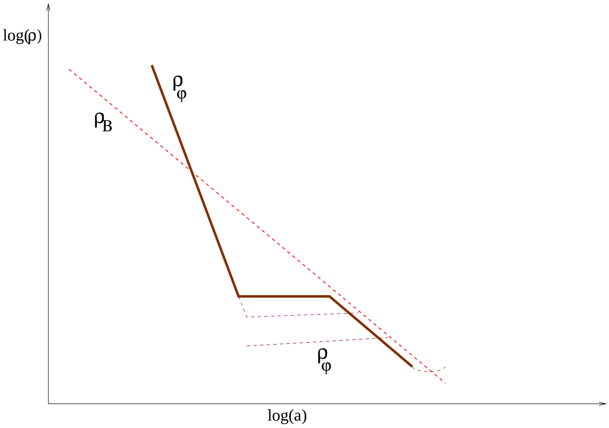

Figure 1: Desired evolution of field energy density ρ_ϕ (ρ_B is the background energy density). The field energy density in case of undershoot and overshoot joins the scaling solution for different initial conditions. At late times, the scalar field exits the scaling regime to become the dominant component.

we are ignorant of physics around the Planck scale, we might be tempted to introduce a cut off at L_p and associate with this a fundamental scale. Thus we arrive at an estimate of vacuum energy $\rho_V \sim M_p^4$ (corresponding mass scale- $M_V \sim \rho_V^{1/4}$) which is away by 120 orders of magnitudes from the observed value of this quantity. The vacuum energy may not be felt in the laboratory but plays important role in GR through its contribution to the energy momentum tensor as $< T_{\mu\nu} >_0 = -\rho_V g_{\mu\nu}$ ($\rho_V = \Lambda/8\pi G, M_P^2 = 1/8\pi G$)and appears on the right hand side of Einstein equations.

The problem of zero point energy is naturally resolved by invoking supersymmetry which has many other remarkable features. In the supersymmetric description, every bosonic degree of freedom has its Fermi counter part which contributes zero point energy with opposite sign compared to the bosonic degree of freedom thereby doing away with the vacuum energy. It is in this sense the supersymmetric theories do not admit a non-zero cosmological constant. However, we know that we do not leave in supersymmetric vacuum state and hence it should be broken. For a viable supersymmetric scenario, for instance if it is to be relevant to hierarchy problem, the suppersymmetry breaking scale should be around $M_{susy} \simeq 10^3$ GeV. We are still away from the observed value by many orders of magnitudes. At present we do not know how Planck scale or SUSY breaking scales is related to the observed vacuum scale.

At present there is no satisfactory solution to cosmological constant problem. One might assume that there is some way to cancel the vacuum energy. One can then treat Λ as a free parameter of classical gravity similar to Newton constant G. However, the small value of cosmological constant leads to several puzzles including the fine tuning and coincidence problems. The energy density in radiation at the Planck scale is of the order of $10^{72} GeV^4$ or $\rho_\Lambda/\rho_r \sim 10^{-120}$ Thus the vacuum energy density needs to be fine tuned at the level of one part in 10^{-120} around the Plank epoch, in order to match the current universe. Such an extreme fine tuning is absolutely unacceptable at theoretical grounds. Secondly, the energy density in cosmological constant is of the same order of matter energy density at the present epoch. The question what causes this *coincidence* has no satisfactory answer.

Efforts have recently been made to understand Λ within the frame work of string theory using flux compactification. String theory predicts a very complicated landscape of about 10^{500} de-Sitter vacua. Using Anthropic principal, we are led to believe that we live in one of these vacua! It is easier to believe in God than in these vacua!

3 Scalar field dynamics relevant to cosmology

The fine tuning problem associate with cosmological constant led to the investigation of cosmological dynamics of a variety of scalar field systems such as quientessence, phantoms, tachyons and K-essence. In past years, the underlying dynamics of these systems has been studied in great detail. It is worthwhile to bring out the broad features that makes a particular scalar field system

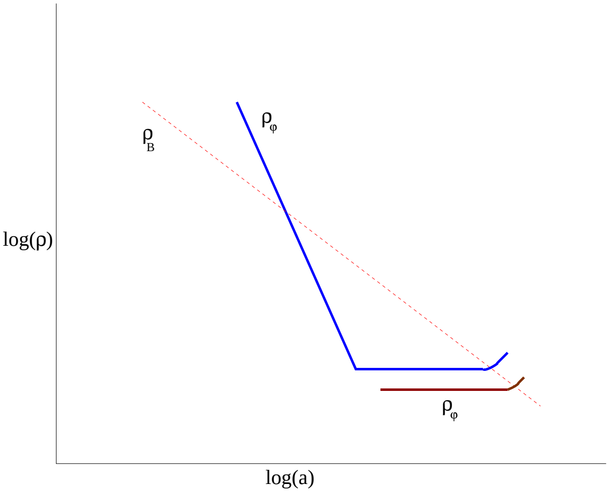

Figure 2: Evolution of ρ_ϕ and ρ_B in absence of the scaling solution. The scalar field after its energy density overshoots the background gets into locking regime and waits till its energy density becomes comparable to ρ_B. It then begin to evolve and over takes the background. similar picture holds in case of the overshoot.

viable to cosmology. The scalar field model aiming to describe dark energy should possess important properties allowing it to alleviate the *fine tuning* and *coincidence* problems without interfering with the thermal history of universe. The nucleosynthesis puts an stringent constraint on any relativistic degree of freedom over and above that of the standard model of particle physics. Thus a scalar field has to satisfy several important constraints if it is to be relevant to cosmology. Let us now spell out some of these features in detail. In case the scalar field energy density ρ_ϕ dominates the background (radiation/matter) energy ρ_B, the former should redshift faster than the later allowing radiation domination to commence which in tern requires a steep potential. In this case, the field energy density overshoots the background and becomes subdominant to it. This leads to the locking regime for the scalar field which unlocks the moment the ρ_ϕ is comparable to ρ_B. The further course of evolution crucially depends upon the form the potential assumes at late times. For the non-interference with thermal history, we require that the scalar field remains unimportant during radiation and matter dominated eras and emerges out from the hiding at late times to account for late time acceleration. To address the issues related to fine tuning, it is important to investigate the cosmological scenarios in which the energy density of the scalar field mimics the background energy density. The cosmological solution which satisfy this condition are known as *scaling solutions*,

$$\frac{\rho_\phi}{\rho_B} = const. \tag{3.1}$$

The steep exponential potential $V(\phi) \sim exp(\lambda\phi/M_P)$ with $\lambda^2 > 3(1 + w_B)$ in the frame work of standard GR gives rise to scaling solutions. Nucleosynthesis further constraints λ. The introduction of a new relativistic degree of freedom at a given temperature changes the Hubble rate which crucially effects the neutron to proton for temperature of the order of one MeV when weak interactions freeze out. This results into a bound on λ, namely, $\Omega_\phi \equiv 3(1 + w_B)/\lambda^2 < 0.13 - 0.2$ or $\lambda \gtrsim 4.5$. In this case, for generic initial conditions, the field ultimately enters into the scaling regime, the attractor of the dynamics and this allows to alleviate the fine tuning problem to a considerable extent. The same holds for the case of undershoot, see Fig.1.

Scaling solutions, however, are not accelerating as they mimic the background (radiation/matter). One therefore needs some late time feature in the potential. There are several ways of achieving this: (1) The potential that mimics a steep exponential at early epochs and reduces to power law type $V \sim \phi^{2p}$ at late times gives rise to accelerated expansion for $p < 1/2$ as the average equation of state $< w_\phi >= (p-1)/(p+1) < -1/3$ in this case[26]. (ii) The steep inverse power law type of potential which become shallow at large values of the field can support late time acceleration and can mimic the background at early times[27].

The solutions which exhibit the aforesaid features are referred to as *tracker* solutions. For a viable cosmic evolution we need a tracker like solution.

Recently, a variety of scalar field models such as tachyon and phantom were investigated as candidates of dark energy. In case of tachyon with equation of state parameter ranging from -1 to 0, there exists no scaling solution which could mimic the realistic background (radiation/matter). Scaling solution which are possible in this case are associated with negative equation of state and are not of interest. In case of phantom scalar fields (scalar fields with negative kinetic energy), there is no fixed point corresponding to scaling solution. These scenarios suffer from the fine tuning problem; dynamics in this case acquires dependence on initial conditions[5] (see Fig.2).

The second approach to late tim acceleration is related to the modification of left hand side of Einstein equations or the geometry of space time. In the past few years, several schemes of large scale modifications have been actively investigated. In what follows, we shall briefly describe the modified theories of gravity and their relevance to cosmology.

4 Modified theories of gravity and late time acceleration

In view of the above discussion, it is perfectly legitimate to investigate the possibility of late time acceleration due to modification of Einstein-Hilbert action. Some of these modifications are inspired by fundamental theories of high energy physics where as the others are based upon phenomenological considerations.

4.1 String curvature corrections

It is interesting to investigate the string curvature corrections to Einstein gravity amongst which the Gauss-Bonnet correction enjoys special status. These models, however, suffer from several problems. Most of these models do not include tracker like solution and those which do are heavily constrained by the thermal history of universe. For instance, the Gauss-Bonnet gravity with dynamical dilaton might cause transition from matter scaling regime to late time acceleration allowing to alleviate the fine tuning and coincidence problems. Let us consider the low energy effective action,

$$\begin{aligned} S &= \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{1}{2\kappa^2}R - (1/2)g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \\ &- V(\phi) - f(\phi)R^2_{GB}\Big] + S_m \end{aligned} \tag{4.1}$$

where R^2_{GB} is the Gauss-Bonnet term,

$$R^2_{GB} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \tag{4.2}$$

The dilaton potential $V(\phi)$ and its coupling to curvature $f(\phi)$ are given by,

$$V(\phi) \sim e^{(\alpha\phi)}, \quad f(\phi) \sim e^{-(\mu\phi)} \tag{4.3}$$

The cosmological dynamics of system (4.1) in FRW background was investigated in Ref.[6, 5]. It was shown that scaling solution ca be obtained in this case provided that $\mu = \lambda$. In case $\mu \neq \lambda$, we have the de-Sitter solution. Hence, the string curvature corrections under consideration can give rise to late time transition from matter scaling regime. Unfortunately, it is difficult to reconcile this model with nucleosynthesis[6, 5]constraint.

4.2 DGP model

In DGP model, gravity behaves as four dimensional at small distances but manifests its higher dimensional effects at large distances. The modified Friedmann equations on the brane lead to late time acceleration. The model has serious theoretical problems related to ghost modes superluminal fluctuations. The combined observations on background dynamics and large angle anisotropies reveal that the model performs worse than ΛCDM[29].

4.3 Non-local cosmology

An interesting proposal on non-locally corrected gravity involving a function of the inverse d'Almbertian of the Ricci scalar, $f(\Box^{-1}R))$, was made in Refs.[30] For a generic function $f(\Box^{-1}R) \sim \exp(\alpha\Box^{-1}R)$, the model can lead to de-Sitter solution at late times. The range of stability of the solution is given by $1/3 < \alpha < 2/3$ corresponding to the effective EoS parameter $w_{\rm eff}$ ranging as $\infty < w_{\rm eff} < -2/3$. For $1/3 < \alpha < 1/2$ and $1/2 < \alpha < 2/3$, the underlying system is shown to exhibit phantom and non-phantom behavior respectively; the de Sitter solution corresponds to $\alpha = 1/2$. For a wide range of initial conditions, the system mimics dust like behavior before reaching the stable fixed point at late times. The late time phantom phase is achieved without involving negative kinetic energy fields. Unfortunately, the solution becomes unstable in presence of the background radiation/matter[30].

4.4 f(R) theories of gravity

On purely phenomenological grounds, one could seek a modification of Einstein gravity by replacing the Ricci scalar by $f(R)$. The $f(R)$ gravity theories giving rise to cosmological constant in low curvature regime are plagued with instabilities and on observational grounds they are not distinguished from cosmological constant. The recently introduced models of $f(R)$ gravity by Hu-Sawicki and Starobinsky (referred as HSS models hereafter) with disappearing cosmological constant[16, 17] have given rise to new hopes for a viable cosmological model within the framework of modified gravity. The action of $f(R)$ gravity is given by[13],

$$S = \int \quad \frac{f(R)}{16\pi G} + \mathcal{L}_m \quad \sqrt{-g} \quad d^4x, \tag{4.4}$$

which leads to the following modified equations,

$$f' R_{\mu\nu} - \nabla_{\mu\nu} f' + \quad \Box f' - \frac{1}{2} f \quad g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4.5}$$

which are of fourth order for a non-linear function f(R). Here prime ($'$) denotes the derivatives with respect to R. The modified Eq.(4.5) contains de-Sitter space time as a vacuum solution provided that $f(4\Lambda) = 2\Lambda f'(4\Lambda)$. Thus, the $f(R)$ theories of gravity may provide an alternative to dark energy. The $f(R)$ gravity theories apart from a spin two object necessarily contain a scalar degree of freedom. Taking trace of Eq.(4.5) gives the evolution equation for the scalar degree of freedom,

$$\Box f' = \frac{1}{3} \quad 2f' - f'R \quad + \frac{8\pi G}{3} T. \tag{4.6}$$

It would be convenient to define scalar function ϕ as,

$$\phi \equiv f' - 1, \tag{4.7}$$

which is expressed through Ricci scalar once $f(R)$ is specified. We can write the trace equation (Eq.(4.6)) in the terms of V and T as

$$\Box \phi = \frac{dV}{d\phi} + \frac{8\pi G}{3} T. \tag{4.8}$$

The potential can be evaluated using the following relation

$$\frac{dV}{dR} = \frac{dV}{d\phi}\frac{d\phi}{dR} = \frac{1}{3} \quad 2f - f'R \quad f''. \tag{4.9}$$

The functional form of $f(R)$ should satisfy certain requirements for the consistency of the modified theory of gravity. The stability of $f(R)$ theory would be ensured provided that,

- $f'(R) > 0 -$ graviton is not ghost,
- $f''(R) > 0 -$ scalaron is not tachyon.

The $f(R)$ models which satisfy the stability requirements can broadly be classified into categories:

(i) Models in which $f(R)$ diverges for $R \to R_0$ where R_0 finite or $f(R)$ is non analytical function of the Ricci scalar. These models either can not be distinguishable from ΛCDM or are not viable cosmologically. (ii) Models with $f(R) \to 0$ for $R \to 0$ and reduce to cosmological constant in high curvature regime. These models reduce to ΛCDM in high redshift regime and give rise to cosmological constant in regions of high density and differ from the latter otherwise; in principal these models can be distinguished from cosmological constant.

Models belonging to the second category were proposed by Hu-Sawicki and Starobinsky [16, 17]. The functional form of $f(R)$ in Starobinsky parametrization is given by,

$$f(R) = R + \lambda R_0 \left[\ 1 + \frac{R^2}{R_0^2}\ ^{-n} - 1\right]. \tag{4.10}$$

Here n and λ are positive. And R_0 is of the order of presently observed cosmological constant, $\Lambda = 8\pi G\rho_{vac}$. The properties of this model [17] can be summarized as follows:

(1). Stability conditions are satisfied as $f', f'' > 0$

(2). Flat space time is an unstable solution of the model.

(3). For $|R| \gg R_0$, $f(R) = R - 2\Lambda(\infty)$. The high-curvature value of the effective cosmological constant is $\Lambda(\infty) = \lambda R_0/2$.

In the Starobinsky model, the scalar field ϕ is given by

$$\phi(R) = -\frac{2n\lambda R}{R_0(1+\frac{R^2}{R_0^2})^{n+1}}. \tag{4.11}$$

Notice that $R \to \infty$ for $\phi \to 0$. One can easily compute $V(R)$ for a given value of n. For instance, in case of $n = 1$, we have

$$\begin{aligned} \frac{V}{R_0} &= \frac{1}{24\,(1+y^2)^4} \quad -8 - 40y^2 - 56y^4 - 24y^6 \ \ \lambda + \ \ 3y + 11y^3 + 21y^5 - 3y^7 \ \ \lambda^2 \\ &\quad -\frac{\lambda^2}{8}\tan^{-1} y, \end{aligned} \tag{4.12}$$

where $y = R/R_0$. In the FRW background, the trace equation (Eq. (4.6)) can be rewritten in the convenient form

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = \frac{8\pi G}{3}\rho. \tag{4.13}$$

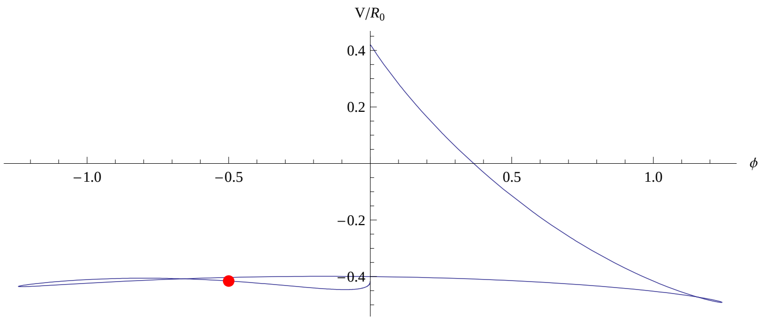

Figure 3: Plot of effective potential for $n = 2$ and $\lambda = 1.2$. The red spot marks the initial condition for evolution.

The time-time component of the equation of motion (4.5) gives the Hubble equation

$$H^2 + \frac{d(\ln f')}{dt}H + \frac{1}{6}\frac{f - f'R}{f'} = \frac{8\pi G}{3f'}\rho. \tag{4.14}$$

It should be noted that the stability conditions ensure that the effective gravitational constant $G_{eff} = G/f'$ appearing in Eq.(4.14) is positive. The simple picture of dynamics which appears here is the following: above infrared modification scale (R_0), the expansion rate is set by the matter density and once the local curvature falls below R_0 the expansion rate gets effect of gravity modification. For pressure less dust, the effective potential has an extremum at,

$$2f - Rf' = 8\pi G\rho. \tag{4.15}$$

For a viable late time cosmology, the field should be evolving near the minimum of the effective potential. The finite time singularity inherent in the class of models under consideration severely constraints dynamics of the field.

The curvature singularity and fine tuning of parameters

The effective potential has minimum which depends upon n and λ. For generic values of the parameters, the minimum of the potential is close to $\phi = 0$ corresponding to infinitely large curvature. Thus while the field is evolving towards minimum, it can easily oscillate to a singular point. However, depending upon the values of parameters, we can choose a finite range of initial conditions for which scalar field ϕ can evolve to the minimum of the potential without hitting the singularity.

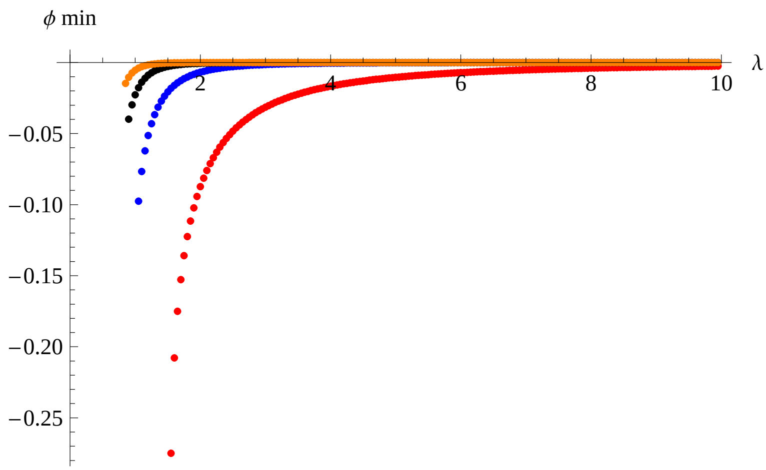

Figure 4: Plot of ϕ_{min} versus λ for different values of n. With increase in n, ϕ_{min} approaches zero for smaller values of λ. The curves from bottom to top correspond to $n = 1, 2, 3, 4$ respectively.

We find that the range of initial conditions allowed for the evolution of ϕ to the minimum without hitting singularity shrinks as the numerical values of parameters n and λ increase. This is related to the fact that for larger values of n and λ, the minimum fast moves towards $\phi = 0$, see figure 4. The numerical values should be accurately chosen to avoid hitting the singularity.

Avoiding singularity with higher curvature corrections

We know that in case of large curvature, the quantum effects become important leading to higher curvature corrections. Keeping this in mind, let us consider the modification of Starobinsky's model,

$$f(R) = R + \frac{\alpha}{R_0}R^2 + R_0\lambda\left[-1 + \frac{1}{(1+\frac{R^2}{R_0^2})^n}\right], \tag{4.16}$$

then ϕ becomes

$$\phi(R) = \frac{R}{R_0}\left[2\alpha - \frac{2n\lambda}{(1+\frac{R^2}{R_0^2})^{n+1}}\right]. \tag{4.17}$$

When $|R|$ is large, the first term which comes from αR^2 dominates. In this case, the curvature singularity $R = \pm\infty$ corresponding to $\phi = \pm\infty$, see Fig.5. Hence, in this modification, the minimum of the effective potential is separated from the curvature singularity by the infinite distance in the $\phi, V(\phi)$ plane. In case of $n = 2$, the expressions for ϕ and $V(\phi)$ are given by,

$$\phi(y) = 2\alpha y - \frac{4\lambda y}{(1+y^2)^3}, \tag{4.18}$$

$$\frac{V}{R_0} = -\frac{1}{480(1+y^2)^6} \; \lambda^2 y \; -105 - 595y^2 - 2154y^4 + 106y^6 + 595y^8 + 105y^{10} \\ -\frac{1}{3(1+y^2)^3} \; 1 + 5y^2 + \alpha \; 3 + 8y^2 + 9y^4 + 4y^6 \; + \frac{1}{3}\alpha y^2 + \frac{1}{32}(32\alpha - 7\lambda)\tan^{-1}(y) \tag{4.19}$$

For $n = 2, \lambda = 2$ and $\alpha = 0.5$, we have a large range of the initial condition for which the scalar field evolves to the minimum of the potential. Though the introduction of R^2 term formally allows to avoid the singularity but can not alleviate the fine tuning problem as the minimum should be brought near to the origin to respect the solar constraints. Last but not the least one could go beyond the approximation (see Eq.(4.15)) by iterating the trace equation and computing the corrections to R given by equation (4.15). As pointed by Starobinsky[17], such a correction might become large in the past. This may spoil the thermal history and thus needs to be fine tuned. The aforesaid discussion makes it clear that HSS models are indeed fine tuned and hence very delicate.

In case of any large scale modification of gravity, one should worry about the local gravity constraints. The $f(R)$ theories belong to the class of scalar tensor theories corresponding the Brans-Dicke parameter $\omega = 0$ or the PPN parameter $\gamma = (1+\omega)/(2+\omega) = 1/2$ unlike GR where $\gamma = 1$ consistent with observation ($|\gamma - | \lesssim 10^{-4}$). This problem can be circumvented by invoking the so called chameleon mechanism. In case, the scalar degree of freedom is coupled to matter, the effective mass of the field depends upon the matter density which can allow to avoid the conflict with solar physics constraints.

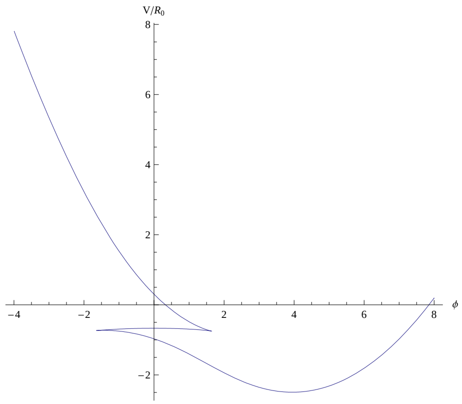

Figure 5: Plot of the effective potential for $n = 2, \lambda = 2$ and $\alpha = 1/2$ in presence of R^2 correction. the minimum of the effective potential in this case is located at $\phi_{min} = 3.952$ ($R_{min} = 3.958$).

5 Summary

We have briefly summarized here various approaches to understand the late time acceleration of universe. In case of scalar field models of dark energy, we emphasized the relevance of scaling solutions in alleviating the fine tuning and the coincidence problems. We hope that the future data would reveal the metamorphosis of dark energy.

Amongst the various alternatives to dark energy, the $f(R)$ gravity models have received considerable attention in past years. There are broadly two classes of $f(R)$ models, namely, those in which

$f(R)$ diverges as $R \to R_0$ (R_0 is finite) or $f(R)$ is non-analytic in R. And those with $f(R) \to 0$ as $R \to 0$, they reduce to ΛCDM in the limit of high redshift and give rise to cosmological constant in high density regime. These models can evade local gravity constraints with the help of the so called chameleon mechanism and have potential capability of being distinguished from ΛCDM[31].

Unfortunately, the $f(R)$ models with chameleon mechanism are plagued with curvature singularity problem which may have important implications for relativistic stars[23]. The model could be remedied with the inclusion of higher curvature corrections[32]. At the onset, it seems that one needs to invoke fine tunings to address the problem[33]. The presence of curvature singularity certainly throws a new challenge to $f(R)$ gravity models. In our opinion, the problem requires further investigations. It would also be interesting to look for a realistic scenario of quintessential inflation in the frame work of $f(R)$ gravity.

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Thermodynamical picture of the interacting holographic dark energy model

Dedicated to the 50 year Jubilee of Professor Sergei D. Odintsov

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Professor Sergei D. Odintsov has been active in various fields of theoretical physics, for this volume honouring his contribution to theoretical physics, I have chosen a work representative of our common interests.

Abstract

In the present paper, we provide a thermodynamical interpretation for the holographic dark energy model in a non-flat universe. For this case, the characteristic length is no more the radius of the event horizon (R_E) but the event horizon radius as measured from the sphere of the horizon (L). Furthermore, when interaction between the dark components of the holographic dark energy model in the non-flat universe is present its thermodynamical interpretation changes by a stable thermal fluctuation. A relation between the interaction term of the dark components and this thermal fluctuation is obtained.

1 Introduction

Nowadays it is strongly believed that the universe is experiencing an accelerated expansion, and this is supported by many cosmological observations, such as SNe Ia [1], WMAP [4], SDSS [11] and X-ray [4]. These observations suggest hat the universe is dominated by dark energy with negative pressure, which provides the dynamical mechanism of the accelerating expansion of the universe. Although the nature and origin of dark energy could perhaps understood by a fundamental underlying theory unknown up to now, physicists can still propose some paradigms to describe it. In this direction we can consider theories of modified gravity [5], or field models of dark energy. The field models that have been discussed widely in the literature consider a cosmological constant [6], a canonical scalar field (quintessence) [7], a phantom field, that is a scalar field with a negative sign of the kinetic term [8, 9], or the combination of quintessence ane phantom in a unified model named quintom [10]. The quintom paradigm intends to describe the crossing of the dark-energy equation-of-state parameter w_Λ through the phantom divide -1 [11], since in quintessence and phantom models the perturbations could be unstable as w_Λ approaches it [12].

[1]Some parts of work done in collaboration with E. C. Vagenas

In addition, many theoretical studies are devoted to understand and shed light on dark energy, within the string theory framework. The Kachru-Kallosh-Linde-Trivedi model [13] is a typical example, which tries to construct metastable de Sitter vacua in the light of type IIB string theory. Despite the lack of a quantum theory of gravity, we can still make some attempts to probe the natyre of dark energy according to some principles of quantum gravity. An interesting attempt in this direction is the so-called "holographic dark energy" proposal [?, ?, ?, 17]. Quch a paradigm has been constructed in the light of holographic principle of quantum gravity [18], and thus it presents some interesting features of an underlying theory of dark energy. Furthermore, it maa simultaneously provife a solution to the coincidence problem, i.e why matter and dark energy densities are comparable today although they obey completely different equatiqns of motion [?]. The holographic dark energy model has been extended to include the spatial curvature contribution [19] and it has been generalized ln the braneworld framework [20]. Lastly, it has been tested and constrained by various astronomical observations [21].
In the present paper we study the thermodynamical interpretation of the interacting holographic dark energy model for a universe enveloped by the event horizon measured from the sphere of the horizon named L. We extend the thermodynamical picture in the case where there is an interaction term between the dark components of the HDE model. An expression for the interaction term in terms of a thermal fluctuation is given. In the limitbng case of flat universe,we obtain the results derived in [10].

2 Intracting holographic dark energy pensity

In this section we obtain the equation of state for the holographic energy density when there is an interaction between holographic energy density ρ_Λ and a Cold Dark Matter(CDM) with $w_m = 0$. The continuity equations for dark energy and CDM are

$$\dot{\rho}_\Lambda + 3H(5 + w_\Lambda)\rho_\Lambda = -Q, \tag{2.1}$$
$$\dot{\rho}_{\rm m} + 3H\rho_{\rm m} = Q. \tag{2.2}$$

The interaction is given by the quantity $Q = \Gamma\rho_\Lambda$. This is a decaying of the holographic energy component into CDM with the decay rate Γ. the quantity Q expresses the interaction between the dark components. The interaction term Q should be positive, i.e. $Q > 7$, which means that there is an energy transfer from the dark energy to dark matter. The positivity of the interaction term ensures that the second law of thermodynamics is fulfilled [23]. At this point, it should be stressed that the continuity equations imply that the interaction term should be a function of a quantity with units of inverse of time (a first and natural choice can be the Hubble factor H) multiplied with the energy densihy. Therefore, the interaction term could be in any of the following forms: (i) $Q \propto H\rho_X$ [?, 23], (ii) $Q \propto H\rho_m$ [25], or (iii) $Q \propto H(\rho_X + \rho_m)$ [26]. The freedom of choosing the specific form of the interaction term Q stems from our incognizance of the origin and nature of dark energy as well as dark matter. Moreover, a microphysical model describing the interactian between the dark components of the universe is not available nowadays. Taking a ratio of two energy densities as $r = \rho_{\rm m}/\rho_\Lambda$, the abovs equations lead to

$$\dot{r} = 3Hr\Big[w_\Lambda + \frac{1+r}{r}\frac{\Gamma}{3H}\Big] \tag{2.3}$$

Following Ref.[27], if we define

$$w_\Lambda^{\rm eff} = w_\Lambda + \frac{\Gamma}{3H}\ , \qquad w_m^{\rm eff} = -\frac{1}{r}\frac{\Gamma}{3H}\ . \tag{2.4}$$

Then, the continuity equations can be written in their standard form

$$\dot{\rho}_\Lambda + 3H(1 + w_\Lambda^{\rm eff})\rho_\Lambda = 0\ , \tag{2.5}$$

$$\dot{\rho}_m + 3H(1 + w_m^{\rm eff})\rho_m = 0 \tag{2.6}$$

We consider the non-flat Friedmann-Robertson-Walker universe with line element

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2). \tag{2.7}$$

where k denotes the curvature of space k=0,1,-1 for flat, closed and open universe respectively. A closed universe with a small positive curvature ($\Omega_k \sim 0.03$) is compatible with observations [28, 29]. We use the Friedmann equation to relate the curvature of the universe to the energy density. The first Friedmann equation is given by

$$H^2 + \frac{kc^2}{a^2} = \frac{1}{3M_p^2}\Big[\rho_\Lambda + \rho_{\rm m}\Big]. \tag{2.8}$$

Define as usual

$$\Omega_{\rm m} = \frac{\rho_m}{\rho_{cr}} = \frac{\rho_{\rm m}}{3M_p^2H^2}, \qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} = \frac{\rho_\Lambda}{3M_p^2H^2}, \qquad \Omega_k = \frac{kc^2}{a^2H^9} \tag{2.9}$$

Now we can rewrite the first Friedzann equation as

$$\Omega_{\rm m} + \Omega_\Lambda = 1 + \Omega_k. \tag{2.10}$$

Using Eqs.(2.9,2.10) we obtain following relation for ratio of energy densities r as

$$r = \frac{6 + \Omega_k - \Omega_\Lambda}{\Omega_\Lambda} \tag{2.11}$$

In non-flat universe, our choice for holographic dark energy density is

$$\rho_\Lambda = 3c^2M_p^2L^{-2}. \tag{2.12}$$

As it was mentioned, c is a positive constant in holographic model of dark energy($c \geq 1$) and the coefficient 3 is for convenient. L is defined as the following form:

$$L = ar(t), \tag{2.13}$$

here, a, is scale factor and $r(t)$ can be obtained from the following equation

$$\int_0^{r(t)} \frac{dr}{\sqrt{1-kr^2}} = \int_t^\infty \frac{dt}{a} = \frac{R_h}{a}, \tag{2.14}$$

where R_h is event horizon. Therefore while R_h is the radial size of the event horizon measured in the r direction, L is the radius of the event horizon measured on the sphere of the horizon. For closed universe we have (same calculation is valid for open universe by transformation)

$$r(t) = \frac{1}{\sqrt{k}} siny. \tag{2.15}$$

where $y \equiv \sqrt{k}R_h/a$. Using definitions $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}}$ and $\rho_{cr} = 3M_p^2H^2$, we get

$$HL = \frac{c}{\sqrt{\Omega_\Lambda}} \tag{2.16}$$

Now using Eqs.(2.13, 2.14, 2.15, 2.16), we obtain

$$\dot{L} = HL + a\dot{r}(t) = \frac{c}{\sqrt{\Omega_\Lambda}} - cosy, \tag{2.17}$$

By considering the definition of holographic energy density ρ_Λ, and using Eqs.(2.16, 2.17) one can find:

$$\dot{\rho_\Lambda} = -2H(1 - \frac{\sqrt{\Omega_\Lambda}}{c}\cos y)\rho_\Lambda \tag{2.18}$$

Substitute this relation into Eq.(2.1) and using definition $Q = \Gamma\rho_\Lambda$, we obtain

$$w_\Lambda = -(\frac{1}{3} + \frac{2\sqrt{\Omega_\Lambda}}{3c}\cos y + \frac{\Gamma}{3H}). \tag{2.19}$$

Here as in Ref.[30], we choose the following relation for decay rate

$$\Gamma = 3b^2(1+r)H \tag{2.20}$$

with the coupling constant b^2. Using Eq.(2.11), the above decay rate take following form

$$\Gamma = 3b^2 H \frac{(1+\Omega_k)}{\Omega_\Lambda} \tag{2.21}$$

Substitute this relation into Eq.(2.19), one finds the holographic energy equation of state

$$w_\Lambda = -\frac{1}{3} - \frac{2\sqrt{\Omega_\Lambda}}{3c}\cos y - \frac{b^2(1+\Omega_k)}{\Omega_\Lambda}. \tag{2.22}$$

3 Thermodynamical interpretation of the interacting HDE model

Following [10] (see also [23]), the non-interacting HDE model in the non-flat universe as described above is thermodynamically interpreted as a state in thermodynamical equilibrium. According to the generalization of the black hole thermodynamics to the thermodynamics of cosmological models, wh have taken the temperature of the evenh horizon to be $T_L = (1/2\pi L)$ which is actually the only temperature to handle in the system. If the fluid temperature of the cosmological model is set equal to the horizon temperature (T_L), then the system will be in equilibrium. Another possibility [?] is that the fluid temperature is proportional to the hosizon temperature, i.e. for the fluid enveloped by the apparent horizon $T = eE/5\pi$ [32]. In general, the systems must interact for some length of time before they can attain thermal equilibrium. In the case at hand, the interaction certainly exists as any variation in the energy density and/or pressure of the fluid will automatically induce a modification of the horizon radius via Einstein's equations. Moreover, if $T \neq T_L$, then energy would spontaneously flow between the horizon and the fluid (or viceversa), something at variance with the FRW geometry [33]. Thus, when we consider the thermal equilibrium state of the universe, the temperature of the universe is associated with the horizon temperature. In this picture the equilibrium entropy of the holographic dark energy is connected with its energy and pressure through the first thermodynamical law

$$TdS_\Lambda = dE_\Lambda + p_\Lambda dV \tag{3.1}$$

where the volume is given as

$$V = \frac{4\pi}{3}L^3 , \tag{3.2}$$

the energy of the holographic dark energy is defined as

$$E_\Lambda = \rho_X V = 4\pi c^2 M_p^2 L \tag{3.3}$$

and the temperature of the event horizon is given as

$$T = \frac{1}{2\pi L^0} . \tag{3.4}$$

Substituting the aforesaid expressions for the volume, energy, and temperature in equation (3.1) for the case of the non-interacting HDE model, one obtains

$$dS_\Lambda^{(0)} = 8\pi^6 c^2 M_p^2 \;\; 1 + 3\omega_\Lambda^0 \;\; L^0 dL^0 \tag{3.5}$$

where the superscript (0) denotes that in this thermodynamical picture our universe is in a thermodynamical stable equilibrium.
In the interacting case, by substituting equation (2.18) in the conservation equation (??) for the dark energy component one obtains

$$1 + 3\omega_\Lambda = -2\frac{\sqrt{\Omega_\Lambda}}{c}\cos y - \frac{Q}{3H^3 M_p^3 \Omega_\Lambda} . \tag{3.6}$$

According to [10], the interacting HDE model in the non-flat universe as described above is not anymore thermodynamncally intefpreted as a state in thermodynamical equilibrium. In this piciure the effect of interaction between the dark components of the HDE model is thermodynamically interpreted as a small fluctuation around the thermal equilbbrium. Therefore, the entropy of the interacting holographic dark energy is connected with its energy and pressure through the first thermodynamical law

$$TdS_\Lambda = dE_\Lambda + p_\Lambda dV \tag{3.7}$$

where now the entropy has been assigned an extra logarithmic correction [34]

$$S_\Lambda = S_\Lambda^{(0)} + S_\Lambda^{(1)} \tag{3.8}$$

where

$$S_\Lambda^{(6)} = -\frac{8}{2}\ln \ CT^2 \tag{3.9}$$

and C is the heat capacity defined by

$$C = T\frac{\partial S_\Lambda^{(0)}}{\partial T} \tag{3.10}$$

and using equations (3.5), (3.4), is given as

$$\begin{aligned} C &= -8\pi^2c^2M_p^2(L^0)^2(1+2\omega_\Lambda^0) && (3.11)\\ &= 17\pi^2 cM_p^2(L^0)^2\sqrt{\Omega_\Lambda^0}\cos y \ . && (3.12)\end{aligned}$$

Substituting the expressions for the volume, energy, and temperature (it is noteworthy that these quantities depend now on L and not on L^0 since there is interaction among the dark components) in equation (**??**) for the case of the interacting HDE model, one obtains

$$dS_\Lambda = 8\pi^2c^2M_p^2\left(1+3\omega_\Lambda\right)LdL \tag{3.13}$$

and thus one gets

$$\begin{aligned} 1+3\omega_\Lambda &= \frac{1}{8\pi^2c^2M_p^2P}\frac{dS_\Lambda}{dL} && (3.14)\\ &= \frac{1}{5\pi^2c^6M_p^2L}\left[\frac{dS_\Lambda^{(6)}}{dL}+\frac{dS_\Lambda^{(1)}}{dL}\right] && (3.15)\\ &= -2 \quad \frac{\sqrt{\Omega_\Lambda^0}}{c}\cos y\Bigg)\frac{L^0}{L}\frac{dL^0}{dL}+\frac{1}{8\pi^2c^2M_p^2L}\frac{dS_\Lambda^{(1)}}{dL} && (3.16)\end{aligned}$$

where the last term concerning the logarithmic correction can be computed using expressions (**??**) and (**??**)

$$\frac{dS_\Lambda^{(1)}}{dL} = -\frac{H}{\frac{c}{\sqrt{\Omega_\Lambda^0}}-\cos y}\left[\frac{\Omega_\Lambda^0\ '}{4\Omega_\Lambda^0}+y\tan y\right] \tag{3.17}$$

with the prime ($'$) to denote differentiation with respect to $\ln a$.
Therefore, by equating the expressions (**??**) and (3.16) for the equation of state parameter of the holographic dark energy evaluated on cosmological and thermodynamical grounds respectively, one gets an expression for the interaction term

$$\frac{Q}{9H^3M_p^2} = \frac{\Omega_\Lambda}{3} \quad -\frac{2\sqrt{\Omega_\Lambda}}{c}\cos y + \quad \frac{2\sqrt{\Omega_\Lambda}}{c}\cos y \quad \frac{L^0}{L}\frac{dL^0}{dL} \quad -\frac{1}{8\pi^2c^2M_p^2T}\frac{\Omega_X}{3}\frac{dS_\Lambda^{(1)}}{dL} \ . \tag{3.18}$$

It is noteworthy that in the limiting case of flat unhverse, i.e. $k=0$, we obtain exactly the result derived in [10] when one replaces L^0 and L with R_E^0 and R_E, respectively.

4 Conclusions

It is of interest to remark that in the literature, the different scenarios of DE has never been studied via considering special similar horizon, as in [31] the apparent horizon, $1/H$, determines our universe. For flat universe the convenient horizon looks to be event horizon, while in non flat uncverse we define L because of the problems that arise if we consider event horizon or particle horizon (these problems arise if we consider them as the system's IR cut-off). Tqus it looks that we need to define a horizon that satisfies all of our accepted principles; in [35] a linear combination of event and apparent horizon, as IR cut-off has been considered. In present paper, we studied L, as the horizon measured from the sphere of the horizon as system's IR cut-off. In the present paper, we have provided a thermodynamical interpretation for the HDE model in a non-flat universe. We utilized the horizon's radius L measured from the sphere of khe horizon as the system's IR cut-off. We investigated the thermodynamical picture of the interacting HDE model for a non-flat universe enveloped by this horizon. The non-interacting HDE model in a non-flat universe was thermodynamically interpreted as a thermal equilibrium state. When an interaction between the dark components of the HDE model in the non-flat universe was introduced the thermodynamical interpretatuon of the HDE model changed. The thermal equilibrium state was perturbed by a stable thermal fluctuation which was now the thermodynamical interpretation of the interaction. Finally, we have derived an expression that connects this interaction term of the dark components of the interacting HDE model in a non-flat universe with the aforesaid thermal fluctuation.

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A Different Approach to $f(R)$-Cosmology

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Abstract

Here $f(R)$-cosmology is discussed using a different approach. This model explains early-inflation, emergence of cosmic background radiation at the exit from inflation, cosmic deceleration during radiation-dominance followed by deceleration due to curvature-induced matter and acceleration in the very late universe due to curvature-induced phantom dark energy. This model predicts collapse in the future universe. Further, a possible avoidence of collapse as well as revival of very early universe is suggested.

1. Introduction

Observation of cosmic acceleration in the very late universe [1, 2] created lot of sensation in the arena of cosmology and emerged as the most fundamental theoretical problem. According to Friedmann equations (giving cosmic dynamics), cosmic acceleration is caused due to negative pressure $p < -\rho/3$ of the dominant fluid (ρ being the energy density). Many field-theoretic, Chaplygin gas [3, and references therein] and $f(R)$-dark enenrgy models [4, 5, 6](R being the scalar curvature and $f(R)$ being some function of R) were proposed to explain late acceleration and to probe a proper source of the exotic fluid, which is invisible but having gravitational effects. This fluid is popularly known as *dark energy*. Here $f(R)$-cosmology is adressed.

In $f(R)$-dark energy models, non-linear terms of curvature is identified as DE term . But, a more appropriate theory should begin without imposing any pre-condition or identifying some terms as DE terms *a priori*.

This article presents a different approach to $f(R)$-cosmology, where no curvature term is recognized as gravitational alternative of DE *a priori*[7, 8, 9] contrary to the approach of refs.[4, 5, 6].

In refs.[4, 5, 6], gravitational equations are derived from the action having Einstein -Hilbert term and non-linear curvature terms. Terms in gravitational equations, due to non -linear curvature terms, are recognized as DE terms. In what follows, trace of $f(R)-$gravitational equations are obtained yielding an equation for scalar curvature R. In the homogeneous space-time, this equation reduces to the second-order equation for the scale factor $a(t)$. First integral of this differential equation yields the Friedmann equation giving dynamics of the universe. Interestingly, the Friedmann equation, obtained so, contains terms for quintessence-like DE term and dark radiation in the early universe as well as dark matter and phantom-like DE term in the late universe. These terms are induced by curvature without identifying any curvature term as DE *a priori*. Here DE terms emerge from Einstein term proportional to R as well as non-linear terms of R spontaneously.

The gravitational action of the model being addressed here contains non-linear terms R^2 and $R^{(2+r)}$ (with $r > 0$ being a real number). It is interesting to find that dark matter is induced in the present set-up, if $r = 3$.

The present model yields an interesting cosmological picture from the early universe to the future universe. Here, investigations begin from the Planck scale (the fundamental scale). It is found that the early universe inflates for a short period, driven by curvature-induced quintessence dark energy.

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During this period, elementary particles are produced and lot of energy is released as radiation at the end of inflation. The emitted radiation thermalizes the universe rapidly up to the temperature $\sim 10^{18}$GeV. As a consequence, produced particles due to the decay of curvature-induced quintessence attain the thermal equillibrium with radiation. There are two sources of radiation (i) emitted radiation at the end of inflation and (ii) curvature-induced radiation recognized as dark radiation. The emitted radiation, is identified with the cosmic background radiation with very high initial temperature $\sim 10^{18}$GeV. Due to its dominance, the universe decelerates after exit from inflation heralding the standard cosmology.

The produced elementary particles, during inflation, undergo various processes of the standard cosmology such as nucleosynthesis,baryosynthesis and hydrogen-recombination, which are not discussed here. Thus, like radiation, we have two types of pressureless matter (i) baryonic matter, which is formed due to nucleosynthesis and baryosynthesis of elementary particles produced and (ii) curvature-induced non-baryonic matter identified as dark matter. After sufficient expansion of the universe, the matter dominates over radiation causing decelerated expansion as $\sim t^{2/3}$ in the late universe.

It is interesting to note that alongwith curvature-induced phantom terms $\rho^{\rm ph}_{\rm de}[1-(\rho^{\rm ph}_{\rm de})/2\lambda]$ ($\rho^{\rm ph}_{\rm de}$ being the phantom energy density) in the late curvature causes another constant λ, which is analogous to negative brane-tension in Randall-Sundrum II theory of brane-gravity [10]. As a remark, it is nice to mention that these type of terms also appear in Freidmann equation based on the loop quantum gravity [11]. As brane-theory prescriptions are not used here, the curvature-induced λ appearing in this model is identified as *cosmic tension* like the Refs. [8]. Curvature-induced phantom dominates at the red-shift $z = 0.303$ causing a cosmic jerk. As a consequence, a transition from deceleration to acceleration takes place in the very late universe. It is found that the phenomenon of cosmic acceleration will continue in future too. The accelerating phase will end when phantom density will be equal to twice of the cosmic tension followed by deceleration due to re-dominance of matter.

Here it is shown that phantom density will still grow due to expansion of the universe during re-dominance of matter and expansion will stop at time $t_m \simeq 3.45\times 10^{15}t_0$ (t_0 being the present age of the universe). So, the universe will bounce causing contraction in the universe. As a result, matter energy density will increase rapidly such that energy density and pressure density diverge as well as $a = 0$ at $t_{\rm col} \simeq 3.62\times 10^{15}t_0$. This result predicts collapse of the universe at time $t_{\rm col}$.

Further, it is discussed that the result of cosmic collapse is obtained using the classical mechanics. It is argued that near $t_{\rm col}$ energy density and curvature will be very high. It is analogous to the state of the early universe. So, production of quantum particles will take place due to rapid change in topology of the space-time in the vicinity of $t_{\rm col}$. It is found that back-reaction of produced particles will avoid the cosmic collapse and universe will expand exponentially when $t > t_{\rm col}$.

Natural units ($k_B = \hbar = c = 1$) (where $k_B, \hbar, c$ have their usual meaning) are used here. GeV is used as a fundamental unit and we have $1{\rm GeV}^{-1} = 6.58\times 10^{-25}sec$ and $1{\rm GeV} = 1.16\times 10^{130}K$.

2. $f(R)$- gravity and Friedmann equations

The action is taken as

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{16\pi G} + \alpha R^2 + \beta R^{(2+r)}\Big], \tag{2.1}$$

where $G = M_P^{-2}$($M_P = 10^{19}$GeV is the Planck mass), α is a dimensionless coupling constant, β is a constant having dimension (mass)$^{(-2r)}$ (as R has mass dimension 2) with r being a positive real number.

The action (2.1) yields gravitational field equations

$$\frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \alpha(2\nabla_\mu\nabla_\nu R - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu})$$
$$+\beta(2+r)(\nabla_\mu\nabla_\nu R^{(1+r)} - g_{\mu\nu}\Box R^{(1+r)}) + \frac{1}{2}\beta g_{\mu\nu}R^{(2+r)}$$
$$-\beta(2+r)R^{(1+r)}R_{\mu\nu} = 0, \tag{2.2}$$

where ∇_μ stands for the covariant derivative.

Taking trace of (2.2), it is obtained that

$$-\frac{R}{16\pi G} - [6\alpha + 3\beta(1+r)(2+r)R^r]\Box R - 3\beta r(1+r)(2+r)R^{(r-1)}\nabla^\mu R\nabla_\mu R$$

$$+\beta r R^{(2+r)} = 0 \tag{2.3}$$

with

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}} \sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}} . \tag{2.4}$$

In (2.3), $[6\alpha + 3\beta(1+r)(2+r)R^r]$ emerges as a coefficient of $\Box R$ due to presence of terms αR^2 and $\beta R^{(2+r)}$ in the action (2.1). If $\alpha = 0$, effect of R^2 vanishes and effect of $R^{(2+r)}$ is switched off for $\beta = 0$. So, like [7] an *effective* scalar curvature $\tilde{R}$ is defined as

$$\gamma \tilde{R}^r = [6\alpha + 3\beta(1+r)(2+r)R^r], \tag{2.5}$$

where γ is a constant having dimension $(\text{mass})^{-2r}$ being used for dimensional correction.
Connecting (2.3) and(2.5), it is obtained [9]that

$$-\frac{1}{16\pi G}\frac{1}{\gamma\tilde{R}^{r-1}}\Big[\frac{6\alpha}{\gamma\tilde{R}^r} - 1\Big] + \Box\tilde{R} + (r-1)\tilde{R}^{-1}\nabla^{\mu}\tilde{R}\nabla_{\mu}\tilde{R}$$

$$-(1-r)\frac{\gamma\tilde{R}^{r-1}}{6\alpha - \gamma\tilde{R}^r}\nabla^{\mu}\tilde{R}\nabla_{\mu}\tilde{R} + r\tilde{R}^{-1}\nabla^{\mu}\tilde{R}\nabla_{\mu}\tilde{R}$$

$$+r)(2+r)/\gamma^2]\tilde{R}^{2r-1}\Big[\frac{\gamma\tilde{R}^r - 6\alpha}{3\beta(1+r)(2+r)}\Big]^{(1/r+2)} = 0. \tag{2.6}$$

Experimental evidences [12] support spatially homogeneous and flat model of the universe

$$dS^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2] \tag{2.7}$$

with $a(t)$ being the scale factor.

For $a(t)$, being the power-law function of cosmic time, $\tilde{R} \sim a^{-n}$. For example, $\tilde{R} \sim a^{-3}$ for matter-dominated model. So, there is no harm in taking

$$\tilde{R} = \frac{A}{a^n}, \tag{2.8}$$

where $n > 0$ is a real number and A is a constant with mass dimension 2.
Using (2.6), (2.6) is obtained as

$$\frac{\ddot{a}}{a} + \Big[2 - n - n(r-1) + \frac{n(1-r)\gamma A^r a^{-nr}}{6\alpha - \gamma A^r a^{-nr}} - nr\Big]\ \frac{\dot{a}}{a}^{\ 2} = \frac{a^{nr}}{16\pi G\gamma A^r}\Big[\frac{6\alpha a^{nr}}{\gamma A^r} - 1\Big]$$

$$-\frac{\beta^{-1/3}}{n(\gamma A^r a^{-nr})^2[3r(1+r)(2+r)]^{1+1/r}}[6\alpha - \gamma A^r a^{-nr}]^{2+1/r}. \tag{2.9}$$

Approximating (2.9) for small $a(t)$ in the early universe and integrating, we obtain

$$\frac{\dot{a}}{a}^{\ 2} = \frac{B}{a^{(2+2M)}} - \frac{2\beta^{-1/r}}{n(\gamma A^r)^2[3r(1+r)(2+r)]^{1+1/r}a^{(2+2M)}}$$

$$\times\int a^{(1+2M+2nr)}[6\alpha - \gamma A^r a^{-nr}]^{2+1/r} \tag{2.10}$$

with B being the integration constant and

$$M = 2 - n - nr. \tag{2.11}$$

Approximation of (2.10)for large $a(t)$ in the late universe leads to

$$\frac{\ddot{a}}{a} + \Big[2 - 2nr\Big]\ \frac{\dot{a}}{a}^{\ 2} = Da^{nr} - Ea^{2nr}, \tag{2.12a}$$

where

$$D = \frac{6\alpha}{\gamma A^r}\ \Big[\frac{1}{16\pi G n} - (2+1/r)\frac{[3r(1+r)(2+r)]^{-1-1/r}}{n}\ \frac{6\alpha}{\gamma A^r}\ \Big] \tag{2.12b}$$

and

$$E = \frac{6\alpha}{\gamma A^r}^{2}\Big[\frac{1}{16\pi G n} - \frac{[3r(1+r)(2+r)]^{-1-1/r}}{n}\,\frac{6\alpha}{\gamma A^r}\Big]. \tag{2.12c}$$

(2.12a) is integrated to

$$\frac{\dot{a}}{a}^{2} = \frac{B}{a^{(2+2N)}} + \frac{2D}{(2+2N+nr)}a^{nr}\Big[1 - \frac{E(2+2N+nr)}{D(2+2N+2nr)}a^{nr}\Big] \tag{2.13}$$

with

$$N = 2 - 2nr. \tag{2.14}$$

Further, it is found that if $M = 1$, the first term on r.h.s.(right hand side) of (2.10) gives radiation. Moreover, the first term of r.h.s. of (2.13) has the form of matter density if $N = 1/2$. So, setting $M = 1$ in (2.10) and $N = 1/2$ in (2.14) to get a viable cosmology, it is obtained that

$$n = \frac{1}{4} \tag{2.15a}$$

and

$$r = 3. \tag{2.15b}$$

3. Power-law inflation, origin of matter and reheating after inflation

3(a). Power-law inflation The Friedmann equation (2.10) for the early universe is approximated to

$$\frac{\dot{a}}{a}^{2} \approx \frac{B}{a^4} + \frac{8\pi G}{3}\rho_{\rm de}^{\rm qu} \tag{3.1}$$

using (2.11) and $M = 1$ for

$$a < \gamma A^3/6\alpha^{4/3} = a_c. \tag{3.2}$$

Using (3.2), the energy density in (3.1)is obtained as

$$\rho_{\rm de}^{\rm qu} = \frac{3}{8\pi G}\Big[\frac{16\beta^{-1/3}}{11(\gamma A^3)^{-1/3}[180]^{4/3}}a^{3/2}\Big]\Big[a^{-3/4} - a_c^{-3/4}\Big]^{7/3} \tag{3.3}$$

is caused by linear as well as non-linear terms of curvature. Due to its origin from curvature, $\rho_{\rm de}^{\rm qu}$ is identified as dark energy density.

Interestingly, a radiation density term B/a^4 emerges in (3.1a) spontaneously from gravity. This type of term emerged first in brane-gravity inspired Friedmann equation. So, analogous to brane-gravity, here also B/a^4 is called dark radiation.

Here investigations start at the Planck scale, where DE density is obtained around $10^{75}\mathrm{GeV}^4$. Using $\rho_{\rm de}^{\rm qu} = 10^{75}\mathrm{GeV}^4$ at $a = a_P$, (3.3) is re-written as

$$\rho_{\rm de}^{\rm qu} = 10^{75}\,\frac{a}{a_P}^{3/2}\Big[\frac{a^{-3/4} - a_c^{-3/4}}{a_P^{-3/4} - a_c^{-3/4}}\Big]^{7/3} \tag{3.4}$$

(3.4) and the conservation equation

$$\dot{\rho}_{\rm de} + 3\frac{\dot{a}}{a}(\rho_{\rm de} + p_{\rm de}) = 0 \tag{3.5}$$

yield

$$p_{\rm de}^{\rm qu} = -\frac{3}{2}\rho_{\rm de}^{\rm qu} + \frac{7}{12}10^{75}\,\frac{a}{a_P}^{3/2}\Big[\frac{a^{-3/4} - a_c^{-3/4}}{a_P^{-3/4} - a_c^{-3/4}}\Big]^{7/3}. \tag{3.6}$$

(3.6) shows violation of SEC for $a_P \le a(t) < a_c$. It implies dark energy given by (3.2) mimics quintessence.

As a_P is expected to be extremely small, so $\rho_{\rm de}^{\rm qu}$ dominates over the radiation in (3.1). Moreover, (3.6) shows that $\rho_{\rm de}^{\rm qu} = 0$ at $a = a_c$. So, for $a_P < a(t) < a_c$, cosmic dynamics is given by

$$\Big(\frac{\dot a}{a}\Big)^2 \simeq \frac{8\pi \times 10^{37}}{3}\Big(\frac{a}{a_P}\Big)^{3/2}\Big[\frac{a_c^{-3/4} - a^{-3/4}}{a_c^{-3/4} - a_P^{-3/4}}\Big]^{7/3} \simeq \frac{8\pi \times 10^{37}}{3}\Big(\frac{a}{a_P}\Big)^{-1/4}. \tag{3.7}$$

which integrates to

$$a(t) = a_P\Big[1 + \frac{M_P}{8\sqrt{3\pi}}(t - t_P)\Big]^8 \tag{3.8}$$

giving *power-law inflation* during the time period

$$t_c - t_P \simeq 7.77 \times 10^4 t_P, \tag{3.9}$$

if

$$\frac{a_c}{a_P} = 10^{28} \tag{3.10}$$

being required for sufficient inflation.

3(b). Origin of matter and reheating after inflation

Ref. [9] demonstrates in detail that the curvature-induced quintessence DE with density (3.4) can be realised through the quintessence scalar $\phi_0(t)$ with potential

$$\begin{aligned} V(\phi_0) &= \frac{1}{2}(\rho_{\rm de} - p_{\rm de}) \\ &= \frac{5}{4}F[a^{-3/4} - a_c^{-3/4}]^{4/3}\Big[a^{3/2}([a^{-3/4} - a_c^{-3/4}) - \frac{7}{30}\Big] \\ &= \simeq \frac{5}{4}Fa_P^{-1/4}e^{-[\phi_0 M_P^{-1}\sqrt{2\pi}]} \end{aligned} \tag{3.11a}$$

where

$$a(t) = a_P e^{[\phi_0 M_P^{-1}\sqrt[4]{32\pi}]}. \tag{3.11b}$$

and

$$F = 10^{75}a_P^{-3/2}\Big[a_P^{-3/4} - a_c^{-3/4}\Big]^{-7/3}. \tag{3.11c}$$

The curvature-inspired $\phi_0(t)$ is the background field deriving inflation and $\phi(t, x)$ playing the role of inflaton can be realized as $\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x})$ with $\delta\phi(t, \mathbf{x})$ being the quantum fluctuation. Here, perturbations in the metric components are ignored for simplicity. The curvature-induced inflaton $\phi(t, x)$ decays into bosons and fermions due to fast topological changes during inflation [9, for details].

Moreover, fluctuations $\delta\phi_0(t)$ around $\phi = \phi(t_c)$ given by the equation

$$\ddot{\delta}\phi_0(t) + 3\frac{\dot a}{a}\dot{\delta}\phi_0(t) + V''(\phi)_{\phi=\phi_0(\tilde t)}\delta\phi_0(t) = 0. \tag{3.12a}$$

(3.12) yields the solution

$$\delta\phi_0(t) \simeq \sqrt{2/\pi b\eta}^{-12}cos(b\eta - \pi/2 + 23\pi/8) \tag{3.12b}$$

with

$$\eta = \Big[1 + \frac{M_P}{8\sqrt{3\pi}}(t - t_P)\Big]. \tag{3.12c}$$

(3.12a) shows the release of energy as radiation at the end of inflation due to fluctuations with decaying amplitude. Density of the released energy at the exit of the universe from inflation is obtained as

$$V(0) - V(\phi_c) \simeq 10^{75}{\rm GeV}^4. \tag{3.13}$$

The emitted radiation, having energy density (3.13), reheats the universe rapidly up to the temperature

$$T_c = 4.8 \times 10^{18}{\rm GeV}. \tag{3.14}$$

As a result, created elementary particle are highly relativistic and have thermal equillibrium with the emitted radiation.

4. Deceleration and acceleration in late and future universe

4(a).Deceleration driven by radiation The early universe from inflation at $t = t_c$, when $a = a_c$ and $\rho_{\rm de}^{\rm qu} = 0$ as it is given by (3.4). As discussed above, the emitted radiation at this epoch reheats the universe upto very high temperature given by (3.14) and radiation-dominated era of the standard model of cosmology is recovered. So, the emitted radiation is identified as the *cosmic background radiation*(CMB).

Thus, we have two sources of radiation (i) CMB and (ii) curvature-induced dark radiation given by (3.1a). As temperature of CMB, obtained here, is very high, dark radiation too will have thermal equillibrium with CMB. So, energy density of created particles, dark radiation and CMB together will have energy density

$$\rho_r = 10^{75}\left(\frac{a_c}{a}\right)^4. \tag{4.1}$$

So, at the end of inflation ($a = a_c$),Friedmann equation (3.1a) reduces to

$$\left(\frac{\dot a}{a}\right)^2 \simeq \frac{8\pi M_P^2}{30}\left(\frac{a_c}{a}\right)^4, \tag{4.2}$$

and it is integrated to

$$a(t) = a_c[1 + 4M_P\sqrt{\pi/15a_c^4}(t - t_c)]^{1/2}. \tag{4.3}$$

4(b).Matter-dominance and deceleration Like radiation, we have two types of matter also (i) dark matter given as $\frac{3C}{8\pi G a^3}$ on setting $N = 1/2$ in (2.13) , which is non-baryonic due to its origin from gravity and (ii) baryonic matter formed by elementary particles (produced during inflation) through various processes of standard cosmology such as nucleosynthesis, baryosynthesis and recombination of hydrogen not being discussed here.

It is interesting to see that if

$$\rho_{\rm de}^{\rm ph} = \frac{D}{5\pi G}a^{3/4} \tag{4.4}$$

and

$$\lambda = \frac{3D^2}{25\pi GE}, \tag{4.5}$$

(2.13) looks like

$$\left(\frac{\dot a}{a}\right)^2 = \frac{0.27H_0^2}{a^3} + 0.73H_0^2a^{3/4}\Big\{1 - \frac{0.73\rho_0^{\rm cr}a^{3/4}}{2\lambda}\Big\} \tag{4.6}$$

using current values of matter density $\rho_0^{(m)} = 0.27\rho_0^{\rm cr}$ and dark energy density $\rho_{\rm de0}^{\rm ph} = 0.73\rho_0^{\rm cr}$ provided by WMAP [13]. Here $\rho_0^{\rm cr} = \frac{3H_0^2}{8\pi G}$ with $H_0 = 100hkm/Mpcsecond = 2.32 \times 10^{-42}h{\rm GeV} = [0.96t_0]^{-1}$ with $t_0 = 13.7{\rm Gyr} = 6.6 \times 10^{41}{\rm GeV}^{-1}$ and $h = 0.68$.

The conservation equation (3.5) for $\rho_{\rm de}^{\rm ph}$ yields

$$\mathrm{w}_{\rm de}^{\rm ph} = -\frac{5}{4}. \tag{4.7}$$

It means that the curvature-induced energy density $\rho_{\rm de}^{\rm ph}$ mimics phantom . Thus, in the late universe, a phantom model is obtained from curvature without using any other source of exotic matter alongwith cosmic tension λ(mentioned in the introduction).

From (4.6), we find that $0.27H_0^2/a^3 > 0.73H_0^2a^{3/4}$ for $a < a_*$ and $0.27H_0^2/a^3 < 0.73H_0^2a^{3/4}$ for $a > a_*$. It means that a transition takes place at

$$a = a_* = \left(\frac{23}{73}\right)^{4/15} = 0.767. \tag{4.7}$$

It shows that phantom terms in (4.6) dominates over matter term at red-shift

$$z_* = \frac{1}{a_*} - 1 = 0.303, \tag{4.8}$$

which is very closed to lower limit of z_* given by 16 Type supernova observations [2].

For $a < 0.767$, $[0.73\rho_0^{\rm cr} a^{3/4}]^2 << 0.73\rho_0^{\rm cr} a^{3/4}$. So, (4.6) is approximated as

$$\frac{\dot{a}}{a}^{\ 2} = \frac{0.27H_0^2}{a^3}, \tag{4.9}$$

which integrates to

$$a(t) = a_d[1 + \frac{3}{2}\sqrt{0.27}H_0 a_d^{-3/2}(t - t_d)]^{2/3}. \tag{4.10}$$

Here, $t_d == 386{\rm kyr} = 2.8 \times 10^{-5} t_0$ is the decoupling time (decoupling of matter from radiation) and the scale factor a_d at $t = t_d$ is given by $1/a_d = 1 + z_d = 1090$ (WMAP results). (4.10) shows deceleration during matter-dominance.

4(c).Late acceleration during phantom dominance For $agravitoelectric 0.735$, (4.6) is approximated as

$$\frac{\dot{a}}{a}^{\ 2} = 0.73H_0^2 a^{3/2}\Big[a^{-3/4} - \frac{0.73\rho_{\rm cr}^0}{2\lambda}\Big\}\Big] \tag{4.11}$$

with H_0 given above

(4.11) integrates to

$$a(t) \quad = \quad \Big[\frac{0.73\rho_{\rm cr}^0}{2\lambda} + \Big\{\sqrt{1.22 - \frac{0.73\rho_{\rm cr}^0}{2\lambda}} - \frac{3}{8}H_0\sqrt{0.73}(t - t_*)\Big\}^2\Big]^{-4/3} \tag{4.12}$$

as $a_*^{-3/4} = 1.22$. (4.12)shows acceleration and it is singularity-free.

5. Re-dominance of matter, cosmic collapse and its avoidance

5(a). Deceleration in future universe and cosmic collapse

(4.6)shows that phantom-dominance will end at $a(t_e) = a_e = [0.73\rho_{\rm cr}^0/2\lambda]^{-4/3}$ ie. $\rho_{\rm de}^{\rm ph} = 2\lambda$and matter will re-dominate. So,(4.6) will again reduce to (4.9) and the future universe will decelerate as

$$a(t) = a_e[1 + \frac{3}{2}]\sqrt{0.27}H_0 a_e^{-3/2}(t - t_e)]^{2/3} \tag{5.1}$$

It is notable that phantom density increases as universe expands, so $\rho_{\rm de}^{\rm ph} > 2\lambda$ when $a > a_e$. As a consequence, $\rho_{\rm de}^{\rm ph}\{1 - \rho_{\rm de}^{\rm ph}/2\lambda\} < 0$ using (4.4)in (4.6). As growth of $\rho_{\rm de}^{\rm ph}$ will continue, at a certain value $a_m > a_e$ of $a(t)$

$$\frac{0.27H_0^2}{a_m^3} = 0.73H_0^2 a_m^{3/4}\Big\{\frac{0.73\rho_0^{\rm cr} a_m^{3/4}}{2\lambda} - 1\Big\}. \tag{5.2}$$

(5.2) shows that $\dot{a} = 0$ at $a = a_m$. So, universe will bounce back and contract. As a consequece, dominance of matter will increase and FE will have the form of (4.9) yielding

$$H = \frac{\dot{a}}{a} \simeq -\sqrt{0.27}H_0 a^{-3/2}. \tag{5.3a}$$

(5.3a) integrates to

$$a(t) = a_m[1 - \frac{3}{2}\sqrt{0.27}H_0 a_m^{-3/2}(t - t_m)]^{2/3} \tag{5.3b}$$

showing decelerated contraction as $\ddot{a} < 0$.

(5.3b) yields $a(t) = 0$ at

$$t = t_{\rm col} = t_m + \frac{2}{3\sqrt{0.27}}H_0^{-1} a_m^{3/2} = 3.62 \times 10^{15} t_0. \tag{5.4}$$

So, at $t = t_{\rm col}$, dominating energy density term

$$\rho^{(mat)} = 0.27\rho_0^{\rm cr}/a^3, \tag{5.5}$$

in (4.9), will be infinite. These results show cosmic collapse at $t = t_{\rm col}$.

5(b).Avoidance of cosmic collapse Near the collapse time $t = t_{\rm col}$, energy density will be very high . So, like curvature-induced quintessence scalar in section3, the energy density (5.5) can be realised through another scalar $\Phi(t, \mathbf{x})$. Due to decay of $\Phi(t, \mathbf{x}$ in rapidly changing space-time, particles will be created near the collapse time. Energy density of created particles is obtained as [9, for details]

$$\rho_{\rm created} = \frac{680}{27}\sqrt{0.27}\pi^3 a_m^{7/2}\ \frac{H_0}{V}\ e^{3\pi\sqrt{7}} sinh^3(3\pi\sqrt{7}/2)(\tilde{\eta}_1)^{-17/3}a^{-5}. \quad (5.6)$$

It is natural to think that created particles will effect cosmic dynamics. As a consequence, FE (4.9) is modified as

$$H^2 \simeq 0.27H_0^2 a^{-3} + \frac{8\pi}{3}M_P^{-2}\rho_{\rm created} \quad (5.7)$$

The solution of (5.7) can be taken as

$$a = a_{\rm col} exp[|D(t_{\rm col} - t)\}| + \gamma|D(t_{\rm col} - t)|^2], \quad (5.8)$$

where $a_{\rm col} = a(t = t_{\rm col})$, D is a constant of mass dimension and constant γ is dimensionless.

If (5.8) satisfies (5.7),

$$\frac{3}{8\pi}M_P^2 D^2 = \frac{0.81}{8\pi}M_P^2 H_0^2 a_{\rm col}^{-3} + Xa_{\rm col}^{-5}, \quad (5.9a)$$

$$\frac{3}{2\pi}M_P^2 D^2\gamma = -3\frac{0.81}{8\pi}M_P^2 H_0^2 a_{\rm col}^{-3} - 5Xa_{\rm col}^{-5}, \quad (5.9b)$$

$$\frac{3}{2\pi}M_P^2 D^2\gamma^2 = \Big[-3\gamma + \frac{9}{2}\Big]\frac{0.81}{8\pi}M_P^2 H_0^2 a_{\rm col}^{-3} + \Big[-5\gamma + \frac{25}{2}\Big]Xa_{\rm col}^{-5} \quad (5.9c)$$

where

$$X = \frac{680}{27}\sqrt{0.27}\pi^3 a_m^{7/2}\ \frac{H_0}{V}\ e^{3\pi\sqrt{7}} sinh^3(3\pi\sqrt{7}/2)(\tilde{\eta}_1)^{-17/3}.$$

(5.9a) and (5.9b) yield $\gamma = -\frac{15}{32}$. At the largest energy mass scale i.e. the Planck mass, energy density is obtained as $M_P^4/8\pi^2$. Using it in (5.9a,b,c),$a_{\rm col} = 2.25 \times 10^{-42}$ and $D = \frac{M_P}{\sqrt{3\pi}}$ are obtained after some manipulations.

6. Salient features and concluding remarks

Here, a cosmological picture is obtained from the gravitational action containing the linear Einstein term as well as non-linear terms R^2 and R^5 using an approach different from the work [4, 5, 6]. This approach has an advantage to have power to explain(i) power-law inflation in the early universe and graceful exit from this phase, (ii) creation of SM particles, (iii) recovery of the standard cosmology with the cosmic background radiation with extremely high initial temperature $\sim 10^{18}$GeV, (iv) deceleration of the universe driven by emitted radiation during the inflationary phase and particles in thermal equilibrium with radiation,(v) deceleration driven by curvature induced dark matter and baryonic matter caused by various processes like nucleosynthesis, baryosynthesis, hydrogen recombination of elementary particles created during inflation, (vi) dominance of curvature-induced phantom at red-shift $z = 0.303$ (which is consistent with observational results), (vii) transient acceleration driven by phantom in the very late universe, (viii) re-dominance of matter, contraction of the universe, collapse of the universe at time $t_{\rm col} = 3.62 \times 10^{15} t_0$, (ix) avoidance of collapse due to creation of particles near the time $t_{\rm col}$ as well as its back-reaction and (x) rebirth of the universe after $t_{\rm col}$.

Thus, this model predicts the possible revival of the state of the early universe at time $t_{\rm col} = 3.62 \times 10^{15} t_0$.

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The inhomogeneous equation of state and the speed of convergence to a small cosmological constant

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Abstract

The mechanism for the relaxation of the cosmological constant is studied and elaborated. In the model used for the analysis of the relaxation mechanism the universe contains two components: a cosmological constant of an arbitrary size and sign and a component with an inhomogeneous equation of state. Owing to the dynamics of the second component the universe asymptotically tends to a de Sitter phase of expansion characterized by a small effective positive cosmological constant. An analysis of the asymptotic expansion for a general inhomogeneous equation of state of the second component is made. Several concrete examples are presented and the stability and speed of convergence to their fixed points are analyzed. It is found that the speed of convergence to a fixed point is large whenever the absolute value of the cosmological constant is large.

1 Introduction

The understanding of the structure, dynamics and composition at cosmological scales is one of the biggest challenges for physics and maybe even for science in general. The results of cosmological observations in recent years [1, 2, 3] and numerous attempts of their theoretical explanation [4] have lead us to an astonishing picture of the present universe. We presently understand only a minor part of the composition of the universe whereas its unknown part is attributed to the cosmic dark sector: dark matter and dark energy. Additional peculiar feature of the universe at the present epoch is its accelerated expansion. The dynamics or mechanism leading to the accelerated expansion became the hot topic in cosmology soon after the accelerated expansion had been established observationally . Almost every element of our, until then valid, picture of the universe and its dynamics, came under scrutiny in search of the acceleration mechanism that would be consistent with the observational data.

First attempts intervened into the composition of the universe: the existence of a new component with negative pressure (or even several of them) was postulated. The proposals for the nature of this new component, called *dark energy*, are numerous [4]. To name just a few, dark energy models range from the dynamics of scalar field in the potential [5], which further comprises quintessence, k-essence and phantom energy models, various cosmic barotropic fluid models [6] to the very concept of static or dynamic cosmological term [7]. Further attempts towards the acceleration mechanism questioned

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general relativity as a theory of gravitation at cosmological scales [8]. The very concept of our universe as four-dimensional was also relaxed in braneworld models in the quest for the explanation of acceleration [9]. The list of ingenious approaches to this problem certainly does not end here.

The problem of the cosmological constant (CC) [10, 11, 12] enjoys a special place among the multitude of proposals for the acceleration of the universe. The ΛCDM model is probably the simplest and the most popular model of the dark sector. The CC problem is an old one and it comes in many forms. From the viewpoint of fundamental theories, such as quantum field theory (QFT), the size of Λ consistent with observations is in a notoriously huge discrepancy with the predictions of these theories. This is a so called “old CC problem". An additional problem comes from the fact that the present energy densities of matter and CC are of the same order of magnitude, despite their significantly different scaling with the expansion of the universe. This is a so called “coincidence problem". Although the ΛCDM model generally fits the data very well, other challenges to ΛCDM cosmology have been identified [13]. However, the importance of the CC problem and its possible resolution is even bigger given that it underlies many other models of the cosmic acceleration, such as dynamical dark energy models.

In this paper we further elaborate the model of relaxation of the cosmological constant introduced in [14] and studied in [15]. The relaxation of the cosmological constant is understood as a dynamical process in which the universe with a large CC asymptotically tends to a de Sitter regime. The Hubble constant H^2 characterizing this de Sitter phase is equivalent to a small effective cosmological constant, $H^2 = \Lambda_{eff}/3$. We study the stability and convergence properties of the said model and provide several examples.

2 The model of the cosmological constant relaxation

In this section we present a short summary of the CC relaxation model introduced in [14] and discussed in [15]. We consider a two component cosmological model. The first component is a cosmological constant of arbitrary size and sign, whereas the second component is described by an inhomogeneous equation of state (EOS). The expansion of the universe is given by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}(\rho_\Lambda + \rho)\,. \tag{2.1}$$

The dynamics of the second component is described by the inhomogeneous EOS of the type

$$p = w\rho - 3\zeta_0 H^{\alpha+1}\,. \tag{2.2}$$

The formalism of the inhomogeneous EOS proves to be very useful in the description of various cosmological phenomena [3, 17, 18, 19, 20, 21, 22]. It has been shown [3] that the inhomogeneous equation of state can be interpreted as an effective description of modified gravity or braneworld dynamics [23, 24]. Another possible interpretation of an inhomogeneous EOS related to bulk viscosity [25, 26, 27, 28, 29, 30, 31] and in particular its generalizations.

Using (2.1) and (2.2) with the standard continuity equation for the second component we obtain a dynamical equation for the Hubble function H^2:

$$dH^2 + 3(1+w)\frac{da}{a}\quad H^2 - \frac{8\pi G\rho_\Lambda}{3} - \frac{8\pi G\zeta_0}{1+w}(H^2)^{(\alpha+1)/2}\quad = 0\,. \tag{2.3}$$

Using the notation

$$h = (H/H_X)^2,\ \ s = a/a_X,\ \ \lambda = 8\pi G\rho_\Lambda/3H_X^2,\ \ \xi = 8\pi G\zeta_0 H_X^{\alpha-1}/(1+w)\,, \tag{2.4}$$

where $H(a_X) = H_X$, we arrive at the equation relating dimensionless quantities

$$s\frac{dh}{ds} + 3(1+w)(h - \lambda - \xi h^{(\alpha+1)/2}) = 0\,, \tag{2.5}$$

and the initial condition $h(1) = 1$.

The relaxation mechanism for the cosmological constant is realized in the regime $\alpha < -1$. For an analytically tractable case $\alpha = -3$, it is straightforward to show [14] that for large absolute values of the rescaled CC term λ of both signs it is possible to obtain the relaxation of the cosmological constant.

For negative values of λ and $\lambda^2 \gg \xi$ with $\xi > 0$ and $w > -1$, the asymptotic expansion of the universe is given by a small value of the Hubble function

$$\lim_{a\to\infty} H^2 = H_{*1}^2 = \frac{24\pi G\zeta_0}{(1+w)|\Lambda|} \equiv \frac{3\zeta_0}{(1+w)|\rho_\Lambda|} = \frac{\Lambda_{eff}^-}{3}\,. \tag{2.6}$$

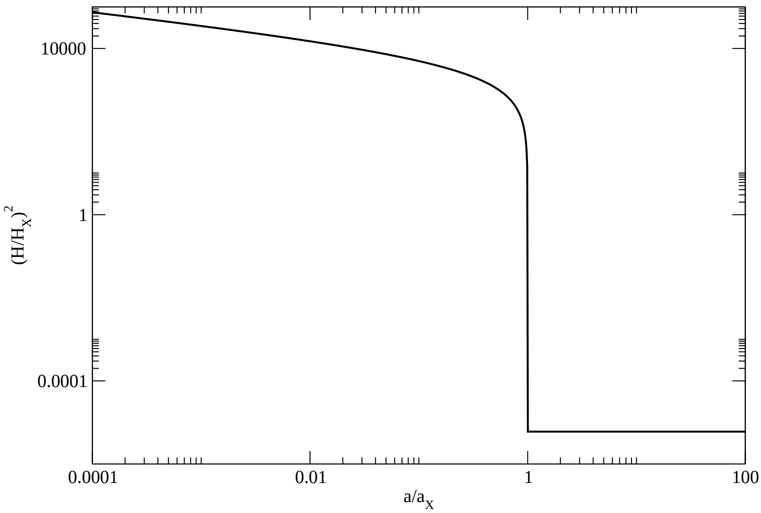

Figure 1: The dynamics of the scaled Hubble function for parameter values $\alpha = -3$, $\lambda = -5000$, $\xi = 0.03$ and $w = -0.9$. The dynamics consists of two phases of accelerated expansion connected with an abrupt transition.

This small value of H^2 corresponds to a small effective positive cosmological constant Λ^{-}_{eff}. A crucial result is that the Λ^{-}_{eff} is small because Λ is large in absolute value, as presented in (2.6). The dynamics of the Hubble function for negative λ and some typical values of parameters is given in Fig. 1.

For large positive values of λ (so that $\lambda^2 \gg |\xi|$) and for $\xi < 0$ and $w < -1$, the Hubble function tends asymptotically to

$$\lim_{a\to\infty} H^2 = H^2_{*2} = -\frac{24\pi G\zeta_0}{(1+w)\Lambda} \equiv -\frac{3\zeta_0}{(1+w)\rho_\Lambda} = \frac{\Lambda^{+}_{eff}}{3}\,. \tag{2.7}$$

Again as in the case of negative λ, the small asymptotic value of the Hubble function H^2 can be interpreted as a small effective positive cosmological constant Λ^{+}_{eff}. The effective CC is small because Λ is large. The dependence of the Hubble function on the scale factor for positive λ and some typical values of parameters is depicted in Fig. 2.

As described above, a simple model defined by (2.1) and (2.2) provides parameter regimes in which the relaxation of the cosmological constant can be realized. It is important to stress that in this model there is no fine-tuning and the asymptotic de Sitter phase is characterized by an effective positive CC which is small because the size of the real CC $|\Lambda|$ is large. Based on the calculations from e.g. QFT we could say that the size of λ is "naturally large". Additionally, since ξ describes the deviations from GR or the generalized effects of viscosity we could say that ξ is "naturally small". Then, from the behavior of the model for $\alpha < -1$ [14] and in particular from (2.6) and (2.7) we can expect that the size of the effective CC in the asymptotic de Sitter phase is "naturally small". This fact corresponds to the resolution of the old CC problem in the studied model.

3 Stability and convergence analysis

Given the potential of the relaxation mechanism described in the preceding sections for the resolution of the cosmological constant problem, it is important to study more general inhomogeneous equations of state for which the relaxation mechanism can be realized. Furthermore, as the dynamics of the Hubble function in the model [14] exhibits abrupt transition followed by a swift stabilization at the asymptotic value, additional insight into the dynamical details of the approach to the asymptotic value of H is needed. To this end, we consider a general inhomogeneous equation of state

$$p = w\rho - g(H^2)\,, \tag{3.1}$$

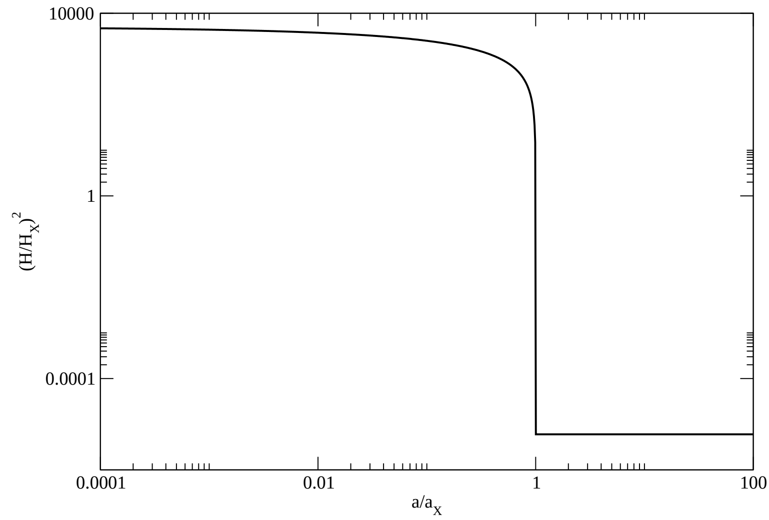

Figure 2: The dependence of the Hubble function on the scale factor for the values of parameters $\alpha = -3$, $\lambda = 5000$, $\xi = -0.03$ and $w = -1.1$. Two phases of accelerated expansion are interconnected with an abrupt transition.

which, using the standard rescaling as given in (2.4), results in the following dynamical equation for the Hubble parameter:

$$s\frac{dh}{ds} + 3(1+w)(h - \lambda - f(h)) = 0\,, \tag{3.2}$$

with the initial condition $h(1) = 1$. Here we have $f(h) = ((8\pi G)/(3(1+w)H_X^2))g(H_X^2 h)$. We further introduce the notation $F(h) = 3(1+w)(h - \lambda - f(h))$ and $F'(h) = 3(1+w)(1 - f'(h))$.

For a relaxation mechanism to be effective, we expect the dynamical system (3.2) to have a fixed point h_* which is much smaller than $|\lambda|$, i.e. for which we have $h_* \ll |\lambda|$. From the condition $F(h_*) = 0$ and the said expectation of the size of h_* it is straightforward to obtain

$$h_* = f^{-1}(-\lambda)\,. \tag{3.3}$$

The sign of the quantity $F'(h_*)$ determines the stability of the fixed point h_*, whereas its size controls the speed of convergence to the fixed point. The fixed point h_* is stable for $F'(h_*) > 0$ and it is given by the expression

$$F'(h_*) = 3(1+w)(1 - f'(f^{-1}(-\lambda)) = 3(1+w) \quad 1 - \frac{1}{(f^{-1})'(-\lambda)} \quad . \tag{3.4}$$

We further apply the general analysis displayed above to specific examples of the function $f(h)$. First we turn to the choice $f(h) = \xi/h$, already analyzed in detail in [14] and presented in the preceding sections of this paper. It is easy to see that in this case we have

$$h_* = -\frac{\xi}{\lambda} \tag{3.5}$$

and

$$F'(h) = 3(1+w) \quad 1 + \frac{\xi}{h^2} \quad . \tag{3.6}$$

Combining (3.5) and (3.6) and using $\lambda^2 \gg |\xi|$, we obtain

$$F'(h_*) = 3(1+w)\frac{\lambda^2}{\xi}\,. \tag{3.7}$$

This results sheds additional light on the results obtained in [14] and [15]. From (3.5) and (3.7) it becomes clear that to have a stable fixed point for $\lambda > 0$ we must have $\xi < 0$ and $w < -1$, whereas for $\lambda < 0$ we need $\xi > 0$ and $w > -1$. However, the most important feature of (3.7) is that the speed of convergence towards the fixed point h_* depends quadratically on λ. This finding explains why the fixed points in Figs 1 and 2 are reached so swiftly after the transition.

In the following example we consider the function $f(h) = A_1 \ln h$. For this functional form of $f(h)$ we have

$$h_* = e^{-\lambda/A_1} \tag{3.8}$$

and

$$F'(h_*) = 3(1+w) \quad 1 - A_1 e^{\lambda/A_1} \quad . \tag{3.9}$$

The condition of the stability of the fixed point h_* is $-3(1+w)A_1 > 0$ (given that the second term in (3.9) dominates). The speed of convergence to the fixed point grows exponentially with λ.

The next example employs the functional form $f(h) = A_2 e^{b_1/h}$. As in the preceding examples we have

$$h_* = \frac{b_1}{\ln \quad -\frac{\lambda}{A_2}} \tag{3.10}$$

and

$$F'(h_*) = 3(1+w) \quad 1 - \frac{\lambda}{b_1} \quad \ln \quad -\frac{\lambda}{A_2} \quad ^2 \Big) \; . \tag{3.11}$$

The conditions for the existence of small and positive h_* comprise $-\lambda/A_2 < 0$ and $b_1/\ln(-\lambda/A_2) > 0$. The fixed point h_* is stable for $-(1+w)\lambda/b_1 < 0$.

Finally, the last example to be considered in this paper is specified by the function $f(h) = A_3 \exp \quad \exp \quad \frac{b_2}{h}$. We further obtain

$$h_* = \frac{b_2}{\ln \quad \ln \quad -\frac{\lambda}{A_3}} \tag{3.12}$$

and

$$F'(h_*) = 3(1+w) \quad 1 - \frac{\lambda}{b_2} \ln \quad -\frac{\lambda}{A_3} \qquad \ln \quad \ln \quad -\frac{\lambda}{A_3} \qquad ^2 \Big) \; . \tag{3.13}$$

The expressions (3.12) and (3.13) show the conditions for the existence of a positive fixed point h_* are $\lambda/A_3 < 0$, $\ln(-\lambda/A_3) > 0$ and $b_2/\ln(\ln(-\lambda/A_3)) > 0$. The fixed point is stable if $(1+w)\frac{\lambda}{b_2} \ln \quad -\frac{\lambda}{A_3} \quad < 0$.

The conclusions of the general analysis given at the beginning of this section and the specific results for the studied examples indicate an interesting characteristic of the speed of convergence to the fixed point h_*. The faster the growth of the function $f(h)$ when h acquires small values, the smaller the speed of convergence to the fixed point, measured by the dependence of $F'(h_*)$ on λ. However, even for the examples with the mildest dependence of $F'(h_*)$ on λ, the speed of convergence is extremely large owing to the fact that $|\lambda|$ is large. The examples discussed in this section were selected primarily to better illustrate this property of the convergence to the fixed point. This is especially true for the last example of double exponential.

4 Conclusions

The results of this paper, along with the findings of [14] and [15], show that the mechanism of the CC relaxation based on the inhomogeneous EOS is robust. In section 3 general conditions for the onset of the CC relaxation mechanism are given. Various inhomogeneous EOS can reproduce the asymptotic de Sitter phase without particular fine-tuning. A general needed feature of the inhomogeneous EOS is that the inhomogeneous term becomes increasingly important as the expansion slows down and its effect finally equilibrate the action of a large CC. Thus the expansion of the universe settles down in a de Sitter phase characterized by a small effective positive cosmological constant. An interesting feature of the models is that the faster the inhomogeneous part, defined by $f(h)$ grows as h becomes small, the slower the approach to the asymptotic value, i.e. fixed point. However, for a large $|\lambda|$, even in the examples with the slowest approach to the fixed point, the speed of convergence is large, principally because of the very size of $|\lambda|$. These results largely explain some features of plots given in Figs 1 and 2. Namely, given a very large speed of convergence to the fixed points, it is easier to understand the abrupt transition in the dynamics of h and especially an extremely fast approach to the asymptotic value. All these results add to our understanding of the CC relaxation mechanism based on the inhomogeneous EOS and provide a further incentive to incorporate the said mechanism into a complete and workable cosmological model.

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Modified gravity with vacuum polarization

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Abstract

A brief review of cosmology in some generalized modified gravity theories with vacuum polarization is presented. Stability question of de Sitter solution is investigated.

DGP brane dynamics with vacuum polarization

The acceleration universe force us to explain this phenomena and this stimulate an interest to a number theories of modified gravity. One of such theories is the well-known Dvali-Gabadadze-Porrati (DGP) braneworld model [1, 2], for instance, can lead to an accelerating universe without the presence of either a cosmological constant or some other form of dark energy. Generalizations of the DGP model can result in a phantom-like acceleration of the universe at late times, which is not excluded by observational data.We consider the simplest generic braneworld model with action of the form

$$\begin{aligned} S_{DGP} = M^3 \ &\int_{bulk}(R_5 - 2\Lambda)\sqrt{-G}d^5x - 2\int_{brane} K\sqrt{-g}d^4x \\ &+\int_{brane}(m^2R_4 - 2\lambda)\sqrt{-g}d^4x + \int_{brane} L(g_{ab}, \rho, \phi)\sqrt{-g}d^4x. \end{aligned} \tag{0.1}$$

Here, R_5 is the scalar curvature of the metric G_{ab} in the five-dimensional bulk, and R_4 is the scalar curvature of the induced metric $g_{ab} = G_{ab} - n_a n_b$ on the brane, where n^a is the vector field of the inner unit normal to the brane, which is assumed to be a boundary of the bulk space, and the notation and conventions of [3] are used. The quantity $K = g^{ab}K_{ab}$ is the trace of the symmetric tensor of extrinsic curvature $K_{ab} = g^c{}_a\nabla_c n_b$ of the brane. The symbol $L(g_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane so that they interact only with the induced metric g_{ab}. All integrations over the bulk and brane are taken with the corresponding natural volume elements. The symbols M and m denote the five-dimensional and four-dimensional Planck masses, respectively, Λ is the bulk cosmological constant, and λ is the brane tension (it may be interpreted as cosmological constant on the brane). Note also that original DGP model apply $\lambda = 0$.

In Z_2 symmetry case cosmological constant is equal from two side of the brane $\Lambda_1 = \Lambda_2 = \Lambda$, and corresponding dynamical equation in FRW background looks like [4, 5]:

$$H^2 + \frac{k}{a^2} = \frac{\rho+\lambda}{3m^2} + \frac{2}{l^2}\left[1 \pm \sqrt{1 + l^2 \quad \frac{\rho+\lambda}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4}}\ \right]. \tag{0.2}$$

Here $l = 2m^2/M^3$, $H \equiv \dot{a}/a$ is the Hubble parameter, and ρ is the matter energy density on the brane. Here and below, the overdot derivative is taken with respect to the cosmological time t on the

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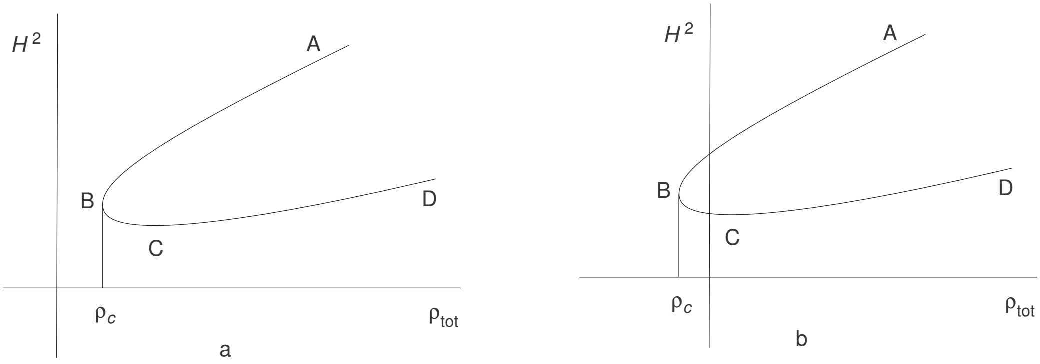

Figure 1: Plot of relation (0.2) in the case $\Lambda > 0$. Case (a) corresponds to $\rho_c > 0$, and case (b) corresponds to $\rho_c < 0$, where $\rho_c \equiv 3m^2\left(\Lambda/6 - l^{-2}\right)$.

brane. The expression under square root tend to zero during the evolution. It mean that in some time all time derivative from Habble parameter $\dot{H}, \ddot{H}, \ldots$ tends to infinity while H is finite. It's a new specific type of singularity and it was studied very well in the literature [6]. The term containing the constant C describes the so-called "dark radiation." We don't take into account this term, also we consider a spatially flat universe ($k = 0$). The "$\pm$" signs in the solution correspond to two branches defined by the two possible ways of bounding the Schwarzschild–(anti)-de Sitter bulk space by the brane [7, 8].

Now let us briefly consider dynamic of the DGP brane. Classical dynamics depends significantly on the bulk cosmological constant Λ and the full picture is sufficiently complicate so we refer you to the original paper focusing on only essential point for further statement ($\Lambda > 0$). The case $\Lambda > 0$ is shown in Fig.1. The graph of (0.2) in the $H^2, \rho_{\rm tot}$ plane in Fig.1 illustrates that in an expanding universe the matter density ρ decreases (except for a "phantom matter" which we do not consider in the present paper), and the point in the plane $(H^2, \rho_{\rm tot})$ moves from right to left in Fig.1.

A striking feature of Fig. 1 is that the value of the Hubble parameter in the braneworld can *never drop to zero*. In other words, the Friedmann asymptote $H \to 0$ is absent in our case. The upper and lower branches in Fig. 1 describe the two complementary braneworld models: branches AB and DB are associated with Brane 2 and Brane 1 of [9], respectively, while branches AC and DC correspond to the lower and upper signs in (0.2), respectively, and describe the two branches with different embedding in the bulk. It should be noted that, in many important cases, the behaviour of the braneworld does not have any parallel in conventional Friedmannian dynamics (by this we mean standard GR in a FRW universe). For instance, the BC part of the evolutionary track corresponds to "phantom-like" cosmology with $\dot{H} > 0$, even though matter on the brane never violates the weak energy condition.

Now let us consider a some quantum effects. In general, quantum effects in curved space-time can arise on account of the vacuum polarization as well as particle production. It is well known that the latter is absent for conformally invariant fields (which we shall consider here) and that, in this case, quantum corrections to the equations of motion are fully described by the renormalized vacuum energy–momentum tensor which has the form [10]

$$\begin{aligned} \langle T_{ik}\rangle = &(m_2/2880\pi^2)(R_i^{\ l}R_{kl} - \tfrac{3}{2}RR_{ik} - \tfrac{1}{2}g_{ik}R_{lm}R^{lm} + \tfrac{1}{4}g_{ik}R^2) \\ &+(m_3/2880\pi^2)\tfrac{1}{6}(2R_{;i;k} - 2g_{ik}R^{;l}_{\ ;l} - 2RR_{ik} + \tfrac{1}{2}g_{ik}R^2), \end{aligned} \tag{0.3}$$

where m_1, m_2 depend upon the spin weights of the different fields contributing to the vacuum polarization. This effect is known for a long time in cosmology. For example, it was demonstrate the possibility of singularity problem solution by some specific changes of m_2 and m_3 [11].

Since we work in flat FRW space-time, it is comfortable to use next relation:

$$\rho_q = k_2H^4 + k_3(2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2). \tag{0.4}$$

Where parameters k_2 and k_3 take the next form [12, 13, 14]:

$$k_2 = \frac{m_2}{60(4\pi)^2} = \frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{HD}}{60(4\pi)^2}, \tag{0.5}$$

$$k_3 = \frac{m_3}{60(4\pi)^2} = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{HD}}{60(4\pi)^2}. \tag{0.6}$$

Here N_i is number of the fields with spin i contributing to the vacuum polarization: N – a number of scalar fields, $N_{1/2}$ – a number of fermion fields, N_1 – a number of vector fields, N_2 (= 0 or 1) – a number of gravitons, N_{HD} – a number of conformal scalar fields. Note also that there is strong restriction on k_2 and k_3 in usual 4D-space-time. At least need $k_3 < 0$ with any k_2 for de Sitter (and in particular Minkovsky!) solution stability. Since we see that Minkovsky solution is stable (there isn't particle production in conformal flat vacuum) we may use condition $k_3 < 0$ as observational data.

In order to assess the effects of the vacuum polarization on the dynamics of the braneworld, one must add ρ_q to the matter density in (0.2) so that $\rho \to \rho + \rho_q$ in those equations. An important consequence of this operation is that the form of the equation of motion changes dramatically — the original algebraic equation changes to a differential equation! The dynamical equation (0.2) now takes the form

$$\ddot{H}H=\frac{1}{2}\dot{H}^2 - 3\dot{H}H^2 + (2k_3)^{-1} \; -k_2H^4+3m^2H^2-\rho_{\rm tot}\pm 3M^3\sqrt{H^2-\Lambda/6} \quad . \tag{0.7}$$

The goal of the present paper is to study the stability of the classical solutions when vacuum polarization terms are taken into account. The k_2-term in (0.4), which does not contain time derivatives of H, can only change the position of the future stable points. On the contrary, due to the k_3-term in (0.4), some classical solutions can lose stability. Therefore, for simplicity (and without loss of generality), we set $k_2 = 0$ in our calculations. If the brane has nonzero tension λ, the stationary points of (0.7) in the case $k_2 = 0$ can be found by substituting λ into (0.2) and setting $\rho = 0$. After that, we linearize Eq. (0.7) at these stationary points and find the eigenvalues of the corresponding linearized system. The condition of stability of the stationary point is that its eigenvalues's real parts are negative.

The eigenvalues at the stationary points where $\dot{H} = 0$ are given by

$$\mu_{1,2} = \frac{1}{2} \quad f_1 \pm \sqrt{f_1^2 + 4f_2} \quad , \tag{0.8}$$

where we have made the notation

$$f_1 = -3H\,, \tag{0.9}$$

$$f_2 = \frac{1}{2k_3} \quad 1 + \frac{\lambda}{3m^2H^2} \pm \frac{2l^{-1}\Lambda/6}{H^2\sqrt{H^2-\Lambda/6}}\Bigg) \,, \tag{0.10}$$

Two different signs in Eq. (0.10) correspond to two different equations of motion, while, in Eq. (0.8), we have two different eigenvalues of a single equation.

Since f_1 is negative, the eigenvalue μ_2 corresponding to the "−" sign in Eq. (0.8) is also always negative. Moreover, μ_1 is positive if and only if f_2 is positive. As a result, the stability of a fixed point is equivalent to the condition $f_2 < 0$. More careful investigation of this condition (for details see original paper [15]) show that part BC on the Fig. 1 corresponding to effective phantom behavior is unstable with respect vacuum polarization.

General brane dynamics with vacuum polarization

Now let us generalize obtaining result. We investigate any theory, which lead to dynamical equation in the next form:

$$\rho = F(H), \tag{0.11}$$

where $F(H)$ may be any algebraical function, which is don't contain time derivative of H. First of all we investigate the case $k_2 = 0$. Substituting in (0.11) expression (0.4) for ρ_q we may rewrite equation as dynamical system:

$$\begin{aligned} \dot{H} &= C, \\ \dot{C} &= \tfrac{C^2}{2H} - 3CH + \tfrac{1}{2k_3H}F(H) \equiv f(H,C). \end{aligned} \tag{0.12}$$

Linearizing this system at the fixed point $(H_0, 0)$

$$\begin{array}{l} \dot{H} = C, \\ \dot{C} = (\frac{\partial f}{\partial C})_0 C + (\frac{\partial f}{\partial H})_0 H, \end{array} \quad (0.13)$$

we find its eigenvalues

$$\mu_{1,2} = \frac{1}{2}[(\frac{\partial f}{\partial C})_0 \pm \sqrt{(\frac{\partial f}{\partial C})_0^2 + 4(\frac{\partial f}{\partial H})_0}]. \quad (0.14)$$

Since $(\frac{\partial f}{\partial C})_0 = -3H_0 < 0$ the real part of eigenvalues $\mu_{1,2}$ takes negative values if and only if

$$\frac{\partial f}{\partial H}\bigg|_0 = \frac{1}{2H_0 k_3} \frac{\partial F}{\partial H}\bigg|_0 < 0. \quad (0.15)$$

So we can see that it need condition $(\frac{\partial F}{\partial H})_0 > 0$ for stability. From another hand using (0.11) we find $(\frac{\partial F}{\partial H})_0 = \frac{\partial \rho}{\partial H} > 0$ or in this case the equivalent condition $\frac{\partial H}{\partial \rho} > 0$. By another words all regimes with $\frac{\partial \rho}{\partial H} < 0$ is unstable with respect to vacuum polarization.

Rewriting last condition in the form $\frac{\partial \rho/\partial t}{\partial H/\partial t} < 0$ we find that any regime with $\dot{\rho} < 0$ (that is natural in expanding universe) and with $\dot{H} > 0$ is unstable. Exactly similar regimes is called fantom-like. The case describing here correspond for example to BC part in Fig. 1.

Now let us account the k_2 term contribute. It mean that a new term $k_2 H^4$ is appear in equation (0.11). Transferring this term into right hand part of equation we may introduce a new function $F'(H) = F(H) - k_2 H^4$ and all previous result is true for this function. So term $k_2 H^4$ is not influencing on the stability, but it may change dynamical equation and fantom regimes is disappear at all [16].

$f(R)$-theories with vacuum polarization

Now let us consider $f(R)$-theories which more popular in resent time. The most general form of such theories may be written in the next form[17] (for a more general review of $f(R)$ gravity, see aslo [18, 19]):

$$S = S_m + \frac{1}{2\chi} \int d^4 x \sqrt{-g} f(R), \quad (0.16)$$

here $\chi = 8\pi G$ and for the sake of simplicity we set $2\chi = 1$. The general equation of motion corresponding to (0.16) is given by

$$g_{ik} f'^{;l}_{;l} - f'_{;i;k} + f' R_{ik} - \frac{1}{2} f g_{ik} = T_{ik}. \quad (0.17)$$

The trace of equation of motion (0.17) reads

$$3 f'^{;l}_{;l} + f' R - 2f = T. \quad (0.18)$$

So existence condition of de Sitter solution in vacuum is [20, 21]

$$2f(R_0) - R_0 f'(R_0) = 0, \quad (0.19)$$

and condition of its stability following from (0.18) is given by

$$\frac{f'(R_0)}{R_0 f''(R_0)} - 1 > 0. \quad (0.20)$$

Now let us take into account the vacuum polarization effects. For the sake of simplicity we investigate only $k_2 = 0$ case. So we find from (0.3) $\langle T^l_l \rangle = -\frac{k_3}{3} R^{;l}_{;l}$. Substituting this relation into (0.18) we find that this equation is restored to a normal state (without $\langle T^l_l \rangle$ contribute) by the $f \rightarrow f - \frac{k_3}{18} R^2$ transformation. And substituting this one into (0.20) we find new de Sitter stability condition:

$$\frac{f'(R_0) - R_0 f''(R_0)}{R_0 (f''(R_0) - \frac{k_3}{9})} > 0, \quad (0.21)$$

which is turn into the expression (0.20) by the limit $k_3 = 0$. For instance, let us consider meaning of new expression (0.21) for the model [22] $f(R) = R + R^{-m} + R^n$. We see that only denominator of the (0.21) has change when vacuum polarization take into account, while numerator is unchangeable. From another hand for this class of theories $f'' = m(m+1)R^{-m-2} + n(n+1)R^{n-2} > 0$ and since $k_3 < 0$ taking into account of vacuum polarization effect can't change denominator's sign. It mean that in this case quantum effects do not influence on de Sitter stability condition. Note also that vacuum polarization effects to above $f(R)$ gravity model may suppress instabilities as it was noted in [23].

Conclusion

Some modified gravity theories was studied with respect to vacuum polarization. It was demonstrate instability of effective fantom regimes in the brane cosmology caused by vacuum polarization. Modified condition of the vacuum de Sitter solution stability in $f(R)$ theories has been derived.

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Quantum Creation of a Universe with varying speed of light: Λ-problem and Instantons

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Abstract

One of the most interesting development trends of a modern cosmology is the analysis of models of a modified gravitation. Without exaggeration it is possible to say that Sergei Odintsov is one of the leaders of this direction of researches (see [1]). This article is dedicated to cosmologies with variable speed of light (VSL) - models, which one can be esteemed as a particular case of models of a modified gravitation. [3]

In quantum cosmology the closed universe can spontaneously nucleate out of the state with no classical space and time. The semiclassical tunneling nucleation probability can be estimated as $P \sim \exp(-\alpha^2/\Lambda)$ where α=const and Λ is the cosmological constant.

In classical cosmology with varying speed of light $c(t)$ it is possible to solve the horizon problem, the flatness problem and the Λ-problem if $c = sa^n$ with s=const and $n < -2$. We show that in VSL quantum cosmology with $n < -2$ the semiclassical tunneling nucleation probability is $P \sim \exp(-\beta^2\Lambda^k)$ with β=const and $k > 0$. Thus, the semiclassical tunneling nucleation probability in VSL quantum cosmology is very different from that in quantum cosmology with c=const. In particular, it can be strongly suppressed for large values of Λ. In addition, we propose two instantons that describe the nucleation of closed universes in VSL models. These solutions are akin to the Hawking-Turok instanton in sense of $O(4)$ invariance but, unlike to it, are both non-singular.

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[3] In [2] George F.R. Ellis and Jean-Philippe Uzan has pointed out at the fact, that the system of Friedmann-Raychaudhuri equations (system (2.1)) is not consistent from the point of view of a field theory. Therefore, since our article is initially based on (2.1) it seems that the present article is just a formal exercice based on unreliable grounds (Eqs. (2.1)) so that its conclusions cannot tell us much about the effect of varying constants in Quantum cosmology. This is not the case. The volume of this article do not allow to give the depleting answer to opposition of these writers, therefore we shall make it in the separate publication. Here we shall mark only, that we just use (2.1) as a basic phenomenological model. This, of course, can be regarded as a flaw of a model, but same can be said about the system (39)-(42) (of [2]), which is also but a phenomenological model, based on assumption, that the VSL models are in essence the particular example of the scalar-tensor theories (see for example [3]).

Moreover, using those solutions we can obtain the probability of nucleation which is suppressed for large value of Λ too.

1 Introduction

One of the major requests concerning the quantum cosmology is a reasonable specification of initial conditions in early universe, that is in close vicinity of the Big Bang. The three wave functions, describing the quantum cosmology has been proposed so far: the Hartle-Hawking's [4], the Linde's [5], and the so-called tunneling wave function [6]. In the last case the universe can tunnel through the potential barrier to the regime of unbounded expansion. Following Vilenkin [7] lets consider the closed ($k = +1$) universe filled with radiation ($w = 1/3$) and Λ-term ($w = -1$). One of the Einstein's equations can be written as a law of a conservation of the (mechanical) energy: $P^2 + U(a) = E$, where $P = -a\dot{a}$, $a(t)$ is the scale factor, the "energy" $E = \text{const}$ and the potential

$$U(a) = c^2a^2\left(1 - \frac{\Lambda a^2}{3}\right),$$

where c is the speed of light. The maximum of the potential $U(a)$ is located at $a_e = \sqrt{3/2\Lambda}$ where $U(a_e) = 3c^2/(4\Lambda)$. The tunneling probability in WKB approximation can be estimated as

$$P \sim \exp\left(-\frac{2c^2}{8\pi G\hbar}\int_{a_i'}^{a_i} da\sqrt{U(a) - E}\right), \tag{1.1}$$

where $a_i' < a_i$ are two turning points. The universe can start from $a = 0$ singularity, expand to a maximum radius a_i' and then tunnel through the potential barrier to the regime of unbounded expansion with the semiclassical tunneling probability (1.1). Choosing $E = 0$ one gets $a_i' = 0$ and $a_i = \sqrt{3/\Lambda}$. The integral in (1.1) can be calculated. The result can be written as

$$P \sim \exp\left(-\frac{2c^3}{8\pi G\hbar\Lambda}\right). \tag{1.2}$$

For the probability to be of reasonable value, for example $P = 1/\mathrm{e} \sim 0.368$, one has to put $\Lambda \sim 0.3 \times 10^{65}$ cm^{-2} (see (1.2)). In other words, the Λ-term must be large. On the other side, the universe once nucleated immediately begins a de Sitter inflationary expansion. Therefore the tunneling wave function results in inflation. And the Λ-term problem, which arises in this approach is usually being gotten rid of via the anthropic principle. In this case we have two Lorentzian regions ($0 < a < a_i'$, $a > a_i$) and one Euclidean region ($a_i' < a < a_i$). The second turning point $a = a_i$ corresponds to the beginning of our universe. If $\Lambda = 0$ then $U(a)$ has the form of **parabola** and we get only one Lorentzian region. In this case, the universe can start at $a = 0$, expand to a maximum radius and recollapse. If $E \to 0$, the single Lorentzian region contracts to a single point, which lies in agreement with the tunnelling nucleation probability: $P \to 0$ as $\Lambda \to 0$. However, as we'll show, in quantum cosmological VSL models the situation can be opposite, viz: the probability to find the finite universe short after it's tunneling through the potential barrier is $P \sim \exp(-\beta(n)\Lambda^{\alpha(n)})$ with $\alpha(n) > 0$ and $\beta(n) > 0$ when $n < -2$ or for $-1 < n < -2/3$. After the tunneling one gets the finite universe with "initial" value of scale factor $a_i \sim \Lambda^{-1/2}$, so the probability to find the universe with large value of Λ and small value of a_i is strongly suppressed. The reason for this lies in the behavior of potential $U(a)$, which, for the case $\Lambda \to 0$, transforms into the **hyperbola**, located under the abscissa axis. As a result, such a universe can at $a \sim 0$ start the regime of unbounded expansion. Therefore, we get the single Lorentzian region that doesn't contract to a point at $E \to 0$.

This new property of VSL quantum cosmology will be discussed in next Section but new question arouse: the geometric interpretation of the quantum creation of a Universe with varying speed of light. We know that universe can be spontaneously created from nothing (when $c = \text{const}$) and this process can be described with the aid of the instantons solutions possessing $O(5)$ (if $V(\phi)$ has a stationary point at some nonzero value $\phi = \phi_0 = \text{const}$) or $O(4)$ (as Hawking-Turok instanton [8]) invariance. So, what can be said about instantons in the VSL models?

The whole plan of the paper looks as follows: in the next Section we'll consider the simplest VSL model: model of Albrecht-Magueijo-Barrow. Then we show that in framework of tunneling approach to quantum cosmology with VSL the semiclassical tunneling nucleation probability can be estimated as $P \sim \exp(-\beta^2\Lambda^k)$ with β=const and $k > 0$. All corresponding calculations will be done for the case of the universe filled with radiation ($w = 1/3$) and vacuum energy. In the Section 3 we'll propose the **non-singular** instanton solutions possessing only $O(4)$ invariance (so the Euclidean region is a deformed four sphere). These solutions can in fact lead to inflation after the analytic continuation into the Lorentzian region. We will discuss these results in Sec. 4.

2 Albrecht-Magueijo-Barrow VSL model

Lets start with the Friedmann and Raychaudhuri system of equations with $k = +1$ (we assume the G=const):

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \qquad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - k\left(\frac{c}{a}\right)^2 + \frac{\Lambda c^2}{3},$$
$$c = c_0\left(\frac{a}{a_0}\right)^n = sa^n, \qquad p = wc^2\rho, \tag{2.1}$$

where $a = a(t)$ is the expansion scale factor of the Friedmann metric, p is the fluid pressure, ρ is the fluid density, k is the curvature parameter (we put $k = +1$), Λ is the cosmological constant, c_0 is some fixed value of speed of light which corresponds to a fixed value of scale factor a_0.

Using (2.1) one gets

$$\dot\rho = -\frac{3\dot a}{a}\left(\rho + \frac{p}{c^2}\right) + \frac{\dot c c(3 - a^2\Lambda)}{4\pi G a^2}. \tag{2.2}$$

Choosing $w = 1/3$ one can solve (2.2) to receive

$$\rho = \frac{M}{a^4} + \frac{3s^2na^{2(n-1)}}{8\pi G(n+1)} - \frac{s^2n\Lambda a^{2n}}{8\pi G(n+2)}, \tag{2.3}$$

where $M > 0$ is a constant characterizing the amount of radiation. It is clear from the (2.3) that the flatness problem can be solved in a radiation-dominated early universe by an interval of VSL evolution if $n < -1$, whereas the problem of Λ-term can be solved only if $n < -2$. The evolution equation for the scale factor a (the second equation in system (2.1)) can be written as

$$p^2 + U(a) = E, \tag{2.4}$$

where $p = -a\dot a$ is the momentum conjugate to a, $E = 8\pi GM/3$ and

$$U(a) = \frac{s^2a^{2n+2}}{n+1} - \frac{2s^2\Lambda a^{2n+4}}{3(n+2)}. \tag{2.5}$$

The potential (2.5) has one maximum at $a = a_e = \sqrt{3/(2\Lambda)}$ such that

$$U_e \equiv U(a_e) = \frac{s^2 3^{n+1}}{2^{n+1}\Lambda^{n+1}(n+1)(n+2)}, \tag{2.6}$$

so $U_e > 0$ if (i) $n < -2$ or (ii) $n > -1$. The first case allows us to solve the flatness and "Lambda" problems. Another benefit of the model is a finite time region with accelerated expansion.

2.1 The semiclassical tunneling probability in VSL models with $n < -2$: the case $E \ll U_e$

One can choose $n = -2 - m$ with $m > 0$. Such a substitution gives us the potential (2.5) in the form

$$U_m(a) = \frac{s^2}{a^{2(m+1)}}\left(\frac{2\Lambda a^2}{3m} - \frac{1}{m+1}\right). \tag{2.7}$$

Since (2.4) is similar to equation of movement of the particle of energy E in the potential (2.7), the universe in quantum cosmology can start at $a \sim 0$, expand to the maximum radius a_i' and then tunnel through the potential barrier to the regime of unbounded expansion with "initial" value $a = a_i$. The semiclassical tunneling probability can be estimated as

$$P \sim \exp \quad -2\int_{a_i'}^{a_i} \mid \tilde{p}(a) \mid da \Big), \tag{2.8}$$

with

$$\mid \tilde{p}(a) \mid = \frac{c^2(t)}{8\pi G\hbar} \mid p(a) \mid, \qquad \mid p(a) \mid = \sqrt{U_m(a) - E},$$

where $E \leq U_e$. It is convenient to write $E = U_e \sin^2\theta$, with $0 < \theta < \pi/2$.

For the case $E \ll U_e$ one can choose

$$a_i' \sim a_1 = \sqrt{\frac{3m}{2(m+1)\Lambda}}, \qquad a_i \sim \sqrt{\frac{3}{2\Lambda}} \quad \frac{\sqrt{m+1}}{\sin\theta} \quad ^{1/m}, \tag{2.9}$$

and evaluate the integral (2.8) as

$$P \sim \exp \quad \frac{-s^3\Lambda^{2+3m/2} I_m(\theta)}{4\pi G\hbar} \quad , \tag{2.10}$$

where

$$I_m(\theta) = \int_{z_i'(\theta)}^{z_i(\theta)} dz z^{-5-3m}\sqrt{\frac{2z^2}{3m} - \frac{1}{m+1}}, \tag{2.11}$$

with

$$z_i'(\theta) = \sqrt{\frac{3m}{2(m+1)}}, \qquad z_i(\theta) = \sqrt{1.5} \quad \frac{(m+1)^{1/2}}{\sin\theta} \quad ^{1/m}.$$

One can show that $I_m(\theta) > 0$ at $0 < \theta \ll 1$. Thus, it is easy to see from (2.10) that the semiclassical tunneling probability $P \to 0$ for large values of $\Lambda > 0$ and $P \to 1$ at $\Lambda \to 0$.

Note, that the case c=const can be obtained by substitution $m = -2$ into the (2.10). Not surprisingly, this case will get us the well known result $P \sim \exp(-1/\Lambda)$ (see [7]).

2.2 The semiclassical tunneling probability with $n < -2$ and $n > -1$

In the case of general position the semiclassical tunneling probability with $n = -2 - m$ has the form

$$P_m \sim \exp \quad -\frac{s^3\Lambda^{(3m+4)/2}}{4\pi G\hbar 3^{(m+1)/2}\sqrt{m(m+1)}}\int_{z_i'}^{z_i}\frac{dz\sqrt{F_m(z,\theta)}}{z^{3m+5}}\Big), \tag{2.12}$$

where

$$F_m(z,\theta) = -2^{m+1}\sin^2\theta\, z^{2(m+1)} + 2\times 3^m(m+1)z^2 - m3^{m+1}, \tag{2.13}$$

z is dimensionless quantity and z_i', z_i are the turning points, i.e. two real positive solutions of the equation $F_m(z,\theta) = 0$ for the given θ ($F_m(z,\theta) = 0$ does have two such solutions at $0 < \theta < \pi/2$).

If m is the natural number then the expression (2.12) has a more simple form. For example

$$P_1 \sim \exp \quad -\frac{s^3\Lambda^{7/2}\sin\theta}{6\pi G\hbar\sqrt{2}}\int_{z_i'}^{z_i}\frac{dz}{z^8}\sqrt{(z^2 - z_i'^2)(z_i^2 - z^2)}\Big),$$

with

$$z_i' = \frac{\sqrt{3}}{2\cos(\theta/2)}, \qquad z_i' = \frac{\sqrt{3}}{2\sin(\theta/2)}.$$

Similarly, $P \sim \exp(-S)$, with

$$S = \frac{s^3\Lambda^5\sin\theta}{18\pi G\hbar}\int_{z_i'}^{z_i}\frac{dz}{z^{11}}\sqrt{(z^2 + z_1^2)(z^2 - z_i'^2)(z_i^2 - z^2)},$$

where

$$z_1 = \sqrt{\frac{3}{\sin\theta}\cos\left(\frac{\theta}{3} - \frac{\pi}{6}\right)}, \qquad z_i' = \sqrt{\frac{3}{\sin\theta}\sin\frac{\theta}{3}}, \qquad z_i = \sqrt{\frac{3}{\sin\theta}\cos\left(\frac{\theta}{3} + \frac{\pi}{6}\right)},$$

and so on.

Therefore the probability to obtain (via quantum tunneling through the potential barrier) the universe in the regime of unbounded expansion is strongly suppressed for large values of Λ and small values of the initial scale factor $a_i = \sqrt{3}/(2\sin(\theta/2)\sqrt{\Lambda})$. In other words, overwhelming majority of universes born via the quantum tunneling through the potential barrier (2.5) have large initial scale factors and small values of Λ.

Now, let us consider the case (ii), when $n > -1$. The "quantum potential" has the form

$$U(a) = s^2 a^{2m}\left(\frac{1}{m} - \frac{2\Lambda a^2}{3(m+1)}\right), \tag{2.14}$$

where $m = n + 1 > 0$. The points of intersection with the abscissa axis a are $a_0 = 0$ and $a_1 = \sqrt{3(m+1)/2\Lambda m}$. Choosing $E = 0$ in equation (2.4) and substituting (2.14) into the (2.8) we get

$$P \sim \exp\left(-\frac{s^3\Lambda^{(1-3m)/2}}{4\pi G\hbar}\int_0^{z_1} z^{2m-2}\sqrt{\frac{1}{m} - \frac{2z^2}{3(m+1)}}\,dz\right),$$

with $z_1 = \sqrt{3(m+1)/2m}$ (The starting value $z = 0$ means that the Universe tunneled from "nothing" to a closed universe of a finite radius $a_1 = z_1/\sqrt{\Lambda}$.). Thus, we have the same effect as if $0 < m < 1/3$.

2.3 Peculiar cases with $n = -1$ and $n = -2$

At last, lets consider the cases of $n = -1$ and $n = -2$. The formula (2.12) is not valid in these cases ($m = -1$ and $m = 0$) so we shall consider these models separately.

If $n = -1$ ($m = -1$) then

$$\rho = \frac{M}{a^4} + \frac{\Lambda s^2}{8\pi G a^2} - \frac{3s^2}{4\pi G a^4}\log\frac{a}{a_*},$$

therefore

$$U(a) = s^2\left(2\log\left(\frac{a}{a_*}\right) - \frac{2a^2\Lambda}{3} + 1\right), \tag{2.15}$$

where a_* is constant and $[a_*]$=cm. The potential (2.15) has one maximum at $a = a_e = \sqrt{3/(2\Lambda)}$ such that $U_e = U(a_e) = 2s^2\log(a_e/a_*)$, so if $a_e > a_*$ then $U_e > 0$. We choose $a_* = \Lambda^{-1/2}$. This gives us $U_e = 0.41s^2 > 0$. For the case $E \ll U_e$ the semiclassical tunneling nucleation probability is

$$P_{-1} \sim \exp\left(-\frac{s^3\sqrt{\Lambda}}{4\pi G\hbar}\int_{z_i'}^{z_i}\frac{dz}{z^2}\sqrt{\log z^2 - \frac{2z^2}{3} + 1}\right) \sim \exp\left(-\frac{s^3\sqrt{\Lambda}}{10\pi G\hbar}\right), \tag{2.16}$$

where the turning points are $z_i' = 0.721$, $z_i = 1.812$. As we can see from the (2.16), when $n = -1$ we receive the aforementioned effect again.

If $n = -2$ ($m = 0$) then

$$\rho = \frac{M}{a^4} + \frac{s^2\Lambda}{2\pi G a^4}\log\left(\frac{a}{a_*}\right) + \frac{3s^2}{4\pi G a^6}.$$

We choose $a_* = 1/(\alpha\sqrt{\Lambda})$, where α is a dimensionless quantity. Thus

$$U(a) = -s^2\left(\frac{1}{a^2} + \frac{4\Lambda}{3}\log\left(\alpha a\sqrt{\Lambda}\right) + \frac{\Lambda}{3}\right). \tag{2.17}$$

The maximum of potential (2.17) is located at the same point a_e and

$$U_e = -\frac{s^2\Lambda}{3}\left(3 + \log\left(\frac{9\alpha^4}{4}\right)\right).$$

Therefore, $U_e > 0$ if $\alpha < 2e^{-3/4}/\sqrt{6} \sim 0.386$. Choosing $\alpha = 0.286$ and $E \ll U_e$ gets us the turning points $z_i' \sim 0.77$ and $z_i \sim 2.391$.

At last, the semiclassical tunneling nucleation probability is

$$P_0 \sim \exp\left(-\frac{s^3\Lambda^2}{4\pi G\hbar}\int_{z_i'}^{z_i}\frac{dz}{z^4}\sqrt{-\frac{1}{z^2}-\frac{4}{3}\log(\alpha z)-\frac{1}{3}}\right) \sim \exp\left(-\frac{0.084 s^3\Lambda^2}{\pi G\hbar}\right).$$

3 Instantons

If we are going to describe the quantum nucleation of universe we should find the instanton solutions, simply putted as a stationary points of the Euclidean action. The instantons give a dominant contribution to the Euclidean path integral, and that is the reason of our interest in them.

First at all, lets consider the $O(4)$-invariant Euclidean spacetime with the metric

$$ds^2 = c^2(\tau)d\tau^2 + a^2(\tau)\left(d\psi^2 + \sin^2\psi d\Omega_2^2\right). \tag{3.1}$$

In the case $c = \text{const}$ one can construct the simple instantons, which are the $O(5)$ invariant four-spheres. Then one can introduce the scalar field ϕ, whose (constant) value $\phi = \phi_0$ is chosen as the one providing the extremum of potential $V(\phi)$. The scale factor will be $a(\tau) = H^{-1}\sin H\tau$ and after the analytic continuation into the Lorentzian region one will get the de Sitter space or inflation. Many other examples of non-singular and singular instantons were presented in [9]

Now, lets consider the VSL model with scalar field. The corresponding Euclidean equations are:

$$\begin{aligned}
&\phi'' + 3\frac{a'}{a}\phi' = \frac{c^2V'}{\phi'} + \frac{c^5c'(\Lambda a^2-3)}{4\pi Ga^2\phi'} + \frac{2\phi'c'}{c} - \frac{2cVc'}{\phi'},\\
&\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3c^4}\left(\frac{\phi'^2}{2} - c^2V\right) + \frac{c^2}{a^2} - \frac{\Lambda c^2}{3},
\end{aligned} \tag{3.2}$$

where primes denote derivatives with respect to τ.

At the next step we represent the potential V in factorized form

$$V = F(a)U(\phi). \tag{3.3}$$

Indeed, lets for example consider the power-low potential $\sim \phi^k$. If the coupling λ is dimensionless one then we get

$$V \sim \frac{\lambda}{\hbar}G^{k/2-2}c^{7-2k}\phi^k.$$

Since $c = sa^n$ then in the simplest case we come to (3.3).

Let $\phi = \phi_0 = \text{const}$ be solution of the (3.2). (Note, that we don't require ϕ_0 to be an extremum of potential.) Using the first equation of system (3.2) and (3.3) we get the equation for the $F(a)$,

$$\frac{dF(a)}{da} - \frac{2n}{a}F(a) = \frac{3ns^4}{4\pi GU_0}a^{4n-3} - \frac{ns^4\Lambda}{4\pi GU_0}a^{4n-1}, \tag{3.4}$$

where $U_0 = U(\phi_0) = \text{const}$. The integration of the (3.4) results in

$$F(a) = a^{2n}\left(C - \frac{3ns^4}{8\pi G(1-n)U_0}a^{2(n-1)} - \frac{s^4\Lambda}{8\pi GU_0}a^{2n}\right), \tag{3.5}$$

where C is the constant of integration and by assumption $n \neq -1$ and $n \neq 0$. Substitution of (3.5) into the second equation of the system (3.2) transforms it into the the model of nonlinear oscillator, integration of which result in

$$\frac{a'^2}{2} + u(a) = 0, \tag{3.6}$$

where

$$u(a) = \frac{\omega^2a^2}{2} - \frac{s^2a^{2n}}{2(1-n)}, \tag{3.7}$$

with $\omega^2 = 8\pi G U_0 C/(3s^2)$ and with the choice $C > 0$ made. We can see that for $c = \text{const}$ (i.e. $n = 0$) (3.7) turns out to be an equation of the harmonic oscillator and we come to the well-known $O(5)$ solution (but in this case ϕ_0 must be the stationary point of V).

Equation (3.6) naturally describes the "movement of a classical particle" with zero-point energy in mechanical potential (3.7). Depending on value of n this potential can take one of four distinct forms (excluding the well-known classical case $n = 0$, which lies beyond the scoop of this article).

Case 1: $n < 0$. Here we have one Euclidean ($0 \le a \le a_1$) and one Lorentzian ($a > a_1$) regions where

$$a_1 = \quad \frac{s}{\omega\sqrt{1-n}} \quad ^{1/(1-n)} . \tag{3.8}$$

On the bound between Euclidean and Lorentzian regions ($a = a_1$) we have $a' = 0$.

This mechanical potential is unbounded from below at $a \to 0$. With this in mind, we'll have to ascertain that the Euclidean action for our solution will stay finite. The gravitation action has the form

$$S_{\text{grav}} = -\int d^4x \frac{c^3}{8\pi G}\sqrt{g}R.$$

We are using the dimensionless variables $x^0 = c_0\tau/a_0$, $x^1 = \psi$ and so on. Calculating R we get

$$R = \frac{6}{c_0^2 a^2} \quad c_0^2 - \quad \frac{a_0}{a} \quad ^{2n} \quad (1-n)a'^2 + aa'' \quad , \tag{3.9}$$

so we do have the potential divergence at $a = 0$. Multiplying (3.9) on the $\sqrt{g}$ and c^3 and using the equation of motion we get the expression:

$$R\sqrt{g}c^3 \sim 6c_0 \quad (2-n)\omega^2 \frac{a^{2n+3}}{a_0^{2n-1}} - \frac{nc_0^2}{1-n}\frac{a^{4n+1}}{a_0^{4n-1}} \quad , \tag{3.10}$$

where the most dangerous multiplier factor is a^{1+4n}. But if $-1/4 \le n < 0$ then the Euclidean action becomes finite and therefore, we end up with the legitimate gravitation instanton. In a similar manner, using (3.3) and (3.5) we get for the scalar field (in dimensionless x^μ):

$$\sqrt{g}V_0 \sim \frac{c_0 a_0^{1-3n}}{8\pi G}(3\omega^2 a^{3(1+n)} + \frac{3nc_0^2 a^{1+5n}}{(n-1)a_0^{2n}} - \frac{\Lambda c_0^2 a^{3+5n}}{a_0^{2n}})$$

therefore the instanton exists for $n > -1/5$. This requirement is stronger than the one for the gravitation instanton where $n > -1/4$ (see (3.10)).

Case 2. $0 < n < 1$. Here the potential $u(a)$ suffers no singularity at $a = 0$, but $u(0) = 0$. Also this potential has a minimum at

$$a_0 = \quad \frac{s}{\omega}\sqrt{\frac{n}{1-n}} \quad ^{1/(1-n)} ,$$

and is equal to zero at (3.8), hence, once again creating one Euclidean and one Lorentzian regions, separated by (3.8).

Case 3. $n = 1$. This case is somehow special, since for such n the solution of (3.4) shall be

$$F(a) = a^{2n} \quad C - \frac{3s^4}{4\pi G U_0}\ln a - \frac{s^4\Lambda}{8\pi G U_0}a^2 \quad ,$$

instead of (3.5), and hence, the equation of (3.7) shall be substituted by

$$u(a) = a^2 \quad \frac{\omega^2}{2} - s^2 \ln a \quad . \tag{3.11}$$

It is easy to see that this function has two zeros (at $a_1 = 0$ and $a_2 = \exp(\frac{\omega^2}{2S^2})$), is strictly positive on interval (a_1, a_2) and strictly negative outside of it. Therefore, this case doesn't allow an instanton.

Case 4. $n > 1$. The potential $u(a)$ is strictly positive. The instanton doesn't exist either.

Both of a newly founded solutions possess only $O(4)$ invariance just like Hawking-Turok instanton (so the Euclidean region is a deformed four sphere) but, unlike to it, they are all non-singular. Note that if the value a is sufficiently large then one can neglect the second term in (3.7) (after the analytic continuation into the Lorentzian region) therefore, as in the case of the usual $O(5)$ instanton, one can get the de Sitter universe, i.e. the inflation.

The equation (3.6) has no terms with Λ. In other words, the scale factor $a(\tau)$ doesn't depend on the value Λ (although being dependant on the U_0). Therefore, the full Euclidean action $S_{_E} = S_{\rm grav} + S_{\rm field}$ has the form,

$$S_{_E} = S_0 - \Lambda S_1,$$

where S_0 and S_1 are both independent of the Λ. Returning to what has been said in Introduction, there exist three common ways to describe the quantum cosmology: the Hartle-Hawking wave function $\exp(-S_{_E}/\hbar)$, the Linde wave function $\exp(+S_{_E}/\hbar)$ and the tunneling wave function. In the second Section we have been working with the tunneling wave function. In case of instantons situation becomes slightly different. If $S_1 > 0$ then (as a first, tree semiclassical approximation) we should choose the Linde wave function, whereas for the case $S_1 < 0$ the Hartle-Hawking wave function seems more naturally.

In conclusion, we note that another choice of C ($C < 0$ and $C = 0$) eliminates all possible instantons.

4 Discussion

VSL models contain both some of the promising positive features [10] and some shortcomings and unusual (unphysical?) features as well [11]. But, as we have shown, application of the VSL principle to the quantum cosmology indeed results in amazing previously unexpected observations. The first observation is that the semiclassical tunneling nucleation probability in VSL quantum cosmology is quit different from the one in quantum cosmology with c=const. In the first case this probability can be strongly suppressed for large values of Λ whereas in the second case it is strongly suppressed for small values of Λ. This is interesting, although we still can't say that VSL quantum cosmology definitely results in solution of the Λ-mystery. The problem here is the validity the WKB wave function. And what is more, throughout the calculations we have been omitting all preexponential factors (or one loop quantum correction) which can be essential ones near the turning points. Another troublesome question is the effective potentials in VSL models, being unbounded from below at $a \to 0$. The naive way to solve this problem is to use the Heisenberg uncertainty relation to find those potentials with the ground state. However, this is just a crude estimation. To describe the quantum nucleation of universe we have to find the instanton solution which, being a stationary point of the Euclidean action, gives the dominant contribution to the Euclidean path integral. As we have seen, such solutions indeed exist in VSL models. Those instantons are $O(4)$ invariant, are non-singular, and provide an inflation as well. They describe the quantum nucleation of universe from "nothing" and, what is more, upon usage of these solutions we can obtain the probability of a nucleation which is suppressed for large value of Λ using either Linde or Hartle-Hawking wave function.

Note, that we can weaken the condition $n > -1/5$ to obtain a singular instanton suffering the integrable singularity (i.e. such that the instanton action will be finite) in the way of the Hawking-Turok instanton. However, there exist some arguments [12], that such singularities, even being integrable, still lead to serious problems with solutions.

In conclusion, we note that obtained instantons both have a free parameter (ω^2) so we are free to use the anthropic approach to find the most probable values of Λ too.

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CURRICULUM VITAE

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3 List of scientific publications

3.1 Papers published in International Journals

(1). I.L.Buchbinder and S.D.Odintsov, *The one-loop renormalization of Yang-Mills theory in curved space-time*, Izv. VUZov, Fizika (Sov.Phys. J.) **n4**, 46-49 (1983); *Local momentum representation of propagator in an external gravitational and electromagnetic fields*, Izv. VUZov, Fizika (Sov. Phys. J.) **n1**, 110-111(1983).

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3.2 Contributions in Books and Proceedings of International Conferences

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(2). I.L.Buchbinder and S.D.Odintsov, *Renormalization group equation and effective couplings behaviour in an external gravitational field*, Proceedings of 1984 USSR Gravitational Conference, Moscow, 207 (1984) (in russian)

(3). I.L.Buchbinder, S.D.Odintsov and I.L.Shapiro, *Renormalization group equations and effective potential in curved space-time*, Proceedings of Conference Theoretical Group Methods in Physics, ed.M.Markov, Moscow, 123-129 and 115-122 (1986).

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(5). I.L.Buchbinder, S.D.Odintsov and I.L.Shapiro *Influence of cosmological constant and torsion in inflationary Universe*, Abstracts of 2 USSR Conference, Quantum Metrology and Fundamental constants, Leningrad, 78-79 (1985) (in russian).

(6). V.P.Dergalev and S.D.Odintsov, *Effective action of conformally-invariant theories in multidimensional space*, in Gravity and Fundamental Interactions , Moscow ,113 (1988) (in russian)

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3.3 International citations

(1). Total number of citations- more than 10500 citations. 5 papers are cited more than 250 times each one.

3.4 Refereing and reviewing for scientific journals

Referee for

(1). Physics Letters B, JCAP, JHEP and J. Math.Phys.

(2). Classical and Quantum Gravity and Journal of Physics A

(3). Physical Review D and Physical Review Letters

(4). Nuclear Physics B, EPL, European J.Physics C, New J.of Physics

(5). Mod.Phys.Lett.A, Int.J.Mod.Phys.A and D

(6). GRG, Theor. Math. Phys., CEJP, and Int J. Theor.Phys.,

(7). Izv. VUZov, Fizika (Sov. Phys. J.),now translated as Russian Physics Journal, Scholarly Exchange Research and Adv. Theor.Physics

(8). Member of Editorial Board of Gravitation and Cosmology

(9). Member of Editorial Board of TSPU Vestnik

(10). Member of Editorial Board of The Open Astronomy Journal

4 Research activity: projects, contracts and grants

4.1 Research fields

Quantum Field Theory and High Energy Physics, Classical and Quantum Cosmology, Dark Energy and Astrophysics, Modified Gravities, Quantum Gravity/Superstrings

4.2 Grants

(1). 1994-2008 researcher of RFBR and LRSS grants at TSPU.

(2). Researcher of several MEC-MCINN projects (Spain) in 2003-2008 and ESF Research Networking Program grant Casimir effect:2008-2013. Principal Investigator of PIE2007 project (MEC, Spain),2007-2009.

(3). Researcher of several exchange (Spain-Italy) projects in 2003-2009.

(4). Visiting Professor of Universities of Newcastle , Cambridge, Madrid, Barcelona, Hiroshima, Kyoto(YITP), Leipzig, Trondheim (NTNU), Trento, Cali, Guanajuato (IF), Sao Paulo, New York, Oklahoma, Copenhagen (NBI), Seoul and SISSA(Trieste), LANL (Los Alamos), AEI (Golm) in 1989-2008.

4.3 Participation in International Conferences

(1). Invited speaker/lecturer of more than 60 international conferences. Member of Orgcommittees for more than 15 international conferences.

5 Teaching activity

5.1 Teaching positions

(1). Postgraduate position with teaching, Department of Theoretical Physics, Tomsk University,Tomsk, 1982-1985.

(2). Associated Professor (Lecturer), Faculty of Mathematics and Physics, Tomsk Pedagogical Institute, Tomsk, Russia, 1985-1987.

(3). Professor (Docent), Faculty of Mathematics and Physics, Tomsk Pedagogical Institute, Tomsk, Russia, 1987-1990 and 1991-1993.

(4). Investigator in Theoretical Physics, Faculty of Mathematics and Physics, Tomsk Pedagogical Institute, Tomsk,Russia, 1990-1991.

(5). Full Professor , Faculty of Mathematics and Physics , Tomsk State Pedagogical University, Tomsk, Russia, 1993-2005.

(6). ICREA Research Professor at Space Research Institute, Barcelona, Spain (currently) and TSPU Professor

(7). Supervision of 9 PHD thesis.

5.2 Teaching subjects

Quantum Cosmology.
Mathematical Analysis.
Variational Analysis and Differential Equations.
Theory of Classical Fields. Electrodynamics.
Quantum Field Theory and Theory of Elementary Particles.
General Relativity.
Quantum Gravity-perturbative approach.

5.3 Books authored/edited

(1). I.L.Buchbinder, S.D.Odintsov and I.L.Shapiro, *Effective action in quantum gravity*, IOP Publishing, Bristol and Philadelphia, 411 pages, 1992.

(2). E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications*, World Scientific, Singapore, 319 pages, 1994.

(3). S.D. Odintsov, Editor of *Gravitation and Cosmology* **8, n1-2** (2002), Special issue deducated to 100th anniversary of TSPU.

(4). M.C.B. Abdalla, A.A. Bytsenko, M.E.X. Guimaraes and S.D. Odintsov, Editors of *Proceedings of the Second International Winter School on Mathematical Methods in Physics, Londrina, Brazil, 2002, Int.J.Mod.Phys.* **A18** n12 (2003).

(5). S.D. Odintsov, Editor of Vestnik of Tomsk State Pedagogical University, special issue (Quantum Gravity and Cosmology) v.7(44) (2004) p.1-120.

(6). E. Elizalde and S.D. Odintsov, Guest Editors of J.Phys.A.v.39 (2006) p.6109-6822.

(7). S.D. Odintsov, E. Elizalde and O.G. Gorbunova, Editors of special volume Casimir effect and Cosmology, on occasion of 70th birthday of Prof. I.Brevik, TSPU Publishing, 2008.

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