The Casimir Effect and Cosmology

A volume in honour of Professor Iver H. Brevik
on the occasion of his 70th birthday.

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The volume contains papers on problems involving the Casimir effect (mainly for electrodynamics) and on problems of cosmology (especially, dark energy).

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Preface

On April 7, 2009 Professor Iver Hakon Brevik, a world-known norwegian scientist in the area of theoretical physics and hydrodynamics, is celebrating his seventieth birthday. This special volume represents the collection of articles devoted mainly to Casimir effect and Cosmology and written by his friends and colleagues who wish to pay tribute to this remarkable event.

It may seem a bit strange why this book appears in Tomsk State Pedagogical University. There is very natural answer to this question. Tomsk State Pedagogical University (TSPU) is famous due to its internationally-recognized Scientific School of Theoretical Physics. The interests of this virtual Institute of Theoretical Physics are very wide. The following directions of leaders of this school may be mentioned: Prof. V.V. Obukhov (General Relativity and Mathematical Physics), Prof. I.L. Buchbinder (Supergravity, Superstrings and Quantum Field Theory), Prof. P.M. Lavrov (Quantization and Gauge Fields), Prof. S.D. Odintsov (Cosmology and Quantum Gravity), Prof. K.E. Osetrin (General Relativity and Mathematical Physics), Prof. V.Ya. Epp (Electrodynamics and Radiation Theory). About a thousand of research articles and half a dozen books on theoretical physics are published by these scientists and their younger colleagues from TSPU. Over fifteen years ago, the scientific contacts between Prof. I.H. Brevik (NTNU, Trondheim) and TSPU School of Theoretical Physics were initiated. Since then, our scientific relations were developed up to very high level. There was published some number of common papers, there was established the cooperation agreement between TSPU, Tomsk and NTNU, Trondheim. Our students and professors are often guests of NTNU at Trondheim. That was the reason why according to the iniciative by TSPU Rector, Prof. V.V. Obukhov it was suggested to publish this special volume Casimir Effect and Cosmology devoted to Prof. Iver Brevik seventieth birthday.

The research activity by Iver Brevik is very wide, as one can see from his brief CV attached at the end of this volume. However, this volume is devoted to only two of the several research directions by Iver Brevik: Casimir Effect and Cosmology. Precisely these two areas overlap with scientific interests of some of TSPU scientists. Needless to say that on his 70th birthday Prof. I.H. Brevik is still very active in science, especially in cosmology and Casimir effect at non-zero temperature. The editors and contributors present this special volume to Iver as a gift on his 70th anniversary. All of them, together with researchers of TSPU Scientific School of Theoretical Physics and TSPU Rector, Prof. V.V. Obukhov wish him excellent health for many years and even more brilliant scientific achievments.

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Spectral zeta function factorization and the multiplicative anomaly

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Abstract

Some basic questions concerning the structure of a generic spectral zeta function (as its poles and the existence of an Euler product) are addressed, starting from specific considerations for the examples of the Riemann and the Hurwitz zeta functions, and covering later higher dimensional Epstein zeta series. Use of the strategy of zeta function factorization —a very useful tool sometimes— allows to give a nice meaning to the multiplicative anomaly of the zeta regularized determinants, alternative to the usual, straightforward one. Finally, the question of the existence of a functional equation for any spectral zeta function is discussed, by taking advantage of the relationships between the momenta generating functions associated with the given zeta function [3].

1 Introduction

In this paper we discuss some particular and general features of a certain kind of zeta functions. We focus on a broad class, usually called spectral zeta functions. By a spectral zeta function we understand a function associated with a numerical sequence $\{\lambda_k\}$, —which will typically be the spectrum of a certain differential operator— of the following kind:

$$\zeta(s) = \sum_n \lambda_n^{-s} \tag{1}$$

This series is analytical for Re $s > s_0$ (s_0 is called the abscissa of convergence), and can be analytically continued to the rest of the complex plane (with the exception of a number of poles). Particular cases of the numerical sequence (such as $\lambda_k = k$ or $\lambda_k = k + q$) lead to

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[3]This article is **dedicated to 70th aniversary of Professor Iver Brevik**

well-known, important functions. Associated to the Riemann zeta function is one of the most famous and long-standing problem of contemporary mathematics: the Riemann hypothesis. While we will not deal with this subject here, it gives us, in fact, some indirect motivation for our quest, since it leads to the following question. Being the Riemann zeta function maybe the most simple case from the spectral point of view, what is the reason why it is, without any doubt, the most important and most studied of all? For many researchers it is the only one they study, except from its natural number-theoretical generalization to Dirichlet L functions, of course. To what essential properties owes the Riemann zeta function its importance? This is probably not hundred per cent clear, but undoubtedly (as identified by Selberg, the great Norwegian mathematician) such properties include: (i) analyticity on the whole complex plane (aside from simple poles), (ii) the existence of a simple functional equation (that is, a reflection formula which is basically multiplication by a gamma function), and (iii) the existence of an Euler product [1].

On the other hand, the mathematical theory associated with spectral zeta functions is rather wide and useful comprising, for example, the theory of pseudodifferential (ΨDO) elliptic operators. The zeta function allows to make sense of the determinant associated to the spectrum, what translated to physical terms opens an incredible amount of possibilities in quantum field theory and the like [2].

From this short summary we guess that, at least from the mathematical point of view, an important question to be asked is the following. Does a given spectral zeta function share the basic mathematical properties of the Riemann zeta function? This means in fact, does it have an Euler product and a functional equation? We can advance a rather generic answer: While it is usually found that a functional equation is indeed satisfied, frequently an Euler product is missing. In this paper we aim at understanding, via some simple examples, why this is so, by making more precise the first statement and producing a number of comments and observations on the second one. We will start with the second issue, illustrated for the case of the Hurwitz zeta function. In addition, our discussion will lead to simple but interesting considerations on the meaning of the multiplicative anomaly of zeta determinants and on the existence of arbitrary factorizations of zeta functions.

2 The Hurwitz zeta function and the multiplicative anomaly

From the spectral point of view, the next to the simplest (Riemann's) example in complexity corresponds to the following spectrum $\lambda_k = k + q$ with q a real parameter, leading to the Hurwitz zeta function

$$\zeta(s,q) = \sum_{n=0}^{\infty} (n+q)^{-s} \tag{2}$$

As for the case of the Riemann zeta function, it has also a unique pole, at $s = 1$, with residue 1. For $q = 1$ one gets back the Riemann zeta function. Both from the physical and from the mathematical viewpoints, there is one value of the parameter q which is singled out. This value is $q = \frac{1}{2}$. In physics, this is because it corresponds to the spectrum of a genuine quantum harmonic oscillator, a system of paramount relevance in nature. In mathematics, it provides the only case of a Hurwitz zeta function that possesses es an Euler product representation (no wonder, since as is well known this Hurwitz zeta function is in fact reducible to the Riemann one by elementary operations). The remarkable fact is that the same value of the parameter is singled out for both the physical and the mathematical point of view.[4] Moreover, we can add that this same value of the constant is the one singled out in the case of quadratic zeta functions (again both for physical and for mathematical reasons), leading to the theory of Epstein zeta functions [3, 4].

This simple observation leads to the question, whether this fact may have indeed a deeper significance, very much in line with the pioneering works that try to connect Number Theory and Physics. In this sense it is worth remarking that, in all these papers the existence of an Euler product is a central issue, in order to be able to consider the zeta function as a possible partition function of an hypothetical quantum system. There are also physical situations where the Hurwitz zeta function with the parameter $\frac{1}{2}$ appears jointly with the Riemann zeta function. One example of this is the computation of vacuum energies in conformal field theories (that define a new grading for the algebra, what has several mathematical implications). The result is that the vacuum energies are essentially given in terms of $\zeta(-1,1)$ and $\zeta(-1,\frac{1}{2})$, depending on the boundary conditions (for fermions, the first corresponds to the Neveu-Schwarz sector and the second to the Ramond sector, while for the bosonic case one has the opposite).

A further observation on the mathematical properties of the Hurwitz zeta function at this preferred value of the constant can be done from the point of view of regularized computations. Again, we find the value $q=\frac{1}{2}$ to have very special features. This is due to the property: $\zeta(0,q)=\frac{1}{2}-q$. To illustrate this point, consider the following spectrum: $\lambda_n=\beta(n+q)$. The associated zeta function is:

$$\zeta(s,q)=\sum_{n=0}^{\infty}[\beta(n+q)]^{-s}=\beta^{-s}\zeta_H(s,q), \tag{3}$$

where the last is Hurwitz's zeta function. To obtain the corresponding determinant, we must take the derivative

$$\frac{d}{ds}\zeta(s,q)=-\zeta_H(s,q)\beta^{-s}\ln\beta+\beta^{-s}\zeta_H'(s,q), \tag{4}$$

and evaluate it at zero, namely,

$$-\zeta'(s,q)\mid_{s=0}=\zeta_H(0,q)\ln\beta-\zeta_H'(0,q)=(\frac{1}{2}-q)\ln\beta-\zeta_H'(0,q). \tag{5}$$

Then, it turns out that

$$\prod_{n=0}^{\infty}(\beta(n+q))=\exp(-\zeta'(s,q)\mid_{s=0})=\beta^{\left(\frac{1}{2}-q\right)}\exp\left(-\zeta_H'(0,q)\right)=\beta^{\left(\frac{1}{2}-q\right)}\prod_{n=0}^{\infty}(n+q). \tag{6}$$

Once more, the value $q=\frac{1}{2}$ is the only one that yields the property that the regularized product of the spectrum under consideration is absolutely independent of the value of the parameter β. This implies that only in this case is our physical system invariant under dilatations of the energy spectrum. This parameter can represent, for instance, a magnetic field. For example, in the case of the Landau problem we have the following spectrum: $E_n=|B|\,(n+\frac{1}{2})$. For this system, the determinant is independent on the intensity of the magnetic field. As soon as we do not have the value $\frac{1}{2}$ this property does not hold any more.

This allows us to make some simple yet interesting remarks regarding the multiplicative anomaly. Taking logarithms in the expression above, it can be easily shown that the zeta

[4] And what has the quantum harmonic oscillator to do with the Euler product?

function calculation of the determinant yields [5] $\prod_{n=0}^{\infty} \beta = \sqrt{\beta}$. Thus, we see immediately that

$$\prod_{n=0}^{\infty} \beta \prod_{n=0}^{\infty} (n+q) = \sqrt{\beta} \prod_{n=0}^{\infty} (n+q)\,. \tag{7}$$

Since $q \neq 0$, it turns out that there is *always* a multiplicative anomaly. Such anomaly (or defect, as termed by I. Singer), is defined by

$$\det(AB) = \det(A)\det(B)e^{a(A,B)} \tag{8}$$

and has the value, in this case, $a(A,B) = -q\ln\beta$. For $q = \frac{1}{2}$, the whole anomaly term $e^{-a(A,B)} = \frac{1}{\sqrt{\beta}}$ cancels exactly with the parameter term, leaving a result which is independent of β.

Thus, the effect of a multiplicative parameter in the spectrum is to multiply the determinant by $\beta^{\zeta(0)}$. Since we have that $\prod_{n=1}^{\infty} \beta = \frac{1}{\sqrt{\beta}}$ and $\prod_{n=0}^{\infty} \beta = \sqrt{\beta}$, already for this very simple example we do have a multiplicative anomaly of the type $(\zeta(0) \pm \frac{1}{2})\ln\beta$.

To emphasize, once again, how simple is to get multiplicative anomalies, let us just consider how $\prod_{n=1}^{\infty} \beta$ is computed:

$$\log \prod_{n=1}^{\infty} \beta = \sum_{n=1}^{\infty} \log\beta = \log\beta \sum_{n=1}^{\infty} 1 = -\frac{1}{2}\log\lambda. \tag{9}$$

Already an operation as simple as performing the following step $\mathrm{Tr}\ [\beta A] \neq \beta \mathrm{Tr}\ [A]$ contains a multiplicative anomaly term (of course one has to take into account that we are dealing all the time with *regularized* traces). It is shown in [6] that, with $\mathrm{Tr}\ \left[A^{1+s}\right] = a_{-1}s^{-1} + a_0 + O(s)$, there is a multiplicative anomaly $a(\beta, A) = \mathrm{Tr}\ [\beta A] - \beta\mathrm{Tr}\ [A] = \beta\ln(\beta)\,a_{-1}$. For completeness, let us also recall that the zeta regularized trace does not either satisfy the additive property, that is, generically $\mathrm{Tr}\ [A+B] \neq \mathrm{Tr}\ [A] + \mathrm{Tr}\ [B]$ (see, e.g., [5]).

3 Zeta function factorization

Let us consider an arbitrary, two-dimensional zeta function, factorized as a product of two one-dimensional zeta functions:

$$\zeta(s) = \zeta_1(s)\zeta_2(s) \tag{10}$$

The determinant associated with $\zeta(s)$ is $\det A = \exp(-\zeta^{'}(0)) = \exp(-\zeta_1'(0)\zeta_2(0) - \zeta_1(0)\zeta_2^{'}(0))$. While the determinants associated with $\zeta_1(s)$ and $\zeta_2(s)$ are, respectively: $\det B = \exp(-\zeta_1^{'}(0))$ and $\det C = \exp(-\zeta_2^{'}(0))$. We see that

$$\det(A) = (\det B)^{\zeta_2(0)}(\det C)^{\zeta_1(0)}. \tag{11}$$

In the particular case when $\zeta_1(s)$ and $\zeta_2(s)$ have the same value at zero: $\zeta_1(0) = \zeta_2(0) = \widetilde{\zeta}(0)$, we have

$$\det(A) = (\det B \det C)^{\widetilde{\zeta}(0)} \tag{12}$$

What does this mean and how does it relate to the multiplicative anomaly issue? We can see this clearly, by way of writing explicit but generic spectra for all the zeta functions involved and those determinants: $\zeta_1(s) = \sum_n e_n^{-s}$ and $\zeta_2(s) = \sum_m \widetilde{e}_m^{-s}$. Thus, $\zeta(s) = \sum_n \sum_m (e_n \widetilde{e}_m)^{-s}$, and also $\det(A) = \prod_n \prod_m (e_n \widetilde{e}_m)$. We see then, that the computation of $\prod_n e_n \prod_m \widetilde{e}_m$ is essentially equivalently to $\prod_n \prod_m (e_n \widetilde{e}_m)$ (the precise relation being $\left(\prod_n e_n\right)^{\zeta_2(0)} \left(\prod_m \widetilde{e}_m\right)^{\zeta_1(0)} = \prod_n \prod_m (e_n \widetilde{e}_m)$), and *not* to $\prod_m (e_m \widetilde{e}_m)$. That is, the object one computes from the product of its parts and that leads to the multiplicative anomaly issue. We see here clearly why: when factorizing the regularized determinant into two regularized products, we are essentially computing the product of all the components in one regularized product, with all the components in the other regularized products, and not just only the "diagonal" ones.

In other words, we see that one is computing essentially the determinant associated with the two-dimensional zeta function $\zeta(s) = \sum_n \sum_m (e_n \widetilde{e}_m)^{-s}$, instead of the determinant of the one-dimensional zeta function $\zeta(s) = \sum_m (e_m \widetilde{e}_m)^{-s}$. Thus, we see that the multiplicative anomaly has to cancel out this non-diagonal products. With this result, the usual procedure of associating the product of two determinants with the determinant of their product is not quite a natural one, since for regularized determinants these quantities cannot be assumed to be naturally linked. This is why the existence of the multiplicative anomaly cannot be excluded, even in the most simple cases. This remark is particularly true in the case $\zeta_1(0) = \zeta_2(0) = \widetilde{\zeta}(0)$, since here the relationship between $\prod_n e_n \prod_m \widetilde{e}_m$ and $\prod_n \prod_m (e_n \widetilde{e}_m)$ is simple: $\prod_n e_n \prod_m \widetilde{e}_m = \left(\prod_n \prod_m (e_n \widetilde{e}_m)\right)^{\frac{1}{\widetilde{\zeta}(0)}}$. Let us consider for this case the usual expression

$$\det(BC) = \det B \det C e^{a(B,C)}, \tag{13}$$

then

$$\exp(a(B,C)) = \frac{\det(BC)}{\det B \det C} = \frac{\det(BC)}{(\det A)^{\frac{1}{\widetilde{\zeta}(0)}}} = \frac{\left(\prod_n e_n \widetilde{e}_n\right)}{\left(\prod_n \prod_m (e_n \widetilde{e}_m)\right)^{\frac{1}{\widetilde{\zeta}(0)}}}, \tag{14}$$

or

$$a(B,C) = \ln \det BC + 2 \ln \det A = \sum_n \ln(e_n \widetilde{e}_n) + \frac{1}{\widetilde{\zeta}(0)} \sum_n \sum_m \ln \widetilde{e}_m e_n, \tag{15}$$

which looks a somewhat artificial quantity (no wonder, since it links two non-directly related quantities).

It is a remarkable property of a regularized product that when computing things like $(e_1 e_2 ...) (\widetilde{e}_1 \widetilde{e}_2 ...)$ we are essentially computing something rather close to $(e_1 \widetilde{e}_1 \widetilde{e}_2 ... \;\; e_2 \widetilde{e}_1 \widetilde{e}_2 ... e_3 \widetilde{e}_1 \widetilde{e}_2 ...)$ and not just simply $(e_1 \widetilde{e}_1 e_2 \widetilde{e}_2 e_3 \widetilde{e}_3 ....)$, that is the quantity that one is used to, from convergent products. In general, if one is dealing with factorizations of the kind $\zeta(s) = \prod_i \zeta_i(s)$, the determinants are related as $\det A = \prod_i (\det A_i)^{\prod_{j \neq i} \zeta_j(0)}$. This can be useful for the computation of determinants of multidimensional zeta functions, once its factorization is known. For a general n-dimensional zeta function we can write its factorization as: $\zeta(s) = \prod_i \zeta_i^{d_i}(s)$, where d_i specifies the dimension of the zeta function, with $n = \sum_i d_i$.

In the simple examples of the previous sections, where we studied the effect of the product of the spectrum by a constant parameter, we were not dealing with a genuine two-dimensional zeta function, since one of the factorized spectra was just the constant parameter. It can be easily checked then, both with the usual method and the methodology of this section that we get the same results. A number of different examples can be worked out with what we have learned up to now. For example, if the factorized zeta functions have all value zero at the origin, then clearly, the associated multidimensional determinant is exactly one. This is clearly what happens to the product of harmonic oscillators : $\prod_{m=0}^{\infty} \cdots \prod_{n=0}^{\infty} (n+\frac{1}{2}) \cdots (m+\frac{1}{2}) = 1$. Or, for example, in a multiple factorization $\zeta^{(N)}(s) = \prod_{i=1}^{N} \zeta_i(s)$, one of the zeta functions evaluated at the origin is zero. Without loosing generality let us suppose that $\zeta_1(0) = 0$. Then, the determinant associated with $\zeta^{(N)}(s)$ is just $\left(e^{-\zeta_1'(0)}\right)^{\prod_{i=2}^{\infty} \zeta_i(0)}$, that is, the determinant of the zeta function which is zero at the origin, up to the product of the other zeta functions at zero. A number of examples of this kind (and also related with the multiplicative anomaly problem) can be worked out in an analogously simple way.

4 Factorizations in the computation of determinants

On a different level, as already indicated above, zeta function factorizations can be useful for the computation of determinants of multi-dimensional zeta functions and for the computation of their functional equations as well. It thus seems a mathematical object to be studied in this context. Of course, the program of zeta function factorization goes much beyond the simple cases considered above (even if they already describe a rather general setting) such as for example the factorization of a two-dimensional Epstein zeta function that can be shown to factorize in terms of the Riemann zeta function and a certain Dirichlet L-function.

In Cartier [7], non-trivial factorizations can be found for zeta functions such as:

$$\zeta(s) = \sum_{m,n}{}' (m^2 + n^2)^{-s}, \tag{16}$$

with the summation extended over the pairs $(m, n) \neq (0, 0)$ in Z^2. The zeta function can be expressed as:

$$\zeta(s) = \zeta_R(s) L(\chi_4, s). \tag{17}$$

where $\zeta_R(s)$ is the Riemann zeta function and $L(\chi_4, s)$ the Dirichlet zeta function corresponding to the character χ_4. Another very interesting factorization is the one for the following particular case of the two-dimensional Epstein zeta function:

$$\zeta(s) = \sum_{m,n}{}' (m^2 + nm + n^2)^{-s} = 6\zeta_R(s) L(\chi_3, s) \tag{18}$$

The two-dimensional Epstein zeta function is very important in Number Theory. Its analytical continuation is given by the celebrated Chowla-Selberg formula [8], which for the Epstein zeta function

$$\zeta_E(s; a, b, c) = \sum_{m,n}{}' (am^2 + bnm + cn^2)^{-s} \tag{19}$$

reads

$$\zeta_E(s;a,b,c) = 2\zeta(2s)\,a^{-s} + \frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\,\Gamma(s-1/2)\zeta(2s-1)$$
$$+\frac{2^{s+5/2}\pi^s}{\Gamma(s)\,\Delta^{s/2-1/4}\sqrt{a}}\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\,\cos(\pi n b/a)\,K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta}\right), \tag{20}$$

where $\sigma_s(n) \equiv \sum_{d|n} d^s$, i.e., sum over the s-powers of the divisors of n. We observe that the rhs's of (20) exhibits a simple pole at $s=1$, with residue: $\mathrm{Res}_{s=1}\zeta_E(s;a,b,c;q) = \frac{2\pi}{\sqrt{\Delta}} = \mathrm{Res}_{s=1}\zeta_E(s;a,b,c;0)$.

This formula has been non-trivially extended to situations of physical interest in recent work [9]. Consider the zeta function (Re $s > p/2$):

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbf{Z}^p}{}' \left[\frac{1}{2}\,(\vec{n}+\vec{c})^T A\,(\vec{n}+\vec{c}) + q\right]^{-s} \equiv \sum_{\vec{n}\in\mathbf{Z}^p}{}' \left[Q\,(\vec{n}+\vec{c})+q\right]^{-s}. \tag{21}$$

As before, the prime on a summation sign means that the point $\vec{n}=\vec{0}$ is to be excluded from the sum. As we shall see, this is irrelevant when q or some component of $\vec{c}$ is non-zero but, on the contrary, it becomes an inescapable condition in the case when $c_1 = \cdots = c_p = q = 0$ (i.e., the case of interest in Number Theory, which strictly corresponds to a multidimensional generalization of the original CS formula). Note that, alternatively, we can view the expression inside the square brackets of the zeta function as a sum of a quadratic, a linear, and a constant form, namely, $Q\,(\vec{n}+\vec{c})+q = Q(\vec{n}) + L(\vec{n}) + \bar{q}$. The end result is a formula that gives (the analytic continuation of) this multidimensional zeta function in terms of an exponentially convergent series, and which is valid in the whole complex plane, exhibiting the singularities (poles) of the meromorphic continuation —with the corresponding residua— explicitly. The only condition on the matrix A is that it correspond to a (non negative) quadratic form, which we call Q. The vector $\vec{c}$ is arbitrary, while q is (to start with) a positive constant. The explicit form of the solution to this problem depends dramatically on the fact that q and/or $\vec{c}$ are zero or not. According to this, one has to distinguish different cases, leading to unrelated final formulas, all to be viewed as different non-trivial extensions of the CS formula (they have been named ECS formulas in [9]).

Writing the dimensions of the submatrices of A as subindices, the result for the multidimensional, pure CS case is

$$\begin{aligned}
\zeta_{A_p}(s) \;\equiv\; \zeta_{A_p,\vec{0},0}(s) &= \frac{2^{1+s}}{\Gamma(s)}\sum_{j=1}^{p}\left(\det A_{p-j}\right)^{-1/2}\Big\{\pi^{(p-j)/2}\left(a_{jj}-\vec{a}_{p-j}^T A_{p-j}^{-1}\vec{a}_{p-j}\right)^{(p-j)/2-s} \\
&\qquad \times\Gamma\left(s-(p-j)/2\right)\,\zeta_R(2s-p+j) \qquad\qquad (22) \\
&+4\pi^s\left(a_{jj}-\vec{a}_{p-j}^T A_{p-j}^{-1}\vec{a}_{p-j}\right)^{(p-j)/4-s/2}\sum_{n=1}^{\infty}\sum_{\vec{m}_{p-j}\in\mathbf{Z}_{1/2}^{p-j}}{}'\cos\left(2\pi\vec{m}_{p-j}^T A_{p-j}^{-1}\vec{a}_{p-j}n\right)\,n^{(p-j)/2-s} \\
&\times\left(\vec{m}_{p-j}^T A_{p-j}^{-1}\vec{m}_{p-j}\right)^{s/2-(p-j)/4} K_{(p-j)/2-s}\left[2\pi n\sqrt{\left(a_{jj}-\vec{a}_{p-j}^T A_{p-j}^{-1}\vec{a}_{p-j}\right)\vec{m}_{p-j}^T A_{p-j}^{-1}\vec{m}_{p-j}}\right]\Big\}.
\end{aligned}$$

With a similar notation as above, here A_{p-j} is the submatrix of A_p composed of the last $p-j$ rows and columns. Moreover, a_{jj} is the j-th diagonal component of A_p, while $\vec{a}_{p-j} = (a_{jj+1},\ldots,a_{jp})^T = (a_{j+1j},\ldots,a_{pj})^T$, and $\vec{m}_{p-j} = (n_{j+1},\ldots,n_p)^T$. Physically, it corresponds to the homogeneous, massless, zero-temperature case. It is to be viewed, in fact, as *the* genuine multidimensional extension of the Chowla-Selberg formula.

When the matirx A is diagonal, one gets the simplified formula

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s} \Gamma(s-j/2)\, \zeta_R(2s-j) \right.$$
$$\left. +4\pi^s a_{p-j}^{j/4-s/2} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbf{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right) \right], \quad (23)$$

with $A_p = \mathrm{diag}(a_1, \dots, a_p)$, $A_j = \mathrm{diag}(a_{p-j+1}, \dots, a_p)$, $\vec{m}_j = (n_{p-j+1}, \dots, n_p)^T$, and ζ_R the Riemann zeta function.

It is immediate to see that the term for $j = 0$ in the sum yields the last term, $\zeta_{A_1}(s)$, of the recurrence, that is:

$$\zeta_{A_1}(s) = \sum_{n_p=-\infty}^{+\infty}{}' \left(\frac{a_p}{2} n_p^2\right)^{-s} = 2^{1+s} a_p^{-s} \zeta_R(2s). \quad (24)$$

It exhibits a pole, at $s = 1/2$ which is spurious —it is actually *not* a pole of the whole function (since it cancels, in fact, with another one coming from the next term, with further cancelations of this kind going on, each term with the next). Concerning the pole structure of the resulting zeta function, as given by Eq. (23), it is not difficult to see that *only* the pole at $s = p/2$ is actually there (as it should). It is in the last term, $j = p-1$, of the sum, and it has the correct residue, namely

$$\mathrm{Res}\ \zeta_{A_p}(s)\big|_{s=p/2} = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A_p)^{-1/2}. \quad (25)$$

The rest of the seem-to-be poles at $s = (p-j)/2$ are not such: they compensate among themselves, one term of the sum with the next, adding pairwise to zero.

Summing up, this formula, Eq. (23), provides the analytic continuation of the zeta function to the whole complex plane, with its only simple pole showing up explicitly. Aside from this, the finite part of the first sum in the expression is quite easy to obtain, and the remainder —an awfully looking multiple series— is in fact an extremely-fast convergent sum, yielding a small contribution, exactly as it happens in the CS formula. It is to be viewed as *the* extension of the original Chowla-Selberg formula —for the zeta function associated with an homogeneous quadratic form in two dimensions— to an arbitrary number, p, of dimensions. There are also formulas which provide extensions of the original CS expression to other cases of physical interest. To summarize, the general case of a quadratic+linear+constant form has been completed in [9], together with the ensuing evaluation of determinants for all these cases.

In fact, as stated above, the factorization method could also be very useful for the evaluation of determinants, in particular, of a higher-dimensional zeta function, starting from the knowledge of the determinants of its factors. In the examples considered above, it seems nevertheless more interesting to compute in this way the determinant of the Dirichlet zeta function, since these functions have been commonly considered in a number of different theoretical contexts and given that the corresponding determinants are not known yet, in many cases (while the other functions involved have been better studied already, under this point of view). Following our previous notation, it is more useful to express the Dirichlet zeta function as $\zeta_2(s) = \frac{\zeta(s)}{\zeta_1(s)}$. Then, from the expression above,

$$\zeta_2'(s) = \frac{\zeta'(s)}{\zeta_1(s)} - \frac{\zeta(s)\zeta_1'(s)}{\zeta_1(s)^2}, \tag{26}$$

and we only need to know the value at $s = 0$ in order to have the determinant. In this, way we can compute the determinant of a Dirichlet L function without the knowledge of any of its special values. Of course, particular values of the L function can be computed too, from the knowledge of the two spectral zeta functions.

Related with that issue, it seems worth to comment that there are some expectations from the study of Dirichlet series. To begin, that given the existence of a functional equation and an Euler product, then some kind of corresponding Riemann hypothesis is expected to hold. Second, that if the function has a simple pole at the point $s = 1$, then it must be a product of the Riemann zeta function and another Dirchlet series with similar properties [1]. Zeta functions with more complex singularity structure are expected to have correspondingly more complex factorizations.

On the other side, a very particular case of zeta factorization has been already considered for the study of certain problems, related with fractal strings. In this context, one usually considers the following factorization:

$$\zeta_F(s) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-s}\lambda_m^{-s} = \sum_{n=1}^{\infty} n^{-s} \sum_{m=1}^{\infty} \lambda_m^{-s}, \tag{27}$$

that is, the Riemann zeta function multiplied by a generic spectral zeta function. In the context of fractal strings, the two-dimensional zeta function is called the frequency counting function and the other is the geometric length function. The full zeta function is the two-dimensional one, that weights all the eigenvalues of the system with all the possible (all integers)degeneracies (excited states). The factorization reduces the problem to the study of the geometric counting function, which is in principle simpler, thanks to the knowledge one already has about the Riemann zeta function. Nevertheless, the zeta functions involved are not spectral in the usual sense and they lead to a very rich singularity pattern, with complex poles. This will make the usual heat-kernel asymptotic study a very rich one, with the presence of oscillating terms due to the presence of complex dimensions.

5 Functional equation of a spectral zeta function

Here we shall deal with the important point of the existence of a functional equation. Recall the functional equation for the Riemann zeta function:

$$\zeta_R(1-s) = \phi(s)\zeta_R(s), \tag{28}$$

where $\phi(s) = 2^{1-s}\pi^{-s}\cos(s\pi/2)\,\Gamma(s)$. We will argue on the basis of some physically inspired arguments. Associated with a sequence, one can define several fundamental spectral functions. In the case that the sequence on hand has a physical meaning —as for example the eigenvalues of a Schrödinger operator— then these spectral functions thoroughly govern the physics of the system. Examples of such functions are:

1. The density of states: $\rho(E) = \sum_n^{\infty} \delta(E - \lambda_n)$.

2. The partition Function: $Z(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}$.

The spectral zeta function is related with these functions in the following way:

1. With the partition function by a Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Z(t) dt. \tag{29}$$

2. With the density of states through the integral

$$\zeta(s) = \int_0^\infty dE E^{-s} \rho(E). \tag{30}$$

3. And, moreover, the partition function is in fact the Laplace transformed of the density of states, namely,

$$Z(t) = \int_0^\infty dE e^{-tE} \rho\left(E\right). \tag{31}$$

We are asuming all the time that the zeta function has been analytically continued to the whole complex plane, and thus that the two integral representations make sense. It is interesting to recall the fact that a Mellin transform encodes all the moments of the function under consideration. In particular, note that

$$\langle Z(t)^s \rangle = \Gamma(s+1)\zeta(s+1) \tag{32}$$

and

$$\langle \rho(E)^s \rangle = \zeta(-s). \tag{33}$$

Furthermore, the inverse problem always exists too, namely, under which conditions —from the knowledge of all the positive integer moments of a function— can one determine and reproduce, in a unique way, the function itself? This problem has already been studied in the context of partition functions and zeta functions and here we just want to point out that $\Gamma(s+1)\zeta(s+1)$ and $\zeta(-s)$ are the s (not necessarily integer) moments of two functions, namely the partition function and the density of states, which are related by a Laplace transform. This naturally implies that

$$\zeta(-s) = \varphi\left[\Gamma(s+1)\zeta(s+1)\right] \tag{34}$$

or, equivalently,

$$\zeta(1-s) = \varphi\left[\Gamma\left(s\right)\zeta\left(s\right).\right] \tag{35}$$

which will be the desired functional equation. To prove that it has the multiplicative form that we have shown before (e.g. $\varphi\left[\Gamma\left(s\right)\zeta\left(s\right)\right] = g(s)\Gamma\left(s\right)\zeta\left(s\right)$) further work is needed. It is likely that the explicit form of the function φ will be different in each particular case. Notice however that we already show here the natural appearance of the Gamma function as a multiplicative factor (cf. the definition of the Selberg class of Dirichlet series [1]). Summing up, we have shown by means of rather simple arguments that, indeed, any spectral zeta function satisfies a functional equation, and have come close to its explicit form.

In a different context, one should also recall the important applications of zeta function methods to quantum vacuum Physics, the Casimir effect, and its possible influence in modern cosmology (the dark energy issue, for a short list of references see [10]). It has been very nice to collaborate with Iver Brevik in some of these problems [11]. We admire him as a scientist and as a person.

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Gauge theory in inflationary Universe from a holographic model

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Abstract

Yang-Mills theory in the inflational Universe, dS_4 space-time, is studied according to the string theory/ gauge theory correspondence which has been proposed by Maldacena. In oeder to set up QCD like theory, fundamental quarks are introduced by embedding D7 brane as a probe in the $\mathrm{AdS}_5 \times S^5$ background. We can see the dynamical effects of the gravity on the gauge fields and quarks through the 4D cosmological constant λ. One of the important facts obtained here is that the confinement force is screened by the gravitational interaction as in the finite temperature gauge theory, namely large cosmological constant corresponds to high temperature [2].

1 Introduction

There have been many approaches to QCD/gravity correspondence based on the superstring theory [2]. After the idea, proposed by Karch and Katz [3], to add light flavor quarks by embedding D7 brane(s), many kinds of analyses have been performed, and various interesting results have been obtained for the properties of quarks and mesons, in the context of the holography [4, 5, 6, 7, 8, 9, 10, 11].

However these analyses are restricted to the gauge theory in 4d Minkowski space-time. On the other hand, some holographic approaches to the theory in the 4d de-Sitter space (dS_4) are seen [12, 13, 14, 15]. This direction is interesting since it would be possible to see the gravitational effect on the the gauge theory through the dependece of the characteristic

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[2]This article is **dedicated to 70th aniversary of Professor Iver Brevik**. Although new points are included, the main contents of this article are the review of our recent work, [1] hep-th/0609152, with M. Ishihara and A. Nakamura.

parameter of the curved space, the 4D cosmological constant, which plays an important role This situation of the positive cosmological constant at the early inflation universe or for the present small acceleration which has been observed recently in our universe. Also from this cosmological viewpoint, it would be important to make clear the non-perturbative properties of the gauge theories at finite cosmological constant or in dS_4. It would be a difficult to see the non-perturbative effects of the gravitational interactions and the gauge fields in the curved space when we consider them within the 4D field theory.

Here we examine this problem from holographic approach which has been useful in the finite temperature case [10]. The bulk solution corresponding to the gauge theory in dS_4 is obtained from type IIB string theory with dilaton and axion under the five form flux. And the D7 brane is embedded in this background as a probe to introduce the flavor quarks. In the bulk, there is a horizon as in the finite temperature case. A phase transition as seen in the high temperature case is also existing for the case of large gauge field condensate.

Through the Wilson-Polyakov loop. We find the existence of a maximum distance between quark and antiquark to maintain the U-shaped string state. Above this maximum length, the quark and antiquark can not make a bound state. In other words, the color force is screened by the gravitational interaction. In this sense, we can say that the theory is in the deconfinement phase. We compare these results with the similar results given for finite temperature theory.

The meson masses through the fluctuation of the D7 brane are examined, and we find that all the states would disappear at large cosmological constant since they becomes unstable and decay to free quarks and ant-quarks. A similar phenomenon for the baryon spectra is also seen by studying the energy of the D5 baryon wrapped on S^5 which is regarded as baryon.

In section 2, we give the setting of our model, and the phase transition is discussed by solving the embedding of the D7 brane. In section 3, the quark properties are studied through the Wilson Polyakov loop. In section 4, the mesons and baryons are discussed. The summary is given in the final section.

2 Background geometry

We consider 10d IIB model retaining the dilaton Φ, axion χ and selfdual five form field strength $F_{(5)}$. Under the Freund-Rubin ansatz for $F_{(5)}$, $F_{\mu_1\cdots\mu_5} = -\sqrt{\Lambda}/2\ \epsilon_{\mu_1\cdots\mu_5}$ [16, 17][3], we obtain [1]

$$ds_{10}^2 = G_{MN}dX^M dX^N$$

$$= e^{\Phi/2}\left\{\frac{r^2}{R^2}A^2\left(-dt^2 + a(t)^2(dx^i)^2\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2\right\} , \tag{1}$$

$$e^{\Phi} = 1 + \frac{q}{r^4}\frac{1-(r_0/r)^2/3}{(1-(r_0/r)^2)^3} , \quad \chi = -e^{-\Phi} + \chi_0 , \tag{2}$$

$$A = 1 - (\frac{r_0}{r})^2, \quad a(t) = e^{2\frac{r_0}{R^2}t} \tag{3}$$

[3]Related solution is seen in [19].

where $M,\ N = 0 \sim 9$ and $R = \sqrt{\Lambda}/2 = (4\pi N)^{1/4}$. The horizon is denoted by r_0, which is related to the 4d cosmological constant λ as

$$\lambda = 4\frac{r_0^2}{R^4}. \tag{4}$$

And q represent the VEV of gauge fields condensate [10].

In the present configuration, the four dimensional boundary represents the $\mathcal{N}$=4 SYM theory in the de Sitter space or in the inflational universe characterized by the 4d cosmological constant λ. We examine the properties of this system by adding flavor quarks by embedding D7 branes in this background.

3 D7 brane embedding and phase transition:

By rewriting the extra six dimensional part of (1) as,

$$\frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 = \frac{R^2}{r^2}\left(d\rho^2 + \rho^2 d\Omega_3^2 + (dX^8)^2 + (dX^9)^2\right)\ , \tag{5}$$

and $r^2 = \rho^2 + (X^8)^2 + (X^9)^2$, we obtain the induced metric for D7 brane,

$$ds_8^2 = e^{\Phi/2}\left\{\frac{r^2}{R^2}A^2\left(-dt^2 + a(t)^2(dx^i)^2\right) + \right.$$

$$\left. \frac{R^2}{r^2}\left((1+(\partial_\rho w)^2)d\rho^2 + \rho^2 d\Omega_3^2\right)\right\}\ , \tag{6}$$

where we set as $X^9 = 0$ and $X^8 = w(\rho)$ without loss of generality due to the rotational invariance in $X^8 - X^9$ plane.

The brane action for the D7-probe is given as

$$S_{\rm D7} = -\tau_7\int d^8\xi\left(e^{-\Phi}\sqrt{-\det\ (\mathcal{G}_{ab}+2\pi\alpha' F_{ab})} - \frac{1}{8!}\epsilon^{i_1\cdots i_8}A_{i_1\cdots i_8}\right)$$

$$+\frac{(2\pi\alpha')^2}{2}\tau_7\int P[C^{(4)}]\wedge F\wedge F\ , \tag{7}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$. $\mathcal{G}_{ab} = \partial_{\xi^a}X^M\partial_{\xi^b}X^N G_{MN}$ $(a,\ b = 0\sim 7)$ and $\tau_7 = [(2\pi)^7 g_s\, \alpha'^{\ 4}]^{-1}$ represent the induced metric and the tension of D7 brane respectively. And $P[C^{(4)}]$ denotes the pullback of a bulk four form potential.

Then, after a calculation, the equation of motion for $w(\rho)$ is obtained as,

$$\begin{aligned}\frac{w}{\rho + w\ w'}\left[\Phi' - \sqrt{1+(w')^2}(\Phi + 4\log A)'\ \right]& \\ +\frac{1}{\sqrt{1+(w')^2}}\left[w'\left(\frac{3}{\rho} + (\Phi+4\log A)'\right) + \frac{w''}{1+(w')^2}\right] &= 0,\end{aligned} \tag{8}$$

where prime denotes the derivative with respect to ρ.

Asymptotic Solution and Chiral Symmetry: Firstly we consider the asymptotic solution of w for large ρ. It is obtained usually as the following form

$$w(\rho) \sim m_q + \frac{c}{\rho^2}, \tag{9}$$

where m_q and c are interpreted from the gauge/gravity correspondence as the current quark mass and the chiral condensate, $-c = \langle\bar{\Psi}\Psi\rangle$, where Ψ denotes the quark field. The ciral condensate is determined by a theory as a function of the quark mass m_q [10]. Then the solutions for w are characterized only by the quark mass m_q, and the vev of chiral condensate is determined uniquely by m_q.

However we must notice that the above asymptotic form (9) is useful for the case of CFT in the 4D Minkowski space-time on the boundary. In the present case, however, the geometry of the 4D boundary is dS_4, then we expect that the form (9) would receive some modification from the gravity.

In order to investigate about c we consider the force between the D3-branes at the horizon and the D7 brane at $X^8 = w$. The force between them is obtained from the potential of w, which is obtained from the D7 action by setting $w' = 0$ and remembering $r^2 = \rho^2 + w^2$ as follows,

$$V(w) = \tau_7 \left(A^4\, e^{\Phi} - C_8\right). \tag{10}$$

In the limit of $r_0 = 0$, 4D boundary is Minkowski space-time $V = \tau_7 =$ constant. So there is noforcee in this case. This is the reflection of the supersymmetry of the solution. As a result, we find $c = 0$ [10].

Sign of c: When the cosmological constant is turned on ($r_0 \neq 0$), we find the non-trivial force,

$$F = -\frac{\partial V}{\partial w} = -8\tau_7 \frac{w\; r_0^2}{r^4}\; A^3\; e^{\Phi}\; < 0, \tag{11}$$

and we can see that the force is attractive at any point of ρ.

Then we can understand that the c must be negative for any solution of w. Then the chiral symmetry is preserved as in the case of $\lambda = 0$. This situation is similar to the case of finite temperature gauge theory, but we find a new feature of c as given below.

UV divergence of c: In the case of $r_0 \neq 0$, we find the following asymptotic form instead of (9),

$$w(\rho) \sim m_q + \frac{c_0 - 4m_q r_0^2 \log(\rho)}{\rho^2}, \tag{12}$$

where m_q and c_0 are constants. This implies that c depends on ρ like $\log(\rho)$ and diverges at $\rho = \infty$. So we need an appropriate regularization, so we can estimate it as

$$c = c_0 - 4m_q r_0^2 \log(\rho_{\mathrm{cutoff}}) \tag{13}$$

by introducing a cutoff ρ_{cutoff}.

Phase transition: For $\lambda > 0$, as in the case of finite temperature, we can see a "topological" phase transion in this theory. For simplicity we consider the case of $q = 0$. It is seen

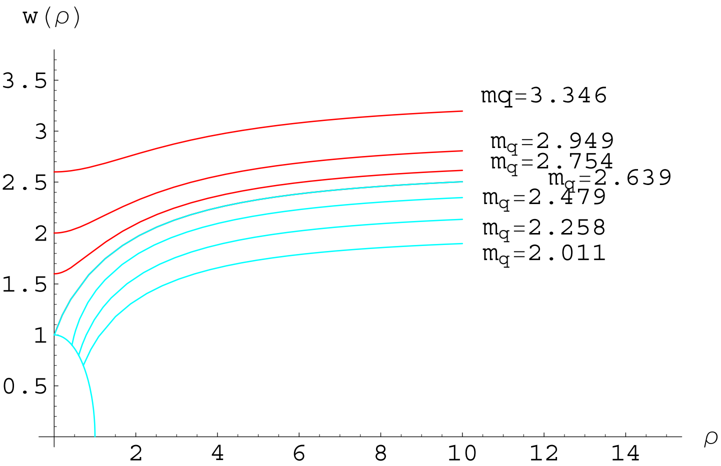

Figure 1: Embedding solutions w and the chiral condensate c for $q = 0$ and $r_0 = R = 1.0$.

through the flip of solution $w(\rho)$ and the D7 regularized energy. Their resuls are shown in the Fig. 1 and Fig. 11. We should notice that an extra reguralization due to the UV divergence of $c(\rho)$ is needed [1]. The details for other parameter case are seen in [1], but the results are similar to the case of $q = 0$.

4 Quark-antiquark potential

Firstly we briefly review how quark-antiquark potentials described in the context of the gauge/gravity correspondence. Consider the Wilson-Polyakov loop, $W = (1/N)\mathrm{Tr}Pe^{i\int A_0 dt}$, in $SU(N)$ gauge theory, then the quark-antiquark potential $V_{q\bar{q}}$ is derived from the expectation value of a parallel Wilson-Polyakov loop as

$$\langle W \rangle \sim e^{-V_{q\bar{q}}\int dt} \sim e^{-S}, \tag{14}$$

in terms of the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-\det h_{ab}}, \tag{15}$$

with the induced metric $h_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, where $X^\mu(\tau,\sigma)$ denotes the string coordinate.

To fix the static string configurations of a pair of quark and anti-quark, we choose $X^0 = t = \tau$ and decompose the other nine string coordinates as $\mathbf{X} = (\mathbf{X}_{||}, r, r\Omega_5)$. Then the Nambu-

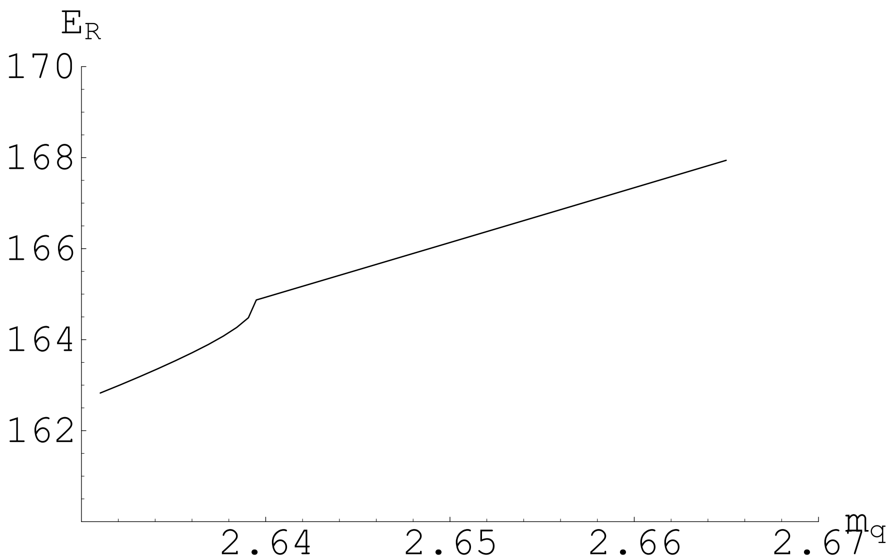

Figure 2: The regularized energy for $q = 0$, and other parameter settings are the same with the above Fig. 1.

Goto Lagrangian in the background (2) becomes

$$L_{\rm NG}=-\frac{1}{2\pi\alpha'}\int d\sigma\ e^{\Phi/2}\sqrt{A(r)^2r'^2+r^2A(r)^2{\Omega_5}'^2+\left(\frac{r}{R}\right)^4A(r)^4a(t)^2{\mathbf{X}_{||}}'^2}, \tag{16}$$

where the prime denotes a derivative with respect to σ. The test string has two possible configurations: (i) a pair of parallel string, which connects horizon and the D7 brane, and (ii) a U-shaped string whose two end-points are on the D7 brane.

Effective quark mass and confinement: Consider the configuration (i) of parallel two strings, which have no correlation each other. The total energy is then two times of one effective quark mass, $\tilde{m}_q$. As mentioned above, it is given by a string configuration which stretches between r_0 and the maximum $r_{\rm max}$, so we can take as $r=\sigma$, $\mathbf{X}_{||}=$ constant, $\Omega_5=$ constant, Then $\tilde{m}_q$ is obtained by substituting these into (16) as follows,

$$\tilde{m}_q=E/2=\frac{1}{2\pi\alpha'}\int_{r_0}^{r_{\rm max}}dr\ e^{\Phi/2}A(r)\ , \tag{17}$$

where $r_{\rm rmax}$ denotes the position of the D7 brane.

The integrand $e^{\Phi/2}A(r)$ is diverges at $r=r_0$ as $1/\sqrt{r-r_0}$, but we find that the contribution of this part to the integration vanishes,

$$\int_{r_0}dr\ \frac{1}{\sqrt{r-r_0}}=2\sqrt{r-r_0}\ |_{r_0}=0\ . \tag{18}$$

Then we find $\tilde{m_q}<\infty$ for finite r_0 or λ. This means that the quark is not confined in this case since single quark could exist. On the other hand, we find $\tilde{m}_q$ diverges for $r_0=0$ and $q>0$ then the quark is confined.

While, in the above discussion, $q\neq 0$ is essential, we find for $q=0$

$$2\tilde{m}_q=\frac{1}{\pi\alpha'}\frac{(r_{\rm max}-r_0)^2}{r_{\rm max}}\ . \tag{19}$$

Then the quark is not confined in this case even if $\lambda=0$ as seen before [10].

U-shaped Wilson-Loop: The U-shaped configuration is given by $\mathbf{X}_{||}=(\sigma,0,0)$, $\Omega_5=$ constant. Since the Lagrangian does not contain *sigma* explicitly, we could use on integration constant. It is set as the value of r at the midpoint r_{min} of the string is determined by $dr/d\sigma|_{r=r_{min}}=0$. Then the distance and the total energy of the quark and anti-quark are given by

$$L=2R^2\int_{r_{min}}^{r_{\rm max}}dr\ \frac{1}{r^2A(r)a(t)\sqrt{e^{\Phi(r)}r^4A(r)^4/\left(e^{\Phi(r_{min})}r_{min}^4A(r_{min})^4\right)-1}}, \tag{20}$$

$$E=\frac{1}{\pi\alpha'}\int_{r_{min}}^{r_{\rm max}}\frac{A(r)e^{\Phi(r)/2}}{\sqrt{1-e^{\Phi(r_{min})}r_{min}^4A(r_{min})^4/\left(e^{\Phi(r)}r^4A(r)^4\right)}}. \tag{21}$$

Here we study the time independent distance $\tilde{L}$ defined as $\tilde{L}\equiv aL$ instead of L given above. The numerical results are shown in the Fig. 3

Two figures show the dependence of the energy E on the distance $\tilde{L}$ at the selected cosmological constant λ and q. For $\lambda=0$, the well-known results are seen for $q=0$ and finite

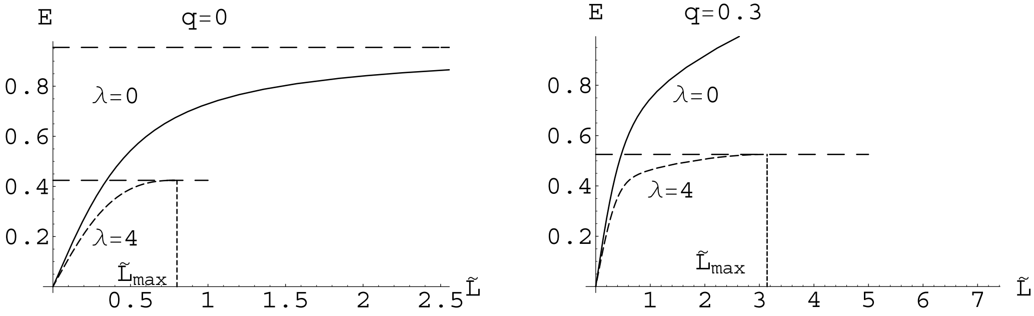

Figure 3: Plots of E vs $\tilde{L}$ at $q = 0$ (the left figure) and $= 0.3$ (the right figure) (GeV^{-4}) for $R = 1\,(\mathrm{GeV}^{-1})$, $r_{\max} = 3\,(\mathrm{GeV}^{-1})$ and $\alpha' = 1\,(\mathrm{GeV}^{-2})$. The solid and dashed curves represent the case of $\lambda = 0$ and $\lambda = 4\,(\mathrm{GeV})$, respectively. The vertical solid and dashed lines represent the energy of two parallel straight strings.

q; (i) For $q = 0$, $E \propto 1/L$ at large L and $E \propto m_q^2\ L$ at small L [4]. And for finite q [10], $E \propto \sqrt{q}L$ at large L and $E \propto m_q^2\ L$ at small L. The behaviors at small L are the common since the same AdS limit is realized there.

For finite λ, there is a maximum bound of L $(= L_{\max})$. In other words, the U-shaped configuration disappears for $L > L$ $(= L_{\max})$. Similar behavior is seen also in the case of the finite temperature [20]. In this sense, the theory in dS_4 is in the quark deconfinement phase as in the finite temperature case. However, we notice the following difference. In the case of finite temperature, there are two possible U-shaped string configurations at the same values of L $(< L_{\max})$, but in the case of finite cosmological constant, U-shaped string configuration is unique at a given value of $\tilde{L}(\tilde{L} < \tilde{L}_{max})$. And at $\tilde{L} = \tilde{L}_{max}$, the energy of this string configuration arrives at $2\tilde{m}_q$. Then, this implies that the U-shaped string configuration is broken for $L > L_{\max}$ to decay to free quark and anti-quark.

On the other hand, an unstable U-shaped string configuration is allowed for the finite temperature case even if $E > 2\tilde{m}_q$, and, just in this energy region, the other U-shaped string configuration is formed.

In our model, $\tilde{L}_{max}$ is obtained as (the details are seen in [1]),

$$\tilde{L}_{max} = \lim_{r_{min}\to r_0} \tilde{L}\ \sim \frac{\pi}{2}\frac{1}{\sqrt{\lambda}}\,. \tag{22}$$

This result implies that the size of the U-shape string configuration is bounded by the length $\lambda^{-1/2}$. So we can observe mesons whose size is smaller than $\lambda^{-1/2}$. At present, we can suppose $\lambda^{1/2} \sim 10^{-3}$ eV, so the size of meson $10^{12}\times$(typical hadoron size) is forbidden. Then the small λ does not work to deconfine the present hadron to free quarks. The real deconfinement phase due to the cosmological constant would be see at very early Universe when $\lambda^{1/2} \sim 1$ GeV. For $\lambda^{1/2} \sim M_{pl}$, any hadron can not exist. This point is assured through numerical calculation of meson mass spectra [1].

5 Summary

The Yang-Mills theory with flavor quarks is investigated in the inflationally expanding 4d space-time, dS_4 space-time. The flavor quarks are introduced by embedding the D7 brane as a probe. The 10d background is deformed from $\mathrm{AdS}_5 \times S^5$ by the dilaton and axion, and its 4d boundary of AdS_5 is set as the time-dependent dS_4 space-time with a 4d cosmological constant.

In this model, the asymptotic form of the profile function $w(\rho)$ of D7 brane includes logarithmic terms coming from the loop-corrections. Actually, we find the following form

$$w(\rho) \sim m_q + \frac{c_0 - 4m_q r_0^2 \log(\rho)}{\rho^2},$$

at the lowest order of large ρ limit, and this implies the vev of the bilinear of quark fields, $\langle\bar{\Psi}\Psi\rangle$, receives the loop correction proportional to the cosmological constant λ and the quark mass m_q as

$$-\langle\bar{\Psi}\Psi\rangle = \frac{c}{R^4} = \frac{c_0}{R^4} - m_q\lambda\log(\rho). \tag{23}$$

This kind of correction would be expected in other quantities also.

And we notice $c < 0$ for any solution of $m_q > 0$ and $c = 0$ for $m_q = 0$. This implies that the chiral symmetry is kept being unbroken for dS_4. The solutions are separated to two groups by their infrared end point whether it is above the horizon or just on the horizon. And when the solution is switched from the one group to the other, a phse transition has been observed. These properties are similar to the finite temperature case.

Another similar property seen in the finite temperature theory is found as the screenig of the quark confinement force. In order to see the potential between quarks, the Wilson-Polyakov loops are studied. Our model shows quark confinement at $\lambda = 0$, but, in dS_4 or for $\lambda > 0$, the potential disappears at large L, where L denotes the distance between quark and anti-quark, for $L > L_{max}$. At $L = L_{max}$, the energy of quark and anti-quark system is equal to the one of two parallel strings, which connect horizon and the D7 brane. This means that the quark and anti-quark do not make the bound state for $L > L_{max}$ since they can move freely. In this sense, we can say that the gauge theory in dS_4 is in the quark deconfinement phase.

While the gauge theory of dS_4 is in the deconfinement phase, we expect that some meson states are stable for small λ. In order to assure this point, the spectra of mesons are examined through the fluctuations of D7 brane [1]. Then, we can show that any meson state for a definite quark mass becomes stable when we take λ to be small enough.

Our previous brane world model has been extended to the dS_4 [21], and the results obtained here would be useful to develop our model such that it could include the gauge fields dynamically coupled with gravity.

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Electromagnetic phenomena in continuous media

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Abstract

After exactly 100 years we still have two competing theories due to Minkowski and Abraham for macroscopic electromagnetism in media. They differ in particular after quantization. A summary of the theoretical and experimental situation is given. When applied to Cerenkov radiation at the quantum level, the Abraham version is ruled out while the Minkowski theory requires free photons with negative energy. Recently an effective field theory has been proposed which avoids these problems by considering the photon as a quasiparticle like any other excitation in condensed matter physics for which the rest frame of the medium is a preferred frame. It relates many different classical and quantum optical phenomena in a unified description [1].

1 Introduction

Nearly 40 years ago Brevik[1] published his first papers on his investigations of electromagnetism in continuous media based on the conflicting theories of Minkowski and Abraham[2][3]. During the subsequent years he continued this line of research and became a central person in the effort to sort out the different problems. Much of his insight was summed up in the well-known review[4]. In parallel to this effort he also pursued investigations of different manifestations of the Casimir effect[5]. In particular, he was one of the first to study systematically the effects of having confining walls not being ideal conductors, but made of realistic materials[6]. For this one needs an understanding of the boundary conditions to impose on the fluctuating field in the vacuum. Again one is faced with the need for a consistent theory for electromagnetism in continuous media.

The most recent step in this line of research was taken this year by Brevik in collaboration with Milton[7]. They calculated the Casimir force between two parallel and ideal plates separated by a distance L and enclosing dielectric matter with a refractive index n. After a rather long and detailed calculation they found the resulting force to be a factor n smaller than the standard vacuum force $F_0 = -\hbar\pi^2/240L^4$ in units where the velocity of light in vacuum is $c_0 = 1$.

[1] This article is **dedicated to 70th aniversary of Professor Iver Brevik**

This simple result is obtained using the Minkowski theory for electromagnetism in media. If one consults the standard text books, one finds instead that the Abraham theory is generally preferred from a theoretical point of view based on the standard theory of relativity[8]. But already in his 1979 review article Brevik concluded that the Minkowski formulation was in agreement with most experiments. This seems to be the situation also today[9]. One might guess that this can be the reason for Jackson in the latest edition of his book argues that the Abraham theory is fundamentally correct, but is modified to become the Minkowski theory when one includes the motion of the particles making up the medium. This point of view reflects a long discussion in the literature[10]. A detailed review of the theoretical situation has recently been given by Obukhov[11].

At the most elementary level it is tempting to think of electromagnetism in a medium to be a straight-forward modification of the standard vacuum theory. A monochromatic wave will have a certain frequency ν and a certain wavelength λ where $\nu\lambda = 1/n < 1$ is the velocity of light in the medium. This simple fact is the basis of geometrical optics. When such an electromagnetic wave is quantized, the corresponding photon is expected to have a momentum $p = h/\lambda$ and energy $E = h\nu$. Introducing the wave vector $\mathbf{k}$ in the direction of the wave, we can then write the momentum vector as $\mathbf{p} = \hbar\mathbf{k}$ when the wave number $k = 2\pi/\lambda$. Similarly, the energy becomes $E = \hbar\omega$ where $\omega = k/n$.

With this simple-minded quantization, one can now look at the Casimir problem investigated by Brevik and Milton[7]. The force results from the zero-point, electromagnetic field energy between the plates which is just $\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} = (\hbar/n)\sum_{\mathbf{k}} |\mathbf{k}|$. Except for the factor $1/n$, this just the standard Casimir energy for vacuum between the plates. We thus have reproduced their result without any calculations.

Another consequence of these elementary ideas is to consider black-body radiation in a cavity filled with the same dielectric matter at temperature T. Standard statistical mechanics says then that the energy density in the large-volume limit is given by

$$u = 2\int \frac{d^3k}{(2\pi)^3} \frac{\hbar\omega_{\mathbf{k}}}{e^{\hbar\omega_{\mathbf{k}}/k_BT} - 1} \tag{1}$$

where k_B is the Boltzmann constant. Again using $\omega_{\mathbf{k}} = k/n$ we find a result which is simply n^3 times the vacuum value $u_0 = \pi^2(k_BT)^4/15\hbar^3$. Interestingly, in the book by Landau and Lifshitz where they also endorse the Abraham description, the same result is derived from consideration of correlators of fluctuating currents in the enclosing cavity walls[12]. At the end of a rather elaborate calculation, they just state without any further comments that the same result can be obtained more directly as done above. It would be interesting and of some importance to verify this experimentally.

These simple ideas thus seem to reproduce some results in a satisfactory way. But can it be part of a consistent theory? What about the photon mass in this picture? In special relativity the squared mass is given by $m^2 = E^2 - p^2$. This gives in our case $(\hbar\omega)^2(1-n^2) < 0$, i.e the photon four-momentum is space-like as for a tachyon. We will in the following see that this is actually the result emerging from the Minkowski theory. Can we live with this? Tachyons in ordinary field theories usually signal some instability which we don't expect to find here. And what about gauge invariance? This fundamental symmetry is in vacuum related to having massless photons.

Recently an attempt has been made to clarify this rather confusing situation[13]. Most of the problems seem to result from forcing the theory into the standard framework of special relativity which is not present as a physical symmetry of the system. Instead one can avoid the problems by considering the electromagnetic field in the macroscopic limit as any other excitation in a medium for which the rest frame is a preferred frame. It can be described as an ordinary, effective field theory. The resulting photons are then quasi-particles on the same footing as quanta in any other field theory with a linear dispersion relation

Starting with the Maxwell equations in the next section, we review the derivation of the energy and momentum of the electromagnetic field in a continuous medium. Gauge invariance is discussed and the equivalent of the covariant Lorenz gauge is found. From the form of the resulting wave equation it follows that the theory is invariant under Lorentz transformations involving the physical speed of light $1/n$ in the medium. The covariant theories of Minkowski and Abraham are briefly described and confronted with a different covariance of the effective theory. Quantization is performed in the rest frame and implications for the three theories discussed.

The following section is devoted to the Cerenkov effect which is most easily understood in the rest frame of the medium. It can also be derived within the Minkowski theory in an arbitrary frame if we are willing to accept free photons with negative energies. On the other hand, the Cerenkov effect at the quantum level is inconsistent with the Abraham formulation. In the last section higher order interactions are added to the free Lagrangian to give an effective field theory which incorporates non-linear dispersion and the Kerr effects in a natural way. Thus it relates many different classical and quantum optical phenomena into a unified and consistent theory.

2 Maxwell theory

Assuming no charges or currents present in the material, the electric fields $\mathbf{E}, \mathbf{D}$ and magnetic fields $\mathbf{B}, \mathbf{H}$ are in general governed by the Maxwell equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \nabla \cdot \mathbf{B} = 0 \tag{2}$$

and

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0, \qquad \nabla \cdot \mathbf{D} = 0 \tag{3}$$

The displacement field $\mathbf{D}$ describes the modification of the electric field $\mathbf{E}$ by the polarization of the atoms in the material, while $\mathbf{H}$ describes the similar modification of the magnetic field $\mathbf{B}$ due to magnetization of the atoms. When the medium can be considered as an isotropic continuum, the relation between these macroscopic fields in the rest frame of the system can be written as $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ as explained in standard text books[8]. These constitutive relations represent very complex phenomena on a microscopic scale involving a large number of atoms. The effective description is therefore only valid on large scales, or equivalently, at sufficiently low energies.

As a first approximation we will take the electric permittivity ε and the magnetic permeability μ to be constants. In the following we will use units so that for the vacuum $\varepsilon_0 = \mu_0 = 1$. It is then straight-forward to show that the above Maxwell equations are Lorentz invariant, but only for transformations involving the physical speed of light $1/\sqrt{\varepsilon\mu}$ in the medium. This should be obvious without any explicit derivation since the theory is identical with the one in vacuum except for this difference in light velocity.

Since the second Maxwell equation in (2) is satisfied by writing $\mathbf{B} = \nabla \times \mathbf{A}$ where $\mathbf{A}$ is the magnetic vector potential, it follows from the first equation that $\mathbf{E} + \partial\mathbf{A}/\partial t$ must be a gradient of a scalar field. One can therefore write

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \tag{4}$$

where Φ is the electric potential. Both the electric and magnetic fields in the medium can therefore be expressed in terms of potentials in the same way as in vacuum. They are invariant under the simultaneous gauge transformations $\mathbf{A} \rightarrow \mathbf{A}+\nabla\chi$ and $\Phi \rightarrow \Phi-\partial\chi/\partial t$ where $\chi(\mathbf{x},t)$ is an arbitrary, scalar function.

Using now these field expressions together with the constitutive relations in the first of equation (3), one obtains the equation of motion

$$\nabla \times (\nabla \times \mathbf{A}) + \varepsilon\mu\frac{\partial}{\partial t}\Big(\frac{\partial \mathbf{A}}{\partial t} + \nabla\Phi\Big) = 0 \tag{5}$$

for the two potentials. Introducing the index of refraction $n = \sqrt{\varepsilon\mu}$, it can be rewritten as

$$\Big(n^2\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\mathbf{A} + \nabla\Big(n^2\dot{\Phi} + \nabla\cdot\mathbf{A}\Big) = 0 \tag{6}$$

Now imposing the gauge condition

$$n^2\dot{\Phi} + \nabla\cdot\mathbf{A} = 0 \tag{7}$$

in the medium, one obtains the standard wave equation

$$\Big(n^2\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\mathbf{A}(\mathbf{x},t) = 0 \tag{8}$$

The electromagnetic propagation velocity is thus $1/n$ as expected. Needless to say, the gauge condition (7) is equivalent to choosing the covariant Lorenz gauge in vacuum.

With the assumption of no free charges, the Maxwell equation $\nabla\cdot\mathbf{E} = 0$ gives the relation $\nabla\cdot\dot{\mathbf{A}} = -\nabla^2\Phi$ with the use of (4). Taking the time derivative of the gauge condition (7), we then see that the scalar potential $\Phi(\mathbf{x},t)$ satisfies the same wave equation (8) as the vector potential. Both of these equations of motion follow from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\varepsilon\mathbf{E}^2 - \frac{1}{2\mu}\mathbf{B}^2 \tag{9}$$

where the potentials $\mathbf{A}$ and Φ are the dynamic fields. On this form it is obviously only valid in the rest frame of the medium.

The energy content of the electromagnetic field in a medium is obtained by standard methods[8]. One takes the scalar products of the first equation in (2) with $\mathbf{H}$ and the first equation in (3) with $\mathbf{E}$. Subtracting the two resulting expressions, the equation

$$\frac{\partial\mathcal{E}}{\partial t} + \nabla\cdot\mathbf{N} = 0 \tag{10}$$

follows. It represents conservation of energy where

$$\mathcal{E} = \frac{1}{2}\big(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\big) \tag{11}$$

is the standard energy density and $\mathbf{N} = \mathbf{E}\times\mathbf{H}$ is the Poynting vector describing the energy current carried by the field.

Momentum conservation can be similarly obtained by forming the vector products of the first equation in (2) with $\mathbf{D}$ and the first equation in (3) with $\mathbf{B}$. Combining the two resulting expressions, one then finds

$$(\nabla\times\mathbf{H})\times\mathbf{B} + (\nabla\times\mathbf{E})\times\mathbf{D} = \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) \tag{12}$$

This can be written on a more compact form using the triple vector product formula $\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. It results in

$$\frac{\partial \mathbf{G}}{\partial t} + \nabla \cdot \mathbf{T} = 0 \tag{13}$$

where $\mathbf{G} = \mathbf{D} \times \mathbf{B}$ and

$$T_{ij} = -(E_i D_j + B_i H_j) + \frac{1}{2}\delta_{ij}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \tag{14}$$

is the Maxwell stress tensor. Using the constitutive equations, it is seen to be symmetric in the rest frame of the medium. It is thus natural to consider the vector $\mathbf{G}$ to represent the momentum density of the field.

3 Covariant formulations

There seems to be no disagreement around the presentation given in the previous section. The difficulties start when one attempts to embed this non-covariant formulation into a four-dimensional framework based on the special theory of relativity. One could then discuss electromagnetic phenomena in a general, inertial frame where the medium could have any velocity below the velocity of light in vacuum. This was first done by Minkowski at the same time as his successfull covariant formulation of the Maxwell theory in vacuum[2].

In the rest frame of the medium the space and time coordinates of an event can be combined into a four-dimensional vector $x^\mu = (t, \mathbf{x})$. The covariant gradient operator is then $\partial_\mu = (\partial/\partial t, \nabla)$. We will raise and lower Greek indices with the standard Lorentz metric $\eta_{\mu\nu}$, taken here to have negative signature. Combining the two potentials Φ and $\mathbf{A}$ into the four-dimensional vector potential $A^\mu = (\Phi, \mathbf{A})$, the antisymmetric field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is then seen to have the components

$$F^{\mu\nu} = \left(\begin{array}{c|c} 0 & -\mathbf{E} \\ \hline \mathbf{E} & -B_{ij} \end{array}\right) \tag{15}$$

in the same frame where $B_{ij} = \varepsilon_{ijk} B_k$. The first two Maxwell equations (2) can then be written as

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \tag{16}$$

Thus this part of the Maxwell theory in a medium is the same as in vacuum.

The problems arise with the remaining fields $\mathbf{D}$ and $\mathbf{H}$. In analogy with the tensor (15) they can be combined into a new, antisymmetric tensor

$$H^{\mu\nu} = \left(\begin{array}{c|c} 0 & -\mathbf{D} \\ \hline \mathbf{D} & -H_{ij} \end{array}\right) \tag{17}$$

with $H_{ij} = \varepsilon_{ijk} H_k$. The two last Maxwell equations (3) can then be simply reduced to $\partial_\nu H^{\mu\nu} = 0$. It also makes it possible to write the Lagrangian (9) on the compact form $\mathcal{L} = -F_{\mu\nu}H^{\mu\nu}/4$ when we make use of the constitutive equations in the rest frame. Despite the covariant form of the Lagrangian, it does not represent a Lorentz-invariant theory in the usual sense. This is so because the phenomenological tensor $H_{\mu\nu}$ must be expressed in terms of the more fundamental tensor $F_{\mu\nu}$ in a frame where the medium is in motion so to generalize the constitutive equations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ valid only in the rest frame of the medium. Such a relation can always be found, but will obviously involve the velocity of the medium[2][14]. In a general frame this velocity will then enter the Lagrangian explicitly and thus signal the lack of physical invariance under vacuum Lorentz transformations as already mentioned. The true invariance of the Maxwell theory in a medium is represented by Lorentz transformations involving the reduced speed of light $1/n$.

But as long as we restrict ourselves to the rest frame of the medium, there are so far no problems. The energy and momentum content of the field derived in the previous section, can then be combined into the four-dimensional energy-momentum tensor

$$T_M^{\mu\nu} = \left(\begin{array}{c|c} \mathcal{E} & \mathbf{N} \\ \hline \mathbf{G} & T_{ij} \end{array}\right) \tag{18}$$

valid in this frame. Minkowski wrote it as

$$T_M^{\mu\nu} = F^\mu{}_\alpha H^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} H^{\alpha\beta} \tag{19}$$

with the intention of making use of it in any inertial frame. A direct derivation can be found in the book by Möller[15]. The two conservation laws can now be expressed on the more compact form $\partial_\nu T_M^{\mu\nu} = 0$. This energy-momentum tensor is in general seen not to be symmetric which implies that the total angular momentum of the field is not conserved. Only in the limit $n \to 1$ where it becomes the electromagnetic energy-momentum tensor of the vacuum, do we recover this desired property.

In order to remedy this lack of symmetry, Abraham proposed the following year that only the symmetric part of the Minkowski tensor should describe the energy-momentum content of the field[3]. The momentum density in the rest frame must therefore equal the Poynting vector $\mathbf{E} \times \mathbf{H}$, i.e. a factor n^2 smaller than the momentum density $\mathbf{D} \times \mathbf{B}$ of the Minkowski theory. The resulting energy-momentum tensor is no longer conserved on either index. Instead there should exist a new volume force which has so far avoided any clear-cut experimental verification.

The most direct way to distinguish between these two formulations, is via the radiation pressure directly related to the electromagnetic momentum density. It should thus be a factor n^2 smaller than in the Minkowski version. This seems to be ruled out by most experiments[16]. But according to Garrison and Chiao[9], there still are a few experiments more consistent with the Abraham theory. Typically of them is that the investigated systems undergo acceleration. Obviously, this makes the interpretation of the measurements more difficult.

Both the Minkowski and the Abraham formulations are based on being valid in any inertial frame related by ordinary vacuum Lorentz transformations. The recently proposed effective theory is instead only valid in the medium rest frame as similar theories in condensed-matter physics[13]. In this frame light moves with the velocity $1/n$. The corresponding light cone is $|\mathbf{x}| = \pm t/n$. As in vacuum, it is desirable to write this on an infinitesemal level as $ds^2 = 0$ with a line element on the form $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. If we now choose $\eta_{\mu\nu}$ to be the Minkowski vacuum metric, the contravariant coordinates in this frame must be $x^\mu = (t/n, \mathbf{x})$. The corresponding covariant derivative is obviously then $\partial_\mu = (n\partial/\partial t, \nabla)$. In a quantum theory this should correspond to the four-momentum $p^\mu = (nE, \mathbf{p})$ for a particle with energy E and three-momentum $\mathbf{p}$. The d'Alembertian $\partial^\mu \partial_\mu = (n^2\partial_t^2 - \nabla^2)$ is invariant under Lorentz transformation corresponding to the light speed $1/n$. It is seen to equal the wave operator we found for the Maxwell theory in a medium.

This theory can now be given a simple covariant formulation. We introduce a four-vector electromagnetic potential $A^\mu = (n\Phi, \mathbf{A})$ so that the electric and magnetic field vectors are again given by the antisymmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It has now the components

$$F^{\mu\nu} = \left(\begin{array}{c|c} 0 & -n\mathbf{E} \\ \hline n\mathbf{E} & -B_{ij} \end{array} \right) \tag{20}$$

instead of (15) for the Minkowski formulation. The rest-frame Lagrangian (9) takes then the standard form $\mu\mathcal{L} = -(1/4)F_{\mu\nu}^2$. The first set of field equations (16) obviously remains unchanged while the second Maxwell equations (3) are replaced by $\partial_\mu F^{\mu\nu} = 0$ when we make use of the constitutive equations. One thus obtains the wave equation $\partial^2 A^\nu - \partial^\nu(\partial \cdot A) = 0$. In the Lorenz gauge defined by $\partial_\mu A^\mu = 0$, it gives the previous wave equation (8). Notice that this covariant gauge condition becomes (7) when written out in terms of components.

From the above invariant Lagrangian the energy-momentum tensor can now be derived as in vacuum, giving

$$\mu T^{\mu\nu} = F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} \tag{21}$$

with components

$$T^{\mu\nu} = \left(\begin{array}{c|c} \mathcal{E} & n\mathbf{N} \\ \hline n\mathbf{N} & T_{ij} \end{array} \right) \tag{22}$$

It is obviously symmetric, traceless and conserved on both indices, i.e. $\partial_\mu T^{\mu\nu} = \partial_\mu T^{\nu\mu} = 0$. In the time direction this gives energy conservation on the form (10) while in the space directions it gives momentum conservation as in (13). The momentum density of the field $\mathbf{G} = \mathbf{D} \times \mathbf{B}$ is therefore the same as in standard Maxwell theory and for the Minkowski description restricted to the rest frame.

4 Quantization

In the rest frame of the system we have the Lagrangian density (9) and the theory can be quantized by standard methods. With no free charges, we can take the scalar potential $\Phi = 0$ and use the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. There are then only two transverse field degrees of freedom governed by Lagrangian

$$L = \int d^3x \left[\frac{1}{2}\varepsilon \dot{\mathbf{A}}^2 - \frac{1}{2\mu}(\nabla \times \mathbf{A})^2 \right] \tag{23}$$

With the system in a volume V with periodic boundary conditions, we can expand the vector potential in plane waves as

$$\mathbf{A}(\mathbf{x}, t) = \sqrt{\frac{1}{V}} \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{24}$$

where each Fourier mode with amplitude $\mathbf{A}_{\mathbf{k}}(t)$ is characterized by a discrete wave vector $\mathbf{k}$. In terms of these complex amplitudes satisfying $\mathbf{A}_{\mathbf{k}}^* = \mathbf{A}_{-\mathbf{k}}$, the Lagrangian becomes

$$L = \frac{1}{2}\varepsilon \sum_{\mathbf{k}} \left(\dot{\mathbf{A}}_{\mathbf{k}} \dot{\mathbf{A}}_{\mathbf{k}}^* - \omega_{\mathbf{k}}^2 \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{\mathbf{k}}^* \right) \tag{25}$$

Each term is seen to describe a harmonic oscillator with frequency $\omega_{\mathbf{k}} = |\mathbf{k}|/n$. Introducing standard creation and annihilation operators for photons with definite polarizations λ, the quantized Hamiltonian thus takes the standard form

$$H = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} \left(a^{\dagger}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \frac{1}{2} \right) \tag{26}$$

where the last term gives the zero-point energy. A single photon with the wave vector $\mathbf{k}$ thus has the energy $E = \hbar\omega_{\mathbf{k}}$. This will also be the photon energy in the Minkowski and Abraham theories as long as they are restricted to the rest frame of the medium.

With the above classical momentum density, we can now derive in a similar way the operator for the total momentum of the quantized field from

$$\mathbf{P} = \int d^3x\, \mathbf{D} \times \mathbf{B} \tag{27}$$

It simplifies to

$$\mathbf{P} = \sum_{\mathbf{k}} \hbar\mathbf{k} (a^{\dagger}_{\mathbf{k}+} a_{\mathbf{k}+} + a^{\dagger}_{\mathbf{k}-} a_{\mathbf{k}-}) \tag{28}$$

when we make use of the same plane-wave expansion and write out explicitly the contributions from the two polarization directions. Thus a photon with wave vector $\mathbf{k}$ has the momentum $\mathbf{p} = \hbar\mathbf{k}$. In the new, covariant formulation the mass-squared of the photon is $(nE)^2 - p^2 = 0$ for $E = \hbar\omega_{\mathbf{k}}$ and $p = \hbar k$. We can therefore say that it is massless also in a medium.

Needless to say, the above energy and momentum of a photon will also result from the Minkowski theory when it is restricted to the rest frame. But this formulation is by construction valid in any inertial frame related to the rest frame by a vacuum Lorentz transformation. The theory can then in principle be quantized in such an arbitrary frame where the medium is in motion. This was first done by Jauch and Watson[14]. As expected, it is much more cumbersome than the above rest-frame quantization and with new problems. This should not come as a surprise since these vacuum Lorentz transformations do not represent a physical symmetry. In particular there are difficulties in the treatment of the longitudinal components of the radiation field. A later attempt by Brevik and Lautrup to to clarify the situation, did not lead to a definite conclusion[17] .

A simple example of such a problem is to consider a photon with the four-momentum $p^{\mu} = (\hbar\omega_{\mathbf{k}}, \hbar\mathbf{k})$ moving along the x-axis in the rest frame. In the Minkowski formulation, it is space-like since $E^2 - p^2 < 0$ and moves with a velocity $1/n < 1$. Now going to a new inertial frame by a vacuum Lorentz transformation moving along the x-axis with a velocity $v > 1/n$, it will be observed to have negative energy[18]. What this means physically, is not clear. One cannot simply assign it a negative frequency. It must in some way be the matter which zooms by in this frame, which imparts upon the photon this negative energy. A similar, strange situation will be found in the next section for the Cerenkov effect when described by the same Minkowski theory.

According to the Abraham description, the photon energy is the same as above while the momentum in the rest frame is reduced to $\mathbf{p} = \hbar\mathbf{k}/n^2$. Its four-momentum is now time-like since $E^2 - p^2 = (\hbar\omega)^2(1 - 1/n^2) > 0$. This is more appropriate for a particle moving with a speed less than the velocity of light in vacuum. But the squared four-momentum is again not an invariant since the theory has no invariance under vacuum Lorentz transformations. In the next section we will see that this formulation of the theory has even greater problems with the Cerenkov effect.

The total angular momentum of the field is given by the classical expression

$$\mathbf{J} = \int d^3x\, \mathbf{r} \times (\mathbf{D} \times \mathbf{B}) \tag{29}$$

Separating out the orbital part, the intrinsic spin part can be quantized and becomes

$$\mathbf{S} = \sum_{\mathbf{k}} \hbar \widehat{\mathbf{k}} (a^\dagger_{\mathbf{k}+} a_{\mathbf{k}+} - a^\dagger_{\mathbf{k}-} a_{\mathbf{k}-}) \tag{30}$$

where $\widehat{\mathbf{k}}$ is a unit vector along the wave vector $\mathbf{k}$. Needless to say, this is exactly the same result as in vacuum. The photon in a medium has spin $S = 1$ with only two helicities $\lambda = \pm$ required for a massless vector particle.

In the Minkowski formulation the photon has a non-zero mass and one should therefore *a priori* expect the spin to have a third direction. This is even more true for the Abraham formulation, but here the magnitude of the photon spin is reduced to $S = 1/n^2$. It was therefore suggested by Brevik in his review paper[4]) that a measurement of the photon spin would offer a clear method to differentiate between these two theories. Some years later such an experiment was performed[19] giving a value very close to $S = 1$. Even if this measurement was not made on free photons as above, but on photons in a wave guide filled with a dielectric liquid, the result should be the same. Again the validity of the Abraham theory seems to be ruled out.

5 Cerenkov radiation of photons

When a charged particle with a speed $v > 1/n$ passes through a medium with index of refraction n, electromagnetic radiation is emitted. This Cerenkov effect is similar to a sonic boom when an object goes through air with a speed larger than the speed of sound.

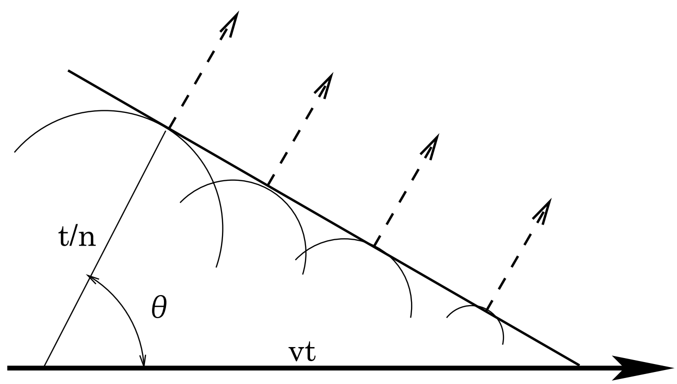

Figure 1: Cerenkov radiation from particle with velocity v during a time t in a medium where velocity of light is $1/n$.

The radiation is emitted in a cone with opening angle given by $\cos\theta = 1/nv$ as shown in Fig.1. As first demonstrated by Frank and Tamm, it is a classical effect following directly from the previous Maxwell equations in the rest frame of the medium[8]. At the microscopic level it corresponds to the incoming particle emitting a photon in a direction θ away from the incoming direction and continuing in a slightly different direction with smaller energy as shown in Fig.2. The quantum mechanical transition rate for this process was calculated by Jauch and Watson in the same frame using the Minkowski formulation[20]. They obtained a radiation rate in agreement with the Frank-Tamm result. This is to be expected from the correspondence principle.

If we denote the energy and momentum of the incoming particle by E and p and similarly primed quantities for the outgoing particle, then energy conservation implies $E = E' + \hbar\omega$. The photon frequency ω is related to its wave number by $\omega = k/n$. Using now the photon momentum $\hbar k$ from the previous section, one has momentum conservation $p = p'_x + \hbar k\cos\theta$ along the incoming x-direction. In the normal y-direction, it similarly follows that $p'_y + \hbar k\sin\theta = 0$. Squaring these two eqations and adding, it follows that

$$p'^2 = p^2 + (\hbar k)^2 - 2\hbar kp\cos\theta \tag{31}$$

Combining this with the squared conservation equation for energy which takes the form

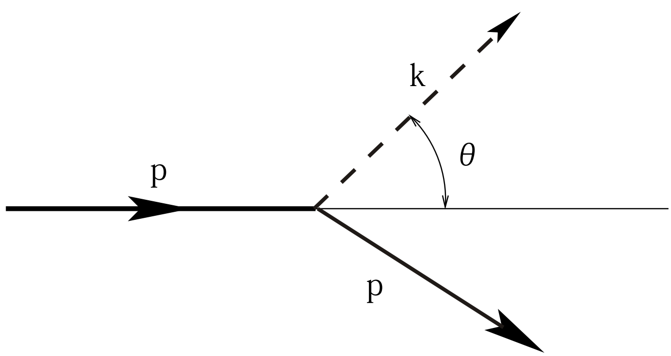

Figure 2: Cerenkov radiation of a photon with wave vector **k** from a charged particle with momentum **p**.

$p'^2 = p^2 - 2\hbar kE/n + (\hbar k/n)^2$, the deflection angle is seen to be determined by

$$\cos\theta = \frac{1}{nv} + \frac{\hbar k(n^2-1)}{2pn^2} \tag{32}$$

where $v = p/E$ is the velocity of the incoming particle. When the particle is relativistic and we consider the emission of visible light, the last, quantum term can be neglected. The angle is then given by the classical expression.

Obviously, this derivation also holds for the Minkowski theory in the rest frame. But in this case we can in principle consider the process in any other inertial frame where the theory should be just as valid. For this reason Jauch and Watson also used the special frame where the incoming particle is at rest. From the kinematics in this frame it then follows that it can then decay into a new particle with a certain three-momentum plus a photon with the opposite momentum. Since the masses of the initial and final particles are assumed to be the same, energy conservation then gives that the photon must have negative energy in this frame. It is therefore a photon with properties very different from all other photons in physics. Although this result seems to be mathematically correct, one must be allowed to ask about its physical validity.

For the Abraham description this process is catastrophic. Going through the same steps as above, but now with the photon momentum $\hbar k/n^2$, it follows immediately that the classical term for the deflection angle gives $\cos\theta = n/v > 1$ for physical velocities. Thus there can be no Cerenkov effect at the quantum level in this case as also noted a long time ago by Brevik and Lautrup[17]. This should come as no surprise since the momentum of a photon with wavelength λ is no longer given by the fundamental de Broglie expression h/λ in this formulation.

6 Including interactions

So far we have only considered the free theory described by the Lagrangian (9) and assuming the phenomenological parameters ε and μ to be constants. It is therefore only valid on very large scales, i.e. at energies so low that no microscopic degrees of freedom are excited. For a physical medium made out of atoms this corresponds to energies much less than a few eV. At higher energies, these effects will start to manifest themselves and must be included some way. In particular we need to incorporate non-linear dispersion in order for the theory to be realistic. And it must be done in such a way that it allows a consistent treatment at the quantum level.

The free theory was formulated along the same lines as for other excitations in condensed matter physics. For many years it has been well known in this field how to incorporate microscopic effects in a macroscopic description by extending the free theory in the rest frame by including higher-order operators in the Lagrangian. The coupling constants of these new terms are determined by the microscopic physics. They must be determined from an underlying, more fundamental theory or from experiments. The resulting Lagrangian describes then an interacting, effective theory. Although it is in general said to be non-renormalizable, finite quantum corrections can be derived from it as long as one restricts oneself to phenomena below a characteristic energy. Such effective field theories have during the last 10-20 years also become of great use in high energy physics[21]. The first well-known theory of this kind was found by Euler and Heisenberg already in 1936 to describe classical electromagnetic effects in strong fields, induced by virtual electron-positron pairs in the vacuum[22]. It is first quite recently that it became clear that it could also be used as an effective, quantum field theory[23]. Now a similar, effective theory for electromagnetic phenomena in media has been proposed[13].

In order to be gauge invariant, higher-order terms or couplings in the Lagrangian can only involve the fields $\mathbf{E}$ and $\mathbf{B}$ and derivatives of them. For the sake of counting, we can use quantum units with $\hbar = 1$ so that these fields have dimension $+2$ and every derivative corresponds to an increase in dimension by $+1$. To be invariant under time-reversal and ordinary rotations, such new couplings must involve at least two spacetime derivatives. For example, one possibility could be the term $\mathbf{E}\cdot\partial^2\mathbf{E}$. It has dimension 6. But the lowest order equation of motion is just $\partial^2\mathbf{E} = 0$ and this term can therefore not contribute. Possible new terms of dimension 8 would be $(\mathbf{E}\cdot\mathbf{E})^2$, $(\mathbf{B}\cdot\mathbf{B})^2$, $\mathbf{E}^2\mathbf{B}^2$ and $(\mathbf{E}\cdot\mathbf{B})^2$. All such terms describe anharmonic interactions involving four fields.

The simplest form of non-linear dispersion follows from dimension-6 interactions when we restrict ourselves to a theory with only rotational invariance. One example of a possible interaction is then $\nabla_i\mathbf{E}\cdot\nabla_i\mathbf{E}$. It is equivalent to $\mathbf{E}\cdot\nabla^2\mathbf{E}$ by a partial integration in the action integral where it appears. The similar term $\partial_t\mathbf{E}\cdot\partial_t\mathbf{E}$ involving two time derivatives is for the same reason equivalent to $\mathbf{E}\cdot\nabla^2\mathbf{E}$ when we use the equation of motion. An interaction like $\mathbf{E}\cdot\nabla^2\mathbf{B}$ is ruled out by parity invariance.

Of most interest are dielectric media for which we can set the permeability $\mu = 1$. In such materials magnetic effects are negligible and it is therefore reasonable to assume that all the terms involving the magnetic field, are absent. The effective Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2}\Big(n^2\mathbf{E}^2 - \mathbf{B}^2\Big) + \frac{d_1}{M^2}\mathbf{E}\cdot\nabla^2\mathbf{E} + \frac{d_3}{M^4}(\nabla^2\mathbf{E})^2 + \frac{a_1}{M^4}(\mathbf{E}\cdot\mathbf{E})^2 \tag{33}$$

when we restrict ourselves to operators with dimension 8 or less. M is a characteristic energy below which the theory should be valid. In addition, it contains only three independent dimensionless parameters d_1, d_3 and a_1. For each material they can therefore be determined by three different measurements when the value of M is known. The Lagrangian should then be able to predict the outcome of other experiments without any more parameter fitting.

The effect of the first new term proportional with d_1 is simplest to analyze since it is quadratic in the field. In the quantum treatment it will give a perturbation ΔE to the energy of a photon with momentum $\hbar k$. It is simple to calculate and the result is found to be $\Delta E = -d_1 k^3/2M^2 n^3$. The resulting total energy $E' = E + \Delta E$ can now be written as $E' = \hbar k/n(\omega)$ where the modified index of refraction is

$$n(\omega) = n\Big(1 - \frac{d_1\omega^2}{2M^2}\Big) \tag{34}$$

where $n = \sqrt{\varepsilon}$ as before. Thus it gives the Cauchy parametrization of non-linear dispersion valid for the longest wavelengths of light[24] when d_1 is negative. Comparing with measured values, we find that $M = 5 - 10$ eV for typical materials if we set the unknown parameter $d_1 = -1$. The operator proportional to d_3 (33) will obviously give a ω^4 correction to this dispersion law. Similarly we can show that the operator $(\mathbf{E}\cdot\mathbf{E})^2$ describes the AC and DC Kerr effects[24]. The mass parameter M is again found to be in the same range as above if we choose $a_1 = 1$. One can therefore instead take M to have the same value for all materials and let the dimensionless parameters d_1, d_3 and a_1 vary from material to material.

7 Conclusion

The Abraham description of electromagnetism in media is inconsistent with both basic theoretical ideas and experimental results. After having been discussed now for 100 years, it is time for it to be laid permanently to rest. While the energy and momentum content resulting from the Minkowski theory in the rest frame of the medium avoid these problems, it still has difficulties with the requirement of being valid in any inertial frame.

Considering instead these fields like any other excitations in condensed matter physics and defined by an effective theory in the medium rest frame, a satisfactory theory can be formulated. Except for the reduced velocity of light, it equals the corresponding theory in vacuum. This new theory thus becomes equivalent to electromagnetism in the ether before 1905. The Maxwell equations were then considered to be valid only in the rest frame of the ether. Einstein's special theory of relativity showed that there is no need for a physical ether and Maxwell equations became valid in all inertial frames. Today we would rather say that there is an ether, but that it is invariant under Lorentz transformations. In contrast, a physical medium is not invariant under these transformations and that makes the whole difference.

A want to thank Iver Brevik for teaching me classical electromagnetism including the Cerenkov effect when I was a student. Throughout the subsequent years he has kept me continually updated on different aspects and problems with the Minkowski and Abraham descriptions of these phenomena in materials.

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Casimir Lifshitz pressure and free energy: exploring a simple model

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Abstract

The Casimir effect, the dispersion force attracting neutral objects to each other, may be understood in terms of multiple scattering of light between the interacting bodies. We explore the simple model in which the bodies are assumed to possess reflection coefficients independent of the energy and angle of incidence of an impinging field and show how much information can be extracted within the geometry of two parallel plates. The full thermal behaviour of the model is found and we discuss how non-analytic behaviour emerges in the combined limits of zero temperature and perfect reflection. Finally we discuss the possibility of a generalised force conjugate to the reflection coefficients of the interacting materials and how, if the materials involved were susceptible to changing their reflective properties, this would tend to enhance the Casimir attraction. The dependence of thi correction on separation is studied for the constant reflection model, indicating that the effect may be negligible under most experimental circumstances [2].

1 Introduction

The Casimir effect was first reported in 1948 [1] as an attractive force between parallel mirrors due to the zero point fluctuations of the electromagnetic field in vacuum. Casimir calculated the formally infinite quantum energy associated with the eigenmodes n of the field between the plates, $\frac{\hbar}{2}\sum_n \omega_n$, subtracted the corresponding energy of free space (infinte plate separation) and obtained after some regularisation the simple result

$$P_C^0 = -\frac{\hbar c\pi^2}{240a^4}; \quad \mathcal{F}_C^0 = -\frac{\hbar c\pi^2}{720a^3} \tag{1}$$

where P_C and $\mathcal{F}_\mathrm{C}$ are the Casimir pressure and free energy per unit plate area respectively and a is the separation between the plates. Here and henceforth a superscript 0 refers to zero temperature. A negative pressure here corresponds to an attractive force. Naturally,

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the relation between pressure and free energy is $P(a) = -\partial\mathcal{F}(a)/\partial a$. In the following we will employ natural units $\hbar = k_{\mathrm{B}} = c = 1$.

In the following section we give a brief review of the understanding of Casimir interactions as a multiple scattering or reflection phenomenon. The remainder of the paper is the beginnings of an exploration of a simple model, first employed in [2] to the author's knowledge. The model is one in which the interacting bodies scatter electromagnetic fields with reflection coefficients $|r| \leq 1$ which are modelled as invariant with respect to the energy and direction of the wave. We do not venture beyond the planar geometry herein, but show that certain closed form solutions exist in this case, and how the model enables simple extraction of key information.

We review in section 3 the derivation of closed form expressions for the Casimir force and free energy in the constant reflection model and in section 4 how this model was used to generalise the frequency spectrum of the Casimir energy to imperfect reflection. In sections 5 through 6 we thereafter calculate the full temperature behaviour of the Casimir-Lifshitz pressure and free energy within the model and demonstrate how one encounters non-analytic behaviour in the limit of perfect reflection, reminiscent of the still ongoing debate over the temperature corrections to the Casimir force. Finally in section 7 we consider the possibility that the Casimir free energy could exhibit a generalised force on the reflective properties of the materials involved, thereby increasing its own magnitude. We lay out the basic theory of such a possibility, not hitherto reported to the author's knowledge, and use the constant reflection model to extract information about how the corresponding correction to Casimir attraction scales with temperature and separation.

2 A brief review of the multiple scattering understanding of Casimir interactions

The beauty and simplicity of Casimir's results (1) stems from the assumption of perfectly conducting plates, that is, the metal plates are perfect mirrors at all frequencies of the electromagnetic field. Drawing on the theory of fluctuations due to Rytov [3], Lifshitz made an important generalisation of Casimir's results to the case of two half-spaces with frequency dependent permittivities $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ [4] (Lifshitz moreover assumed the slabs be immersed in a third medium which we assume to be vacuum here for simplicity). The calculation was rather involved and the result at zero temperature was found to be:

$$P^0 = -\frac{1}{2\pi^2}\int_0^\infty d\zeta \int_\zeta^\infty d\kappa\kappa^2 \sum_{\sigma=s,p} \frac{r_\sigma^{(1)} r_\sigma^{(2)} e^{-2\kappa a}}{1 - r_\sigma^{(1)} r_\sigma^{(2)} e^{-2\kappa a}} \tag{2a}$$

$$\mathcal{F}^0 = \frac{1}{4\pi^2}\int_0^\infty d\zeta \int_\zeta^\infty d\kappa\kappa \sum_{\sigma=s,p} \ln\left[1 - r_\sigma^{(1)} r_\sigma^{(2)} e^{-2\kappa a}\right] \tag{2b}$$

where the quantities $r_\sigma^{(i)}$ pertaining to medium i are

$$r_s^{(i)} = \frac{\kappa - \kappa_i}{\kappa + \kappa_i}; \quad r_p^{(i)} = \frac{\epsilon_i(i\zeta)\kappa - \kappa_i}{\epsilon_i(i\zeta)\kappa + \kappa_i} \tag{3}$$

and $\kappa_i = \kappa_i(\kappa, i\zeta) = \sqrt{\kappa^2 + [\epsilon_i(i\zeta) - 1]\zeta^2}$.

By noting that $i\kappa = k_z$, $\hat{z}$ being the axis normal to the plates one may recognise $r_s^{(i)}$ and $r_p^{(i)}$ as the standard Fresnel reflection coefficients of a single interface for the TE and

TM polarisation respectively, as well known from classical optics. Thus the Casimir-Lifshitz force (2a) does not depend directly on the bulk properties of the materials of the slabs as is ostensible from the original Lifshitz derivation, but only on the reflection properties of the *surfaces* of the material half-spaces. Kats [5] may have been the first to point this out explicitly in 1977, and the point has been given widespread attemtion more recently [6, 7, 8, 9]. It is a simple exercise to show that inserting $(r_\sigma^{(i)})^2 = 1,\ \ \forall i, \sigma$ into (2a) and (2b) yields the Casimir limits (1).

The trait that the Casimir-Lifshitz pressure (2a) is a function of reflection properties only is a tell-tale that the effect may be thought of as the result of multiple scattering of light between boundaries. Another hint is the recognition of the fraction in (2a)

$$\frac{r_\sigma^{(1)} r_\sigma^{(2)} e^{-2\kappa a}}{1 - r_\sigma^{(1)} r_\sigma^{(2)} e^{-2\kappa a}} = \sum_{k=1}^{\infty} \left(r_\sigma^{(1)} r_\sigma^{(2)} e^{2ik_z a} \right)^k \tag{4}$$

as a sum of contributions from waves which are reflected off both interfaces k times before returning to whence it originated.

This implies that the Casimir interaction between much more general materials than bulk dielectrics (as considered by Lifshitz) may be calculated, if one is able to obtain an expression for the reflection properties of the surfaces involved and how light is transmitted between the bodies. This fact was used, among other things, to calculate the effect of spatial dispersion [5, 10, 11, 12] and interaction between (magneto)dielectric multilayers [13, 14, 15, 16, 17] based on Green's function methods [18]. Some further considerations were given in [19].

In recent years, the understanding of Casimir problems in terms of multiple scattering has become widespread and makes way for what is presently perhaps the most powerful techniques for calculating Casimir energies in non-trivial geometries. Within such a general scattering formalism the Lifshitz formula (2b) may be seen as a special case of the much more general formula

$$\mathcal{F}^0 = \int_0^\infty \frac{d\zeta}{2\pi} \mathrm{Tr}\ \ \ln\left[1 - \mathbb{T}_1 \mathbb{G}_{12}^0 \mathbb{T}_2 \mathbb{G}_{21}^0\right] \tag{5}$$

where $\mathbb{T}_i$ is the T-matrices (operators) of two arbitrary interacting bodies and $\mathbb{G}_{ij}^0$ is a vacuum propagator (Green's function) from object i to object j. The energy expression (5) was recently dubbed the TGTG formula and is written here as derived in [20, 21], but the use of less general embodiments of essentially the same multiple scattering technique goes back at least to the 1970s [22, 23]. The recent acceleration of progress towards understanding the role of geometry in Casimir interactions has brought much attention to this technique in recent years (e.g. [24, 25, 26, 27, 28]; for a review see [29] and the introduction to [27]).

To see somewhat roughly how the Casimir-Lifshitz free energy (2b) is a special case of (5) let the propagators be simply that of a plane wave along the $\hat{z}$ direction over a distance a, $\mathbb{G}^0 \to \exp(ik_z a)$ and let the T matrices represent specular scattering at the surfaces, $\mathbb{T}_i \to \mathrm{diag}(r_s^{(i)}, r_p^{(i)})$. Take the trace operation in (5) to include an integral over the transverse momentum $\mathbf{k}_\perp$ plane (isotropic due to rotational symmetry) and one obtains (2b) with minimal manipulation. See e.g. [30] for details.

For reasons of simplicity much of the recent research on geometry effects has been made for the massless scalar field satisfying the Klein-Gordon equation rather than the vectorial electromagnetic field. Historically, Dirichlet and Neumann boundary conditions have been employed together with path integral methods of quantum field theory to mimic the two electromagnetic polarisations (note that the sum of the Dirichlet and Neumann scalar solutions of the wave equation only reproduces the ideally conducting electromagnetic case in special geometries where the electromagnetic modes decouple, such as the original Casimir geometry).

In order to model semi-transparent bodies in this formalism, the introduction of delta-function potentials into the Klein-Gordon equation has been common (see review in [30]). A delta potential $V(\mathbf{r}) = \lambda\delta^3(f(\mathbf{r}))$ models a body whose surface solves $f(\mathbf{r}) = 0$ and where the coupling constant λ determines the "transparency". Dirichlet boundary conditions are regained in the strong coupling limit $\lambda \to \infty$, and it has turned out that several non-trivial geometries are exactly solvable to linear order in λ in the weak coupling case $\lambda \ll 1$ [26, 27].

The model of constant reflection coefficients is a somewhat similar idea and constitutes another model of semi-transparency where some physicality is traded for mathematical manageability.

3 Closed form expression using polylogarithms

It is straightforward to obtain a closed form expression for the Casimir pressure and energy in the constant reflection model. The mathematical formalism which enters is that of polylogarithmic functions. The νth order polylogarithm of x is defined as

$$\mathrm{Li}_\nu(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^\nu}. \tag{6}$$

It is related to the Riemann zeta functions (as is obvious for $\nu > 1$) by $\mathrm{Li}_\nu(1) = \zeta(\nu)$ and obeys the recursion relation $(d/dx)\mathrm{Li}_\nu(x) = (1/x)\mathrm{Li}_{\nu-1}(x)$, which in particular implies that for $|\Re\mathrm{e}A| < 1$

$$\int dx \mathrm{Li}_\nu(Ae^{-bx}) = -\frac{1}{b}\mathrm{Li}_{\nu+1}(Ae^{-bx}) + C \tag{7}$$

where A, b, C are constants. We recognise the polylogarithms which enter into (2a) and (2b),

$$\mathrm{Li}_1(x) = -\ln(1-x); \quad \mathrm{Li}_0(x) = \frac{x}{1-x}. \tag{8}$$

The polylogarithms of interest herein are all of real and integer order.

In the Wick rotated formalism in Euclidean space where the time axis is imaginary, it follows from the general properties of causal response functions that the reflection coefficients are necessarily real quantities [31]. Now, assuming the reflection coefficients are constants with respect to κ and ζ the integrals are easily solved with partial integration using (7) and yields for the pressure and free energy at zero temperature, respectively[3],

$$P^0 = -\frac{3}{16\pi^2 a^4} \sum_{\sigma=p,s} \mathrm{Li}_4(r_\sigma^{(1)} r_\sigma^{(2)}); \tag{9}$$

$$\mathcal{F}^0 = -\frac{1}{16\pi^2 a^3} \sum_{\sigma=p,s} \mathrm{Li}_4(r_\sigma^{(1)} r_\sigma^{(2)}). \tag{10}$$

In the ideal limit $|r_\sigma| \to 1$, $\mathrm{Li}_4(r_\sigma^{(1)} r_\sigma^{(2)}) \to \zeta(4) = \pi^4/90$ and Casimir's results (1) are regained. The Casimir pressure as a function of the squared reflection coefficient r^2 (assuming both materials equal and the same coefficient for both polarisations) is plotted in figure 1. A similar graph for the free energy would obviously be exactly identical.

[3] If the calculation is performed for real frequencies, reflection coefficients are generally complex and the real part of the Li_4 functions should be taken [2].

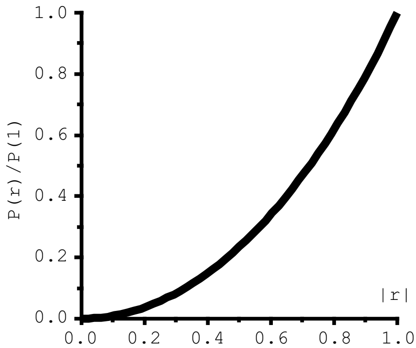

Figure 1: Casimir pressure as a function of a constant reflection coefficient relative to the ideal conductor Casimir result. Materials are assumed similar and the reflection coefficient equal for both polarisations for simplicity.

4 Real-frequency spectrum

The model of constant reflections was introduced in [2] in order to slightly generalise considerations of the real-frequency spectrum of the Casimir force due to Ford [32]. He showed from quantisation of the vacuum how the Lifshitz frequency integrand is equal to the vacuum energy spectrum, which in the case of perfect mirrors studied by Ford turns out to be an oscillating function of frequency with discontinuities at $\omega = n\pi/a,\ n \in \mathbb{N}$. The Lifshitz pressure formula for real frequencies at zero temperatures reads [4]

$$P^0(a) = -\frac{1}{2\pi^2}\Re\mathrm{e}\int_0^\infty d\omega\omega^3 \int_\Gamma dpp^2 \times \sum_{\sigma=s,p} \frac{r_\sigma^2 \exp(2ip\omega a)}{1 - r_\sigma^2 \exp(2ip\omega a)} \tag{11}$$

where the Lifshitz variable p is the positive real part root of $p = \sqrt{1-(\mathbf{k}_\perp/\omega)^2}$. In the following we will assume the materials equal for simplicity; the generalisation to different reflectivity is $r_\sigma^2 \to r_\sigma^{(1)} r_\sigma^{(2)}$. Replacing an isotropic integral over all $\mathbf{k}_\perp$ the integration contour Γ therefore runs from 1 to 0 (propagating modes) and thence to $i\infty$ (evanescent modes).

By assuming reflection coefficients to be constant with $|\Re\mathrm{e}\{r_\sigma^2\}| \le 1$, the frequency spectrum can be found. Defining

$$P^0 = \int_0^\infty d\omega \sum_{\sigma=s,p} P^0_{\omega,\sigma} \tag{12}$$

one finds the spectrum

$$P^0_{\omega,\sigma} = \frac{-1}{16\pi^2 a^3}\left[-\xi^2 \Im\mathrm{mLi}_1(r_\sigma^2 e^{i\xi}) - 2\xi\Re\mathrm{eLi}_2(r_\sigma^2 e^{i\xi}) + 2\Im\mathrm{mLi}_3(r_\sigma^2 e^{i\xi})\right] \tag{13}$$

where we have defined the shorthand dimensionless quantity $\xi = 2\omega a$. The spectrum (13) is plotted for a few different r_σ in figure 2. Note how the discontinuous behaviour seen in the ideal case $r_\sigma^2 = 1$, which stems from the term

$$\Im\mathrm{mLi}_1(e^{i\xi}) = \arctan\left(\frac{\sin\xi}{1-\cos\xi}\right) \tag{14}$$

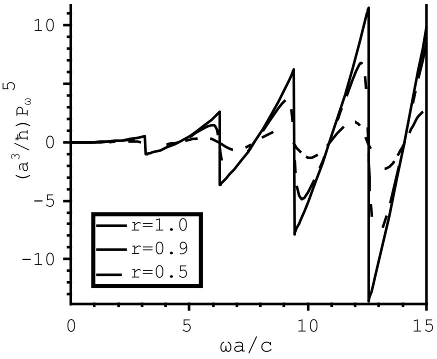

Figure 2: Casimir-Lifshitz frequency spectrum for real constant reflection coefficients. This figure generalises figure 2 of [32].

becomes smooth for $r_\sigma^2 < 1$. This is one example of how the Lifshitz formulae exhibit non-analytic behaviour in the perfectly reflecting limit, a fact which is closely related to the ongoing dispute about the temperature correction to the Casimir force as explained in the following.

5 Thermal behaviour

We start by generalising the closed form result (10) to include finite temperature corrections. It is easiest to work within the imaginary frequency formalism. When going to finite temperature the real frequency integrand of (11) and the corresponding free energy expression receives an additional factor $\coth(\omega/2T)$ from the Bose-Einstein distribution. By use of Cauchy's theorem the real frequency integral can be written as a sum over the poles of this factor at $\omega/2T = m\pi i$, $m \in \mathbb{N}$. Thus the Lifshitz formula for free energy of polarisation mode σ (letting $\mathcal{F} = \mathcal{F}_p + \mathcal{F}_s$) at temperature T is

$$\mathcal{F}_\sigma^T = \frac{T}{2\pi} {\sum_{m=0}^{\infty}}' \int_{\zeta_m}^{\infty} d\kappa\kappa \ln(1 - r_\sigma^2 e^{-2\kappa a}) \tag{15a}$$

$$= -\frac{T}{8\pi a^2} {\sum_{m=0}^{\infty}}' \big[2a\zeta_m \mathrm{Li}_2(r_\sigma^2 e^{-2\zeta_m a}) + \mathrm{Li}_3(r_\sigma^2 e^{-2\zeta_m a})\big] \tag{15b}$$

where $\zeta_m = 2\pi m T$ are the Matsubara frequencies and the prime on the sum means the $m = 0$ term is taken with half weight. In the last form we use that $\ln(1 - x) = -\mathrm{Li}_1(x)$, and partial integration by use of (7).

In the high temperature limit $2\zeta_1 a \gg 1$ the $m = 0$ term dominates (other terms are exponentially small) and we immediately obtain the free energy in this limit:

$$\mathcal{F}_\sigma^T \sim -\frac{T}{16\pi a^2} \mathrm{Li}_3(r_\sigma^2); \quad \zeta_1 a \gg 1, \tag{16}$$

in accordance with the well known high-temperature free energy between ideal plates, $\mathcal{F}_C \approx -\zeta(3)T/(8\pi a^2)$ known at least since the 1960s [33].

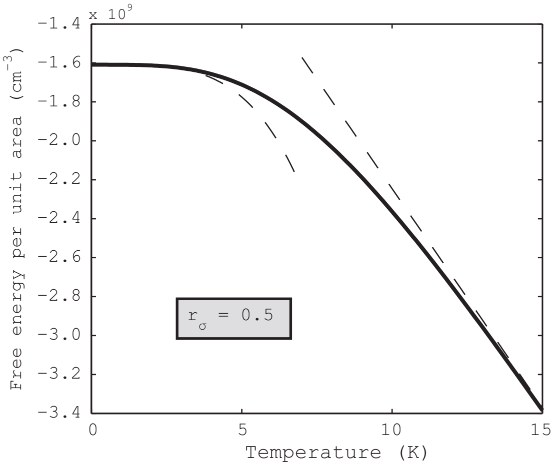

Figure 3: Casimir-Lifshitz free energy as a function of temperature for $r_\sigma = 1/2$ and the high and low temperature asymptotics, (16) and (18) respectively.

By using the definition (6) and changing the order of summation, (15b) can be written

$$\mathcal{F}_\sigma^T = \frac{-T}{16\pi a^2}\sum_{k=1}^{\infty}\frac{r_\sigma^{2k}}{k^3}\left[\frac{\zeta_k a}{\sinh^2(\zeta_k a)} + \coth(\zeta_k a)\right] \tag{17}$$

This is a generalisation of equation (3.12) of [34], which is for ideal conductors. One may note that the expression between the square brackets equals the Wronskian $\mathcal{W}(\coth x, x)$ with $x = \zeta_k a$. For numerical purposes (17) is useful for having a summand which converges geometrically and consists of standard functions only.

We go on to find the asymptotic behaviour for small T. When aT is small and $r_\sigma^2 < 1$ only small values of the quantity $\zeta_k a$ are of importance to the sum (17) because for a given r_σ the temperature may be chosen so small that the sum has converged due to the factor r_σ^{2k} before $\zeta_k a$ becomes of order unity. Then a Laurent expansion

$$x\sinh^{-2}(x) + \coth(x) = 2x^{-1} + 2x^3/45 + ...$$

gives the low temperature expansion assuming $r_\sigma^2 < 1$:

$$\begin{aligned}\mathcal{F}_\sigma^T \sim &-\frac{1}{16\pi^2 a^3}\mathrm{Li}_4(r_\sigma^2) - \frac{\pi^2 a T^4}{45}\frac{r_\sigma^2}{1-r_\sigma^2}\\ &+\mathcal{O}(T^6); \quad T \to 0\end{aligned} \tag{18}$$

where we use $\mathrm{Li}_0(x) = x/(1-x)$. The thermal behaviour of $\mathcal{F}_\sigma$ is plotted in figure 3 together with the high and low temperature asymptotics.

One may note a couple of peculiar traits about this low-temperature behaviour. Firstly, all finite temperature coefficients are singular in the ideal limit $r_\sigma^2 \to 1$; there are only even order terms, and the temperature correction of order T^{2n} diverges as $(1-r_\sigma^2)^{3-n}$ for $n \geq 2$ as we will show below. This is an indication that $\mathcal{F}_\sigma^T$ is not analytic in the double limit where T vanishes and $r_\sigma^2 \to 1$.

Secondly, note the contrast with the corresponding ideal result $r_\sigma^2 = 1$ derived in [34, 35],

$$\frac{1}{2}\mathcal{F}_C^T \sim -\frac{\pi^2}{1440a^3} - \frac{\zeta(3)T^3}{4\pi} + \frac{\pi^2 a T^4}{90} + ...; \quad T \to 0. \tag{19}$$

where further corrections are exponentially small (see also [36]). Mathematically the change of sign and coefficient of the T^4 term from (18) to (19) can be naïvely explained by

$$\frac{r_\sigma^2}{1-r_\sigma^2} = \mathrm{Li}_0(r_\sigma^2) \stackrel{r_\sigma^2\to 1}{\longrightarrow} \zeta(0) = -\frac{1}{2}, \tag{20}$$

yet there appears a hitherto unseen term $\propto T^3$ which is independent of a and therefore does not contribute to the Casimir pressure.

Mathematically, the reason for this fundamental change of temperature behaviour at $r_\sigma^2 = 1$ is due to the fact that the summand of (15a) becomes a non-analytical function of m at $m = 0$ when $r_\sigma^2 = 1$, but is analytical whenever $r_\sigma^2 < 1$. It was demonstrated in [37] that a term $\propto T^3$ in the low temperature expansion of $\mathcal{F}$ appears when the summand of (15a) contains a term proportional to $m^2 \ln(m)$.

Before elaborating this further, we will work out the full asymptotic series expansion of $\mathcal{F}$ in powers of T by use of the method developed in [37]. We define the function $g_\sigma(\mu)$

$$\mathcal{F}_\sigma^T \equiv -\frac{T}{8\pi a^2}\sum_{m=0}^{\infty}{}' g_\sigma(\mu) \tag{21}$$

where $\mu = mT$ and $g_\sigma(\mu)$ is the expression inside the square brackets of (15b). When $g_\sigma(\mu)$ is analytical at $\mu = 0$, g_σ can be written as a Taylor series $g_\sigma(\mu) = \sum_{k=0}^\infty c_k^\sigma \mu^k$. By zeta regularisation the temperature correction $\Delta\mathcal{F}_\sigma(T) = \mathcal{F}_\sigma^T - \mathcal{F}_\sigma^0$ can be written[37]

$$\begin{aligned}\Delta\mathcal{F}_\sigma(T) &\sim -\frac{1}{8\pi a^2}\sum_{k=1}^{\infty} c_{2k-1}^\sigma \zeta(1-2k)T^{2k} \\ &= \frac{1}{8\pi a^2}\sum_{k=1}^{\infty} c_{2k-1}^\sigma \frac{B_{2k}}{2k}T^{2k}; \quad T\to 0,\end{aligned} \tag{22}$$

where B_n are the Bernoulli numbers as defined in [38]. Only odd orders of μ from the Taylor expansion contribute since $\zeta(-2k) = 0$; $k \in \mathbb{N}$, thus there are only even orders of T.

Since

$$\left(\frac{d}{dx}\right)^k \mathrm{Li}_n(Ae^{-bx}) = (-b)^k \mathrm{Li}_{n-k}(Ae^{-bx})$$

and since for $\Re\mathrm{e}A < 1$,

$$\mathrm{Li}_{-k}(A) \propto (1-A)^{-(k+1)}, \quad k \geq 0,$$

it is clear that the summand of (15a) is analytic if and only if $r_\sigma^2 < 1$, since the higher derivatives of the Li_3 term become divergent at $m = 0$. The asymptotic series on the form (22) is therefore valid for all $r_\sigma^2 < 1$ but not in the perfectly reflecting limit.

When $r_\sigma^2 < 1$ it is obvious that

$$\mathrm{Li}_n(r_\sigma^2 e^{-\alpha}) = \sum_{l=0}^{\infty} \frac{(-\alpha)^l}{l!}\mathrm{Li}_{n-l}(r_\sigma^2),$$

which automatically gives the Taylor expansion of $g_\sigma(\mu)$. Inserted into $g_\sigma(\mu)$ from (15b) we find

$$g_\sigma(\mu) = \mathrm{Li}_3(r_\sigma^2) - \sum_{k=1}^{\infty} \frac{k-1}{k!}(-4\pi a\mu)^k \mathrm{Li}_{3-k}(r_\sigma^2). \tag{23}$$

It is thus clear that $c_1^\sigma = 0$, in accordance with (18) where the lowest correction to zero temperature was found to be T^4. With (22) the full temperature expansion to arbitrary order is thus

$$\mathcal{F}_\sigma^T = \frac{1}{16\pi^2 a^3} \sum_{k=0}^{\infty} \frac{(k-1)B_{2k}}{(2k)!} \mathrm{Li}_{4-2k}(r_\sigma^2)(4\pi a T)^{2k}. \tag{24}$$

One may easily verify that this generalises (18), noting that $\mathrm{Li}_0(x) = x/(1-x)$. One may show that this series has zero convergence radius, that is, it does not converge for any finite T.

6 Asymptotic temperature expansion for perfect conductors revisited

The fact that the naïve transition (20) yields the correct T^4 term for ideal conductors leads one to speculate that the even-power terms of the asymptotic T-series for ideal conductors may be given by simply letting $\mathrm{Li}_{4-2k}(r_\sigma^2) \to \zeta(4-2k)$ in (24). Since the Riemann zeta function with even negative integer arguments is zero, this would if so truncate the series beyond order T^4. This does not explain the appearence of the T^3 term in (19), however, and does not preclude the emergence of other additional terms of higher non-even order.

The answer is readily found using the above mentioned method developed in [37]. From (15b) and (21) we see that for ideal conductors

$$g_\sigma(\mu) = \tau \mathrm{Li}_2(e^{-\tau}) + \mathrm{Li}_3(e^{-\tau}) \tag{25}$$

where we have defined the shorthand $\tau = 4\pi a\mu$. The asymptotic behaviour of $\mathrm{Li}_n(e^{-\tau})$ for small τ was found by Robinson [39] who studied the function[4]

$$\phi(s,\tau) = \frac{1}{\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^{x+\tau}-1} = \mathrm{Li}_s(e^{-\tau}).$$

For integer $s = n$ the Robinson formula is

$$\begin{aligned} \mathrm{Li}_n(e^{-\tau}) = & \frac{(-\tau)^{n-1}}{(n-1)!}\left[\sum_{k=1}^{n-1}\frac{1}{k} - \ln(\tau)\right] \\ & + \sum_{\substack{k=0 \\ k \neq n-1}}^{\infty} \frac{\zeta(n-k)}{k!}(-\tau)^k \end{aligned} \tag{26}$$

which gives

$$\begin{aligned} g_\sigma(\mu) = & \zeta(3) - \frac{\tau^2}{4} + \frac{1}{2}\tau^2 \ln(\tau) \\ & - \sum_{k=3}^{\infty} \frac{k-1}{k!}\zeta(3-k)(-\tau)^k. \end{aligned} \tag{27}$$

[4] For this integral representation of the polylogarithm see e.g. [40] equation (2.4).

It is shown in [37] that, as defined in (21), a term in $g_\sigma(\mu)$ of the form $c^\sigma_{2l}\mu^2\ln\mu$ gives a term in the free energy

$$\mathcal{F}_{2l} = -\frac{1}{8\pi a^2}\frac{\zeta(3)}{4\pi^2}c^\sigma_{2l}T^3.$$

From (27) one recognises $c^\sigma_{2l} = 8\pi^2a^2$, wherewith the T^3 term of (19) is regained.

Terms of $g_\sigma(\mu)$ which are constant or proportional to μ^2 give no contribution to the temperature correction to free energy and a comparison of (23) and (27) to order μ^3 and higher shows that for all orders of T above cubic the expansion of $\mathcal{F}^T_C$ is the same as (24) with $\mathrm{Li}_{2-2k}(r_\sigma^2) \to \zeta(2-2k) = -\frac{1}{2}, 0, 0, \ldots$ for $k = 1, 2, 3, \ldots$ Thus the series is terminated at fourth order and the expansion (19) is in fact the full temperature behaviour modulo exponentially small corrections:

$$\mathcal{F}^T_C \sim -\frac{\pi^2}{720a^3} - \frac{\zeta(3)T^3}{2\pi} + \frac{\pi^2aT^4}{45}; T \to 0. \tag{28}$$

This result was found by different methods in [34, 35, 36] and is consistent with Mehra's early considerations [33].

6.1 Relation to the temperature debate

In connection with an ongoing debate concerning the temperature correction to the Casimir force, a point which has been raised is that the application of certain reflectivity models lead to apparent inconsistencies with the third law of thermodynamics, the Nernst heat theorem (c.f. [41] and references therein), that is, entropy does not vanish with vanishing temperature as it should. It was recently concluded that these formal violations of Nernst's theorem stem from non-analytical behaviour in the combined limit of zero frequency (where reflection coefficients approach unity for metal models) and zero temperature [42, 43]. Indeed, violation can only occur due to particular types of non-analyticities causing abrupt change of reflectivity at the point $\omega = T = 0$ [44]. The nonzero entropy at zero temperature would then stem from the fact that the summand of the free energy sum such as (15a) became discontinuous at $m = 0$.

The transition from imperfect to perfect reflection in the previous paragraph is reminiscent of the anomalous entropy at some level. In [42, 43, 44] the situation is one in which the reflection coefficients and thus the free energy summand is discontinuous when frequency and temperature are taken continuously to zero. Here the second temperature *derivative* of the free energy integrand (15a) is discontinuous (indeed divergent) as reflection coefficient and temperature are taken continuously to zero. The former discontinuity leads to a change in free energy leading temperature dependence from quadratic to linear, the linear dependence which implies nonzero entropy at zero temperature since $S = -\partial\mathcal{F}/\partial T$. The $r_\sigma \to 1$ transition considered above changes the temperature correction from quartic to cubic. No anomalous entropy at $T = 0$ stems from this transition, yet its mathematical dynamics are very similar.

7 A generalised force on reflectivity?

We conclude with a few remarks on the possibility of a generalised force whose generalised coordinate is the reflectivity of one of the materials. In most calculations of Casimir forces between real materials the material is treated as inert and it is assumed that its reflection properties do not change due to the Casimir interaction across the gap. One could remark, however, that were it possible, the system could lower its free energy by increasing its reflectivity. Such a mechanism was in fact suggested as a possible explanation of the energetics of the high temperature superconducting transition in which a ceramic multilayer can decrease its total free energy by becoming superconducting, thus a better reflector [45].

In the following a few notes are made on this possibility. A determination of the question of whether such an effect could be measurable is only possible subsequent to calculating the material's free energy as a functional of its reflection coefficients and determining to which extent variation of reflectivity is a degree of freedom. This is complicated task we shall not pursue herein.

One is reminded at this point of the previously mentioned dispute over the thermal dependence of the Casimir effect between real materials (reviews include [46, 41]). Puzzlingly, recent high accuracy experiments which have measured the Casimir force between good metals ([47] and references therein) report a measured Casimir pressure significantly larger than that predicted by several theoretical groups [34, 46, 48, 49].

Our calculations indicate that the Casimir self-enhancing effect is negligible under most circumstances yet it might be worth investigating it further taking into account specific material characteristics for a quantitative treatment. Here we shall content ourselves with laying out the very basic theory and using the constant reflection model as a tool to extract the dependence on temperature and separation in two limits.

Consider the Lifshitz free energy on yet another form,

$$\begin{aligned}\mathcal{F}^T_\sigma[r^{(1)}_\sigma, r^{(2)}_\sigma] =& \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\coth\frac{\omega}{2T}\\ &\times\int\frac{d^2k_\perp}{(2\pi)^2}\ln(1-r^{(1)}_\sigma r^{(2)}_\sigma e^{-2\kappa a})\end{aligned} \tag{29}$$

with $\kappa = \sqrt{\mathbf{k}_\perp^2 - \omega^2}$ with $\Re\mathrm{e}\{\kappa\} > 0$ and reflection coefficients functions of $\mathbf{k}_\perp$ and ζ. In the special case of a single interface between vacuum and a dielectric, $r^{(i)}_\sigma$ take the form (3). Note that the integrand of (29) is complex but only the imaginary part contributes due to symmetry properties so that the expression as a whole is real (see e.g. [43]). The logarithm is understood as its principal value.

The total free energy of the system per unit transverse area should be well approximated by

$$\mathcal{F}^{\mathrm{tot}}_\sigma = \mathcal{F}^{(1)}_\sigma[r^{(1)}_\sigma] + \mathcal{F}^{(2)}_\sigma[r^{(2)}_\sigma] + \mathcal{F}^{\mathrm{L}}_\sigma[r^{(1)}_\sigma, r^{(2)}_\sigma]$$

where the first two terms on the right hand side pertain to the two media on either side of the gap and the last term is the Lifsthiz free energy, now with a superscript L for distinction (we assume finite temperature throughout this section except as explicated). We define the generalised force acting on material i:

$$\begin{aligned}\Phi^{(i)}_\sigma(\omega, \mathbf{k}_\perp) =& -\frac{\delta\mathcal{F}^{\mathrm{L}}_\sigma[r^{(i)}_\sigma, r^{(j)}_\sigma]}{\delta r^{(i)}_\sigma(\mathbf{k}_\perp, \omega)}\\ =&\frac{1}{2i}\coth\left(\frac{\omega}{2T}\right)\frac{r^{(j)}_\sigma e^{-2\kappa a}}{1-r^{(i)}_\sigma r^{(j)}_\sigma e^{-2\kappa a}}\end{aligned} \tag{30}$$

where $i,j = 1,2$; $i \neq j$ and $\delta/\delta r^{(i)}_\sigma$ denotes the functional derivative. The dependence of reflection coefficients on ω and $\mathbf{k}_\perp$ has been suppressed on the right hand side. The generalised force can take either sign but always acts so as to increase the attraction between the plates, an observation which is self evident from the fact that the negative Casimir-Lifshitz free energy (15a) increases in magnitude with increasing reflectivity [5].

[5] One may note that if one were to have a dielectric and a magnetic material, repulsion can in principle be effectuated. In this case Φ_σ acts to decrease repulsion.

A given material i will have a generalised susceptibility which determines its ability to alter its reflective properties in response to Φ_σ,

$$\chi_\sigma^{(i)}(\omega,\omega',\mathbf{k}_\perp,\mathbf{k}'_\perp) = \frac{\delta r_\sigma^{(i)}(\omega,\mathbf{k}_\perp)}{\delta\Phi_\sigma^{(i)}(\omega',\mathbf{k}'_\perp)} \tag{31}$$

$$= \left[\frac{\delta^2\mathcal{F}_\sigma^{(i)}[r_\sigma^{(i)}]}{\delta r_\sigma^{(i)}(\omega,\mathbf{k}_\perp)\delta r_\sigma^{(i)}(\omega',\mathbf{k}'_\perp)}\right]^{-1} \tag{32}$$

and a Taylor expansion in Φ_σ gives

$$\Delta r_\sigma^{(i)}(\omega,\mathbf{k}_\perp) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int\frac{d^2k'_\perp}{(2\pi)^2}\chi_\sigma^{(i)}(\omega,\omega',\mathbf{k}_\perp,\mathbf{k}'_\perp) \times \Phi_\sigma^{(i)}(\omega',\mathbf{k}'_\perp) + ...$$

At finite temperature we may close the ω' integral path around the upper half complex plane and invoke the Cauchy theorem. Since $\chi_\sigma^{(i)}(\cdots)$ does not have any singularities in the upper ω' plane [31], the integral over ω' then gives a sum over the poles of $\coth(\omega'/2T)$, and by letting $\omega \to i\zeta$ we obtain

$$\Delta r_\sigma^{(i)}(i\zeta,\mathbf{k}_\perp) = T\sum_{m=0}^{\infty}{}'\int\frac{d^2k'_\perp}{(2\pi)^2}\chi_\sigma^{(i)}(i\zeta,i\zeta_m,\mathbf{k}_\perp,\mathbf{k}'_\perp) \times \Phi_\sigma^{(i)}(i\zeta_m,\mathbf{k}'_\perp) + ... \tag{33}$$

where

$$\Phi_\sigma^{(i)}(i\zeta,\mathbf{k}_\perp) = \frac{r_\sigma^{(j)}(i\zeta,\mathbf{k}_\perp)e^{-2\kappa a}}{1 - r_\sigma^{(i)}(i\zeta,\mathbf{k}_\perp)r_\sigma^{(j)}(i\zeta,\mathbf{k}_\perp)e^{-2\kappa a}}. \tag{34}$$

On the imaginary frequency axis all quantities in (33) and (34) are real.

Since $\chi_\sigma^{(i)}(\cdots)$ depends on $r_\sigma^{(i)}$ and $\Phi_\sigma^{(i)}$ depends on both reflection coefficients, quation (33) defines a set of integral equations for the new reflection coefficients. Note that $\Phi_\sigma^{(i)}$ always has the same sign as $r_\sigma^{(i)}$ and increases in magnitude with increasing $|r_\sigma^{(i)}|$, so equation (33) implies that given time, $|r_\sigma^{(i)}|$ will flow to ever higher values until the fixed point

$$\chi_\sigma^{(i)}(i\zeta,i\zeta,\mathbf{k}_\perp,\mathbf{k}_\perp) = 0 \tag{35}$$

is reached for both materials. If one is able to calculate $\chi_\sigma^{(i)}(\cdots)$ for a given $r_\sigma^{(i)}$, (33) with (34) may be invoked iteratively for a simple numerical scheme to obtain the new reflection coefficients.

An approximation of the change in reflectivity is provided by use of (33) using the 'first order' estimate

$$\Phi_{\sigma,0}^{(i)} = \frac{r_{\sigma,0}^{(j)}e^{-2\kappa a}}{1 - r_{\sigma,0}^{(i)}r_{\sigma,0}^{(j)}e^{-2\kappa a}} \tag{36}$$

where $r_{\sigma,0}^{(i)}$ are the reflection coefficients without any Casimir interaction, which satisfy $\delta\mathcal{F}_\sigma^{(i)}/\delta r_{\sigma,0}^{(i)} = 0$. To first order in Δr the change in Lifshitz free energy is

$$\Delta\mathcal{F}_\sigma^{\mathrm{L}} = -T\sum_{m=0}^{\infty}{}'\int\frac{d^2k_\perp}{(2\pi)^2}\left(\frac{\Delta r_\sigma^{(1)}}{r_{\sigma,0}^{(1)}} + \frac{\Delta r_\sigma^{(2)}}{r_{\sigma,0}^{(2)}}\right) \times \frac{r_{\sigma,0}^{(1)}r_{\sigma,0}^{(2)}e^{-2\kappa a}}{1 - r_{\sigma,0}^{(1)}r_{\sigma,0}^{(2)}e^{-2\kappa a}}. \tag{37}$$

which, upon comparison with (36) gives the 'one-loop' approximation

$$\Delta\mathcal{F}^{\mathrm{L}}_{\sigma} \approx -T^2 \sum_{m,m'=0}^{\infty}{}' \int \frac{d^2k_\perp}{(2\pi)^2}\frac{d^2k'_\perp}{(2\pi)^2} \sum_{i=1,2} \Phi^{(i)}_{\sigma,0}(i\zeta_m,\mathbf{k}_\perp)\chi^{(i)}_{\sigma}(i\zeta_m,i\zeta_{m'},\mathbf{k}_\perp,\mathbf{k}'_\perp)\Phi^{(i)}_{\sigma,0}(i\zeta_{m'},\mathbf{k}'_\perp). \tag{38}$$

It is understood that $\chi^{(i)}_{\sigma}(\cdots)$ is evaluated assuming unperturbed reflection.

7.1 Constant reflection model

Assuming constant reflection coefficients as before it is easy to see that Φ_σ scales with distance like $\mathcal{F}_\sigma$:

$$\begin{aligned}\Phi^{(i)}_{\sigma} &= T\sum_{m=0}^{\infty}{}'\int \frac{d^2k'_\perp}{(2\pi)^2}\Phi^{(i)}_{\sigma}(i\zeta_m,\mathbf{k}_\perp) = -\frac{\partial\mathcal{F}_\sigma}{\partial r^{(i)}_\sigma}\\ &\sim \begin{cases}(16\pi^2a^3r^{(i)}_\sigma)^{-1}\mathrm{Li}_3(r^{(1)}_\sigma r^{(2)}_\sigma), & T\to 0\\ T(16\pi a^2 r^{(i)}_\sigma)^{-1}\mathrm{Li}_2(r^{(1)}_\sigma r^{(2)}_\sigma), & aT\gg 1\end{cases}\end{aligned} \tag{39}$$

where only the last form is specific to the constant reflection model.

The one-loop correction (38) now simplifies to

$$\begin{aligned}\Delta\mathcal{F}^{\mathrm{L}}_{\sigma} \approx &-\frac{T^2}{4\pi^2}\sum_{m,m'=0}^{\infty}{}'\int_{\zeta_m}^{\infty} d\kappa\kappa\int_{\zeta_{m'}}^{\infty} d\kappa'\kappa'\\ &\times\sum_{i=1,2}\Phi^{(i)}_{\sigma,0}(\kappa)\chi^{(i)}_{\sigma}(\kappa,\kappa')\Phi^{(i)}_{\sigma,0}(\kappa').\end{aligned}$$

The dependence of $\chi^{(i)}_\sigma$ on κ,κ' is of course unknown, but it is in the spirit of our simple model to assume it constant with respect to these arguments (dependent on $r^{(1)}_\sigma$ and $r^{(2)}_\sigma$ only) as a first approximation so as to extract some information as to how the corrections to Casimir force and free energy depend on distance. In this model the simple result is

$$\Delta\mathcal{F}^{\mathrm{L}}_{\sigma} \approx -\chi^{(i)}_\sigma\left[\Phi^{(i)}_\sigma\right]^2 \propto \begin{cases} a^{-6}, & T=0\\ T^2a^{-4}, & aT\gg 1\end{cases} \tag{40}$$

with $\Phi^{(i)}_\sigma$ from (39).

The indication is thus that the change in the Casimir pressure will fall off as a^{-7} and a^{-5} in the two regimes respectively, much faster than the Casimir pressure, which falls off as a^{-4} and a^{-3} respectively. Although tentative and subject to restrictive assumptions, the above calculation indicates that the effect of the generalised force on reflectivity is likely to be negligible under most circumstances. It is notable, however, that the effect increases as T^2 in the high aT limit, whereas the Casimir force is a linear function of temperature in this regime.

Conclusions

We have reviewed how the Casimir effect can be thought of as a multiple scattering phenomenon, an observation which inspires the use of a simple model in which the reflection coefficients of interacting bodies (the relative amplitude of reflected vs. incoming field) are assumed to be independent of the direction and energy of the field. We review how this simple model yields some closed form results in the planar geometry famoyusly considered by Casimir and Lifshitz, and how much important information may be extracted with relatively simple methods within the confines of the model.

We review how the frequency spectrum of the Casimir effect is generalised from perfect reflection and becomes analytic and continuous upon introducing non-unity reflection coefficients. The full asymptotic behaviour of the Casimir-Lifshitz free energy in powers of temperature is found, and it is demonstrated how the transition to the perfectly reflecting case is not smooth. This is another demonstration of the non-analytic behaviour of the Lifshitz formalism in the double limit of zero temperature and perfect reflection which has given rise to debate over the thermodynamic consistency of various reflection models in connection with the temperature behaviour of the Casimir force.

We finally discuss the idea of a generalised "Casimir" force conjugate to the reflection coefficients of the interacting bodies. If there exist mechanisms by which the materials involved could be susceptible to changing their reflective properties, the generalised force initiates a back reaction effect by which the reflection coefficients tend towards their maximal available values, increasing the Casimir interaction. The indication is, however, that the effect would be small and fall off faster with interplate separation than the Casimir force itself.

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Some results on dynamical black holes

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Abstract

We give indications that outer future trapping horizons play a role in the particular semi-classical instability of an evolving black hole that produces the Hawking's radiation. These are obtained with the use of the Hamilton-Jacobi tunneling method. It automatically selects one special expression for the surface gravity of a changing horizon, the one defined a decade ago by Hayward using Kodama's theory of spherically symmetric gravitational fields. The method also applies to point masses embedded in an expanding universe and to general, spherically symmetric black holes. The local surface gravity solves a puzzle concerning the charged stringy black holes, namely that it vanishes in the extremal limit, whereas the Killing global gravity does not[2].

1 Introduction

It has long been felt that the usual semi-classical treatment of stationary black holes (abbrev. BHs) should be extended to cover at least slowly changing, or evolving black holes. By this expression we mean black holes that can still be described in terms of few multiplole moments such as mass, angular momentum and the charges associated to local gauge symmetries, except that the parameters and the causal structure change with time either because matter and gravitational radiation fall in, or because there operate a Hawking's process of quantum evaporation or finally because the hole is actually immersed in a slowly expanding universe. A technical definition of a "slowly varying BH" can be given in some cases, an example being the Booth-Fairhurst slowly evolving horizon, but in general it depends on the actual physical processes involved. For example, in the case of Hawking's evaporation, conditions for slowness in the presence of a near-horizon viscous fluid have been given by Brevik [1] in an interesting attempt to generalize 't Hooft's model of the self-screening Hawking atmosphere (quantum corrections to this model can be found in [2]). In general it is understood that the black hole temperature is to be be much smaller than the Planck mass, or equivalently the

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[2]This article is **dedicated to 70th aniversary of Professor Iver Brevik**

mass $M \gg M_P = G^{-1/2} \sim 10^{19}$ Gev, while in order to study the effects of the expansion the Hubble rate H^{-1} should dominate over the black hole emission/absorption rate.

One surprising aspect of the semi-classical results obtained so far, is that the radiation caused by the changing metric of the collapsing star approaches a steady outgoing flux for large times, implying a drastic violation of energy conservation if one neglects the back reaction of the quantum radiation on the structure of spacetime. But the back reaction problem has not been solved yet in a satisfactory way. As pointed out by Fredenhagen and Haag long ago [3], if the back reaction is taken into account by letting the mass of the black hole to change with time, then the radiation will possibly originate from the *surface* of the black hole at all times after its formation.

This poses the question: *what is and where is the surface of a dynamical black hole*? This issue always baffled scientists from the very beginning and produced several reactions during the nineties, which eventually culminated with the notion of outer trapping horizons by Hayward[4] and the isolated [5] and dynamical horizons of Ashtekar and Krishnan [6, 7] (a fine review is in [8]). Thus one is concerned to show, in the first place, what kind of a surface a dynamical horizon can be and also which definition can capture a useful local notion of such a surface, and then what sort of instability, if any, really occurs near the horizon of the changing black hole. This question was non trivial since a changing horizon is typically embedded in a dynamical space-time and it is not even expected to be a null hypersurface, although it is still one of infinite red shift.

We analyze this question for a class of dynamical black hole solutions that was inspired by problems not directly related to black hole physics, although these were subsequently reconsidered in the light of the black hole back reaction problem in the early Eighties. The metrics we shall consider are the Vaidya radiating metric [9], as revisited by J. Bardeen [10] and J. York [11], together with what really is a fake dynamical black holes, the McVittie solution representing, in author's mind, a point mass in cosmology [12]. We shall indicate how the results can be extended to all dynamical, spherically symmetric solutions admitting a possibly dynamical future outer trapping horizon.

2 Horizons

After the time lasting textbook definition of the event horizon (abbr. EH) to be found in the Hawking & Ellis renowned book [13], several quasi-local notions of dynamical horizons appeared in the literature (a nice review is in [14]) , perhaps starting with the perfect horizons[3] of Hájiček [15] and the apparent horizons (AHs, boundaries of trapped 3-dimensional space-like regions) of Hawking-Ellis themselves. But the former only applied to equilibrium BHs and the existence of the latter is tie to a partial Cauchy surface so it represents only a "localization in time". Moreover it has proven not possible to formulate thermodynamical laws for AHs akin to those holding for event horizons.

The first succesfull attempt to go beyond the limitations imposed either by the istantaneous character of the apparent horizons or by the global, teleological nature of the event horizons is due to S. Hayward. His concept of a future outer trapping horizon (FOTH) then evolved either into some less constrained definition, like the Ashtekar-Krishnan dynamical horizons (DH), or some specialization like the Booth-Fairhurst slowly evolving FOTH [16]; so an updated (but perhaps partial) list of locally or quasi-locally defined horizons would contain:

[3] These are null hypersurfaces whose rays have zero expansion and intersect space-like hypersurfaces in compact sets. All stationary horizons are perfect, but the converse is not true.

(a) Trapping horizons (Hayward [4])

(b) Dynamical horizons (Ashtekar & Krishnan [7, 6])

(c) Non expanding and perfect horizons (Hájiček [15])

(d) Isolated and weakly isolated horizons (Ashtekar et al. [5])

(e) Slowly evolving horizons (Booth & Fairhurst [16])

In contrast to the old fashioned apparent horizons, these newly defined horizons do not require a space-like hypersurface, no notion of interior and exterior and no conditions referring to infinity (all are non local conditions). Moreover they are not teleological and, given a solution of Einstein equations, one can find whether they exist by purely local computations. Finally, unlike EHs they are related to regions endowed by strong gravitational fields and absent in weak field regions.

All quasi-local horizons rely on the local concept of trapped (marginally trapped) surface: this is a space-like closed 2-manifold $\boldsymbol{S}$ such that $\theta_{(\ell)}\theta_{(n)} > 0$, where ℓ, n are the future-directed null normals to $\boldsymbol{S}$, normalized to $\ell \cdot n = -1$, and $\theta_{(\ell)}$, $\theta_{(n)}$ are the respective expansion scalars. We write the induced metric on each $\boldsymbol{S}$ in the form

$$q_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a \tag{1}$$

and put $q^{ab} = g^{ab} + \ell^a n^b + \ell^b n^a$, not an inverse. Then q^a_b is the projection tensor to $T_*(\boldsymbol{S})$, the tangent space to $\boldsymbol{S}$. To cover BHs rather than white holes it is further assumed that both expansions are negative (non positive).

The most important quantities associated with the null vector fields ℓ and n are the projected tensor fields $\Theta_{ab} = q^a_n q^b_m \nabla_a l_b$ and $\Phi_{ab} = q^a_n q^b_m \nabla_a n_b$ and their decomposition into symmetric, anti-symmetric and trace part. Their twists are zero since they are normal to $\boldsymbol{S}$. Finally, the expansions are given by

$$\theta_{(\ell)} = q^{ab}\nabla_a \ell_b, \qquad \theta_{(n)} = q^{ab}\nabla_a n_b \tag{2}$$

Let us describe the listed horizons in turn, adding comments where it seems appropriate. A black triangle down ▼ will close the definitions.

Future Outer Trapping Horizon: A future outer trapping horizon (FOTH) is a smooth three-dimensional sub-manifold H of space-time which is foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, $t \in R$, with future-directed null normals ℓ and n such that (i) the expansion $\theta_{(\ell)}$ of the null normal ℓ vanishes, (ii) the expansion $\theta_{(n)}$ of n is negative and (iii) $/calL_n \theta_{(\ell)} < 0$. ▼

Condition (i) requires strong fields since certainly $\theta_{(\ell)} > 0$ in weak fields. Condition (ii) is related to the idea that H is of the future type (i. e. a BH rather than a WH); (iii) says that H is of the outer type, since a motion of $\boldsymbol{S}_t$ along n^a makes it trapped. It also distinguishes BH horizons from cosmological ones.

One can always found a scalar field C on H so that

$$V^a = \ell^a - Cn^a \quad \text{and} \quad N_a = \ell_a + Cn_a\,, \tag{3}$$

are respectively tangent and normal to the horizon. Note that $V \cdot V = -N \cdot N = 2C$. Hayward [4] has shown that if the null energy condition holds, then $C \geq 0$ on a FOTH. Thus, the horizon must be either space-like or null, being null iff the shear $\sigma^{(\ell)}_{ab}$ as well as $T_{ab}\ell^a\ell^b$ both vanish across H. Intuitively, H is space-like in the dynamical regime where gravitational radiation and matter are pouring into it and is null when it reaches equilibrium.

The second law of trapping horizon mechanics follows quite easily. If $\sqrt{q}$ is the area element corresponding to the metric q_{ab} on the cross-sections, then

$$\mathcal{L}_{\mathcal{V}}\sqrt{\Pi} = -\mathcal{C}\theta_{(\backslash)}\sqrt{\Pi}\,. \tag{4}$$

By definition $\theta_{(n)}$ is negative and we have just seen that C is non-negative, so we obtain the local second law: If the null energy condition holds, then the area element $\sqrt{q}$ of a FOTH is non-decreasing along the horizon. Integrating over $\boldsymbol{S}_t$, the same law applies to the total area of the horizon sections. It is non-decreasing and remains constant if and only if the horizon is null.

Dynamical Horizon: a smooth three-dimensional, *space-like* sub-manifold H of space-time is a dynamical horizon (DH) if it can be foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, with future-directed null normals ℓ and n such that (i) on each leaf the expansion $\theta_{(\ell)}$ of one null normal ℓ^a vanishes, (ii) the expansion $\theta_{(n)}$ of the other null normal n is negative.▼

Like FOTHs, a DH is a space-time notion defined quasi-locally, it is not relative to a space-like hypersurface, it does not refer to ii, it is not teleological. A space-like FOTH is a DH on which $\mathcal{L}_\backslash\theta_{(\ell)} < /$; a DH which is also a FOTH will be called a space-like future outer horizon (SFOTH). The DH cannot describe equilibrium black holes since it is space-like by definition, but is better suited to describe how a BH grow in general relativity. Suitable analogues of the laws of black hole mechanics hold for both FOTHs and DHs. Our main interest in the following will be precisely for these local horizons, but for the time being we continue our description.

Perfect and Non-Expanding Horizons: a perfect horizon is a smooth three-dimensional *null* sub-manifold H of space-time with null normal ℓ^a such that $\theta_{(\ell)} = 0$ on H and which intersect space-like hypersurfaces in compact sets. ▼

If in the last clause H is topologically $R \times S^2$ and moreover the stress tensor T_{ab} is such that $-T^a_b\ell^b$ is future causal for any future directed null normal ℓ^a, then H is called a non-expanding horizon.▼

If X, Y are tangent to a non-expanding horizon we can decompose the covariant derivative

$$\nabla_X Y = D_X Y + N(X,Y)\ell + L(X,Y)n$$

where D_X is the projection of the vector $\nabla_X Y$ onto the spheres $\boldsymbol{S}_t$ in H. If X is tangent to the spheres then D_X is the covariant derivative of the induced metric q_{ab}, and if X is tangent to H one may regard the operator $\widehat{\nabla}_X = D_X + N(X,\cdot)\ell$, acting on vector fields, as a connection on H. If this connection is "time independent" then the geometry of H is time independent too and we have Ashtekar et al. notion of an horizon in isolation.

Isolated Horizon: a non-expanding horizon with null normal ℓ^a such that $[\mathcal{L}_\ell, \widehat{\nabla}_{\mathcal{X}}] = /$ along H.▼

These horizons are intended to model BHs that are themselves in equilibrium but possibly in a dynamical space-time. For a detailed description of their mathematical properties we refer the reader to Ashtekar-Krishnan's review [8].

The new horizons just introduced all have their own dynamics governed by Einstein eq.s. There exist for them existence and uniqueness theorems [17], formulation of first and second laws [8, 4, 18] and even a "membrane paradigm" analogy. In particular, they carry a momentum density which obey a Navier-Stokes like equation generalizing the classical Damour's equations of EHs, except that the bulk viscosity $\zeta_{FOTH} = 1/16\pi > 0$ [19, 20]. We think Iver would be amused by that.

The new horizons are also all space-like or null, hence it remains to see what is the role they play in the problem of black hole quantum evaporation. In this connection the following notion can be useful.

Time-like Dynamical Horizon: a smooth three-dimensional, *time-like* sub-manifold H of space-time is a time-like dynamical horizon (TDH) if it can be foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, with future-directed null normals ℓ and n such that (i) on each leaf the expansion $\theta_{(\ell)}$ of one null normal ℓ^a vanishes, (ii) the expansion $\theta_{(n)}$ of the other null normal n is strictly negative.▼

Surface Gravities

The surface gravity associated to an event horizon is a well known concept in black hole physics whose importance can be hardly overestimated. Surprisingly, a number of inequivalent definitions beyond the historical one appeared recently (over the last 15 years or so) in the field with various underlying motivations. We have collected the following (we rely on the nice review of Nielsen and Yoon [28]):

1. The historical Killing surface gravity (Bardeen et al. [21], textbooks)
2. Hayward's first definition [4]
3. Mukohyama-Hayward's definition [22]
4. Booth-Fairhurst surface gravity for the evolving horizons [16]
5. The effective surface gravity appearing in Ashtekar-Krishnan [8]
6. The Fodor et al. definition for dynamical spherically symmetric space-times [23]
7. The Visser [24] and Nielsen-Visser [25] surface gravity
8. Hayward's definition [26] using Kodama's theory [27].

We will not spend much time on the various definitions and their motivations except for the last item, which is what the tunneling approach leads to, among other things.

1. The Killing surface gravity is related to the fact that the integral curves of a Killing vector are not affinely parametrized geodesics on the Killing horizon H. Hence

$$K^a \nabla_b K_a \cong \kappa K_a$$

defines the Killing surface gravity κ on H, where $\cong$ means evaluation on the horizon. The Killing field is supposed to be normalized at infinity by $K^2 = -1$. The definition can be extended to EHs that are not Killing horizons, by replacing K with the null generator ℓ of the horizon. However there is no preferred normalization in this case, and this is one reason of the debating question regarding the value of the surface gravity in dynamical situations.

2. Hayward's first definition was motivated by the desire to get a proof of the first law for THs. It is defined without appeal to inaffinity of null geodesics as

$$\kappa \cong \frac{1}{2}\sqrt{-n^a \nabla_a \theta_{(\ell)}} \tag{5}$$

and is independent on the parametrization of ℓ^a integral curves, since the evaluation is on a marginal outer surface where $n \cdot \ell = -1$ and $\theta_{(\ell)} = 0$.

3. We leave apart the Mukohyama-Hayward and the Booth-Fairhurst definitions (see **4.**) as they are somewhat more technical and complicated than it is necessary, so we refer the reader to the original papers.

5. Given a weakly isolated horizon H, Ashtekar and Krishnan showed that for any vector field t^a along H with respect to which energy fluxes across H are defined, there is an area balance law that takes the form

$$\delta E^t = \frac{\bar{\kappa}}{8\pi G}\delta A_S + \text{work terms}$$

with an effective surface gravity given by

$$\bar{\kappa} = \frac{1}{2R}\frac{dr}{dR}$$

R is the areal radius of the marginally trapped surfaces, i.e. $A_S = 4\pi R^2$, the function r is related to a choice of a lapse function and finally E^t is the energy associated with the evolution vector field t^a. For a spherically symmetric DH a natural choice would be $r = R$ so $\bar{\kappa} = 1/2R$, just the result for a Schwarschild BH. To illustrate the naturalness of this definition, consider a slowly changing spherically symmetric BH with mass $M(v)$, where v is a time coordinate. Defining the horizon radius at each time by $R = 2M(v)$ and $A_S = 4\pi R^2$, we can differentiate M to obtain

$$\dot{M} = \frac{\dot{R}}{2} = \frac{1}{2R}\frac{\dot{A}_S}{8\pi} \implies \delta M = \frac{\bar{\kappa}}{8\pi}\delta A_S$$

which is the usual area balance law with surface gravity $\bar{\kappa} = 1/2R = 1/4M$. Consider, however, the more general possibility where the horizon is at $R = 2M(v, R)$, as it happens for example in the Vaidya-Bardeen metric. The same computation leads to

$$\dot{M} = \frac{1}{2R}(1 - 2M^{'})\frac{\dot{A}_S}{8\pi} \implies \kappa \cong \frac{1}{4M}(1 - 2M^{'}) \tag{6}$$

a prime denoting the radial derivative. Thus naturalness is not a decisive criterion in this case.

6. The Fodor et al. definition looks like the Killing form of surface gravity in that $\kappa\ell^b = \ell^a\nabla_a\ell^b$, where now ℓ^a is an outgoing null vector orthogonal to a trapped or marginally trapped surface. This is because, as a rule, such null vectors are not affinely parametrized, although they can always be so parametrized that $\kappa = 0$. So one needs to fix the parametrization: Fodor et al. choose

$$\kappa = -n^a\ell^b\nabla_b\ell_a$$

with n^a affinely parametrized and normalized to $n \cdot t = -1$ at space-like infinity, t^a being the asymptotic Killing field. Note that this definition is non local but looks like as a natural generalization of the Killing surface gravity.

7. We postpone the discussion of the Visser and Visser-Nielsen surface gravity to the next section.

8. Finally we have a local geometrical definition of this quantity for the trapping horizon of a spherically symmetric black hole as [26] follows. One can introduce local null coordinates $x^{\pm}$ in a tubular neighborhood of a FOTH such that $n = -g^{+-}\partial_-$ and $\ell = \partial_+$: then (shortening $\theta_{(\ell)} = \theta_+$, $\theta_{(n)} = \theta_-$)

$$\kappa = \frac{1}{2}(g^{+-}\partial_-\theta_+)_{|\theta_+=0} \tag{7}$$

Later we will show that this κ fixes the expansion of the metric near the trapping outer horizon along a future null direction. The definition may look somewhat artificial, but in fact it is very natural and connected directly with what is known for the stationary black holes. To see this one notes, following Kodama [27], that any spherically symmetric metric admits a unique (up to normalization) vector field K^a such that $K^a G_{ab}$ is divergence free, where G_{ab} is the Einstein tensor; for instance, using the double-null form, one finds $K = -g^{+-}(\partial_+ r\partial_- - \partial_- r\partial_+)$. The defining property of K shows that it is a natural generalization of the time translation Killing field of a static black hole. Moreover, by Einstein equations $K_a T^{ab}$ will be conserved so for such metrics there exists a natural localizable energy flux and its conservation law. Now consider the expression $K^a\nabla_{[b}K_{a]}$: it is not hard to see that on H it is proportional to K_b. The proportionality factor, a function in fact, is the surface gravity: $K^a\nabla_{[b}K_{a]} = -\kappa K_b$. For a Killing vector field $\nabla_b K_a$ is anti-symmetric so the definition reduces to the usual one.

3 Two examples: Vaidya and McVittie's metrics

We consider first spherically symmetric spacetimes which outside the horizon (if there is one) are described by a metric of the form

$$ds^2 = -e^{2\Psi(r,v)}A(r,v)dv^2 + 2e^{\Psi(r,v)}dvdr + r^2 dS^2\,. \tag{8}$$

where the coordinate r is the areal radius commonly used in relation to spherical symmetry and v is intended to be an advanced null coordinate. In an asymptotically flat context one can always write (we use geometrized units in which the Newton constant $G = 1$)

$$A(r,v) = 1 - 2m(r,v)/r \tag{9}$$

which defines the active mass. This metric was first proposed by Vaidya [9], and studied in an interesting paper during the classical era of black hole physics by Lindquist et al [29]. It has been generalized to Einstein-Maxwell systems and de Sitter space by Bonnor-Vaidya and Mallet, respectively [30]. It was then extensively used by Bardeen [10] and York [11] in their semi-classical analysis of the back reaction problem. We will call it the Vaidya-Bardeen metric. A cosmological constant can be introduced by setting

$$A(r,v) = 1 - 2m(r,v)/r - r^2/L^2 \tag{10}$$

where $L^{-2} \propto \Lambda$. If one wishes the metric can also be written in double-null form. In the (v,r)-plane one can introduce null coordinates $x^{\pm}$ such that the dynamical Vaidya-Bardeen space-time may be written as

$$ds^2 = -2f(x^+,x^-)dx^+dx^- + r^2(x^+,x^-)dS^2_{D-2}\,, \tag{11}$$

for some differentiable function f. The remaining angular coordinates contained in dS^2 do not play any essential role. In the following we shall use both forms of the metric, depending on computational convenience. The field equations of the Vaidya-Bardeen metric are of interest. They read

$$\frac{\partial m}{\partial v} = 4\pi r^2 T^r_{\ v}, \quad \frac{\partial m}{\partial r} = -4\pi r^2 T^v_{\ v}, \quad \frac{\partial \Psi}{\partial r} = 4\pi r e^{\Psi} T^v_{\ r} \tag{12}$$

The stress tensor can be written as

$$T_{ab} = \frac{\dot m}{4\pi r^2}\nabla_a v\nabla_b v - \frac{m'}{2\pi r^2}\nabla_{(a}r\nabla_{b)}v \tag{13}$$

If m only depends on v it describes a null fluid and obeys the dominant energy condition if $\dot m > 0$.

The second example we are interested in is the McVittie solution [12] for a point mass in a Friedmann-Robertson-Walker flat cosmology. In D-dimensional spacetime in isotropic spatial coordinates it is given by [31]

$$ds^2 = -A(\rho,t)dt^2 + B(\rho,t)(d\rho^2 + \rho^2 dS^2_{D-2}) \tag{14}$$

with

$$A(\rho,t) = \left(\frac{1-(\frac{m}{a(t)\rho})^{D-3}}{1+(\frac{m}{a(t)\rho})^{D-3}}\right)^2, \qquad B(\rho,t) = a(t)^2(1-(\frac{m}{a(t)\rho})^{D-3})^{2/(D-3)}. \tag{15}$$

When the mass parameter $m = 0$, it reduces to a spatially flat FRW solution with scale factor $a(t)$; when $a(t) = 1$ it reduces to the Schwarzschild metric with mass m. In four dimensions this solution has had a strong impact on the general problem of matching the Schwarzschild solution with cosmology, a problem faced also by Einstein and Dirac. Besides McVittie, it has been extensively studied by Nolan in a series of papers [32]. To put the metric in the general form of Kodama theory, we use what may be called the Nolan gauge, in which the metric reads

$$ds^2 = -(A_s - H^2(t)r^2)dt^2 + A_s^{-1}dr^2 - 2A_s^{-1/2}H(t)r\,drdt + r^2 dS^2_{D-2} \tag{16}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter and, for example, in the charged 4-dimensional case,

$$A_s = 1 - 2m/r + q^2/r^2 \tag{17}$$

or in D dimension $A_s = 1 - 2m/r^{D-3} + q^2/r^{2D-6}$. In passing to the Nolan gauge a choice of sign in the cross term $drdt$ has been done, corresponding to an expanding universe; the transformation $H(t) \to -H(t)$ changes this into a contracting one. In the following we shall consider $D = 4$ and $q = 0$; then the Einstein-Friedmann equations read

$$3H^2 = 8\pi\rho, \qquad 2A_s^{-1/2}\dot{H}(t) + 3H^2 = -8\pi p. \tag{18}$$

It follows that $A_s = 0$, or $r = 2m$, is a curvature singularity. In fact, it plays the role that $r = 0$ has in FRW models, namely it is a big bang singularity. When $H = 0$ one has the Schwarzschild solution. Note how the term H^2r^2 in the metric strongly resembles a varying cosmological constant; in fact for H a constant, it reduces to the Schwarzschild-de Sitter solution in Painlevé coordinates. As we will see, the McVittie solution possesses in general both apparent and trapping horizons, and the spacetime is dynamical. However, it is really not a dynamical black hole in the sense we used it above, since the mass parameter is strictly constant: for this reason we called it a fake dynamical BH. This observation prompts one immediately for an obvious extension of the solution: to replace the mass parameter by a function of time and radius, but this will not be pursued here.

The study of black holes requires also a notion of energy; the natural choice would be to use the charge associated to Kodama conservation law, but this turns out to be the Misner-Sharp energy, which for a sphere with areal radius r is the same as the Hawking mass [33], given by $E = r(1 - 2^{-1}r^2g^{+-}\theta_+\theta_-)/2$. Using the metric (8) an equivalent expression is

$$g^{\mu\nu}\partial_\mu r\partial_\nu r = 1 - 2E/r \tag{19}$$

In this form it is clearly a generalization of the Schwarzschild mass. As we said, E is just the charge associated to Kodama conservation law; as showed by Hayward [34], in vacuum E

is also the Schwarzschild energy, at null infinity it is the Bondi-Sachs energy and at spatial infinity it reduces to the ADM mass.

Let us apply this general theory to the two classes of dynamical BH we have considered. Using Eq. (8), we have $\theta_{(\ell)} = A(r,v)/2r$. The condition $\theta_{(\ell)} = 0$ leads to $A(r_h,v) = 0$, which defines a curve $r_h = r_h(v)$ giving the location of the horizon; it is easy to show that $\theta_- < 0$, hence the horizon is of the future type. Writing the solution in the Vaidya-Bardeen form, that is with $A(r,v) = 1 - 2m(r,v)/r$, the Misner-Sharp energy of the black hole is $E = m(r_h(v),v)$, and the horizon will be outer trapping if $m^{'}(r_h,v) < 1/2$, a prime denoting the radial derivative. The geometrical surface gravity associated with the Vaidya-Bardeen dynamical horizon is

$$\kappa(v) \cong \frac{A^{'}(r,v)}{2} = \frac{m(r_h,v)}{r_h^2} - \frac{m^{'}(r_h,v)}{r_h} = \frac{1}{4m}(1-2m^{'}) \tag{20}$$

the same as Eq. (6), where $m \equiv m(r_h,v)$. We see the meaning of the outer trapping condition: it ensures the positivity of the surface gravity.

As a comparison, Hayward's first definition would give $\kappa = \sqrt{1-2m'}/4m$, while Fodor et al. expression is

$$\kappa = \frac{2^{\Psi}}{4m}(1-2m^{'}) + \dot{\Psi} \tag{21}$$

The effective surface gravity of Ashtekar-Krishnan simply is $\kappa = 1/4m$, everything being evaluated on the horizon. Note that some of them are not correct for the Reissner-Nordström black hole. We also note that κ (20) is inequivalent to the Nielsen-Visser surface gravity, which in these coordinates takes the form

$$\tilde{\kappa} = \frac{1}{4m}\left(1-2m^{'} - e^{-\Psi}\dot{m}\right) \tag{22}$$

though they coincide in the static case. Also, both are inequivalent to the Visser surface gravity $e^{\Psi}\tilde{\kappa}$, which was derived as a temperature by essentially the same tunneling method as discussed below, but in Painlevé-Gullstrand coordinates. Part of the difference can be traced to a different choice of time.

In the case of McVittie BHs, we obtain

$$\theta_{\pm} = \pm(\sqrt{A}_s \mp Hr)/2rf_{\pm}$$

where the functions $f_{\pm}$ are integrating factors determining null coordinates $x^{\pm}$ such that $dx^{\pm} = f_{\pm}[(\sqrt{A}_s \pm Hr)dt \pm A_s^{-1/2}dr]$. One may compute from this the dual derivative fields $\partial_{\pm}$. The future dynamical horizon defined by $\theta_+ = 0$, has a radius which is a root of the equation $\sqrt{A}_s = Hr_h$, which in turn implies $A_s = H^2 r_h^2$. Hence the horizon radius is a function of time. The Misner-Sharp mass and the related surface gravity are

$$E = m + \frac{1}{2}H(t)^2 r_h^3 \tag{23}$$

$$\kappa(t) = \frac{m}{r_h^2} - H^2 r_h - \frac{\dot{H}}{2H} = \frac{E}{r_h^2} - \frac{3}{2}H^2 r_h - \frac{\dot{H}}{2H} \tag{24}$$

Note that $E = r_h/2$. In the static cases everything agrees with the standard results. The surface gravity has an interesting expression in terms of the sources of Einstein equations and the Misner-Sharp mass. Let $\tilde{T}$ be the reduced trace of the stress tensor in the space normal to the sphere of symmetry, evaluated on the horizon H. For the Vaidya-Bardeen metric it is, by Einstein's equations (12),

$$\tilde{T} = T^v_{\ v} + T^r_{\ r} = -\frac{1}{2\pi r_h}\frac{\partial m}{\partial r}_{|r=r_h}$$

For the McVittie's solution, this time by Friedmann's equations (18) one has

$$\tilde{T} = -\rho + p = -\frac{1}{4\pi}(3H^2 + \frac{\dot{H}}{Hr_h})$$

We have then the mass formula

$$\frac{\kappa A_H}{4\pi} = E + 2\pi r_h^3 \tilde{T} \tag{25}$$

where $A_H = 4\pi r_h^2$. It is worth mentioning the pure FRW case, i.e. $A_s = 1$, for which $\kappa(t) = -(H(t) + \dot{H}/2H)$. One can easily see that (25) is fully equivalent to Friedmann's equation. We feel that these expressions for the surface gravity are non trivial and display deep connections with the emission process. Indeed it is the non vanishing of κ that is connected with the imaginary part of the action of a massless particle, as we are going to show in the next section.

4 Tunneling and instability

The essential property of the tunneling method is that the action I of an outgoing massless particle emitted from the horizon has an imaginary part which for stationary black holes is $\Im I = \pi\kappa^{-1}E$, where E is the Killing energy and κ the horizon surface gravity. The imaginary part is obtained by means of Feynman $i\epsilon$-prescription, as explained in [35, 36]. As a result the particle production rate reads $\Gamma = \exp(-2\Im I) = \exp(-2\pi\kappa^{-1}E)$. One then recognizes the Boltzmann factor, from which one deduces the well-known temperature $T_H = \kappa/2\pi$. Moreover, an explicit expression for κ is actually obtained in terms of radial derivatives of the metric on the horizon.

Let us consider now the case of a dynamical black hole in the double-null form [37]. We have for a massless particle along a radial geodesic the Hamilton-Jacobi equation $\partial_+ I \partial_- I = 0$. Since the particle is outgoing $\partial_- I$ is not vanishing, and we arrive at the simpler condition $\partial_+ I = 0$. First, let us apply this condition to the Vaidya-Bardeen BH. One has then

$$2e^{-\Psi(r,v)}\partial_v I + A(r,v)\,\partial_r I = 0\,. \tag{26}$$

Since the particle will move along a future null geodesic, to pick the imaginary part we expand the metric along a future null direction starting from an arbitrary event $(r_h(v_0), v_0)$ on the horizon, i.e. $A(r_h(v_0), v_0) = 0$. Thus, shortening $r_h(v_0) = r_0$, we have $A(r,v) = \partial_r A(r_0,v_0)\Delta r + \partial_v A(r_0,v_0)\Delta v + \cdots = 2\kappa(v_0)(r - r_0) + \ldots$, since along a null direction at the horizon $\Delta v = 0$, according to the metric (8); here $\kappa(v_0)$ is the surface gravity, Eq. (20). >From (26) and the expansion, $\partial_r I$ has a simple pole at the event (r_0, v_0); as a consequence

$$\Im I = \Im\int \partial_r I dr = -\Im\int dr \frac{2e^{-\Psi(r,v)}\partial_v I}{A'(r_0,v_0)(r - r_0 - i0)} = \frac{\pi\omega(v_0)}{\kappa(v_0)}\,. \tag{27}$$

where $\omega(v_0) = e^{-\Psi(r_0,v_0)}\partial_v I$, is to be identified with the energy of the particle at the time v_0. Note that the Vaidya-Bardeen metric has a sort of gauge invariance due to conformal reparametrizations of the null coordinate v: the map $v \to \tilde{v}(v)$, $\Psi(v,r) \to \tilde{\Psi}(\tilde{v},r) + \ln(\partial\tilde{v}/\partial v)$ leaves the metric invariant, and the energy is gauge invariant too. Thus we see that the Hayward-Kodama surface gravity appears to be relevant to the process of particles emission. The emission probability, $\Gamma = \exp(-2\pi\omega(v)/\kappa(v))$, has the form of a Boltzmann factor, suggesting a locally thermal spectrum.

For the McVittie's BH, the situation is similar. In fact, the condition $\partial_+ I = 0$ becomes

$$\partial_r I = -F(r,t)^{-1}\partial_t I$$

where

$$F(r,t) = \sqrt{A_s(r)}(\sqrt{A_s(r)} - rH(t))$$

As before, we pick the imaginary part by expanding this function at the horizon along a future null direction, using the fact that for two neighbouring events on a null direction in the metric (16), one has $t - t_0 = (2H_0^2 r_0^2)^{-1}(r - r_0)$, where $H_0 = H(t_0)$. We find the result

$$F(r,t) = (\frac{1}{2}A_s^{'}(r_0) - r_0 H_0^2 - \frac{\dot{H}_0}{2H_0})(r - r_0) + \cdots = \kappa(t_0)(r - r_0)\ldots \tag{28}$$

where this time $r_0 = r_h(t_0)$. >From this equation we see that $\partial_r I$ has a simple pole at the horizon; hence, making use again of Feynman $i\epsilon$-prescription, one finds $\Im I = \pi\kappa(t_0)^{-1}\omega(t_0)$, where $\omega(t) = \partial_t I$ is again the energy at time t, in complete agreement with the geometric evaluation of the previous section. Obviously, if κ vanishes on the horizon there is no simple pole and the black hole should be stable[4]. The kind of instability producing the Hawking flux for stationary black holes evidently persists in the dynamical arena, and so long as the evolution is sufficiently slow the black hole seems to evaporate thermally. Note that the imaginary part, that is the instability, is attached to the horizon all the time, confirming the Fredenhagen-Haag suggestion quoted in the introduction. It is worth mentioning the role of κ in the analogue of the first law for dynamical black holes (contributions to this problem for Vaidya black holes were given in [39]). Using the formulas of the projected stress tensor $\tilde{T}$ given above, and the expression of the Misner-Sharp energy, one obtains the differential law

$$dE = \frac{\kappa\, dA_H}{8\pi} - \frac{\tilde{T}}{2}\, dV_H \tag{29}$$

provided all quantities were computed on the trapping horizon. Here $A_H = 4\pi r_H^2$ is the horizon area and $V_H = 4\pi r_H^3/3$ is a formal horizon volume. If one interprets the "d" operator as a derivative along the future null direction one gets Hayward's form of the first law. But one can also interpret the differential operation more abstractly, as referring to an ensemble. Indeed, to obtain Eq. (29) it is not necessary to specify the meaning of the "d". It is to be noted that the same law can be proved with other, inequivalent definitions of the surface gravity, even maintaining the same meaning for the energy. Thus other considerations are needed to identify one: the tunneling method has made one choice.

As thoroughly discussed in Hayward et al. [40], Eq. (8) is actually the most general form of a spherically symmetric metric, so the above calculations works throughout. Of course $\kappa > 0$ if the trapping horizon is of the outer type. Thus the method has derived a positive temperature if and only if there is a future outer trapping horizon.

[4]However, charged extremal black holes can radiate [38].

Extremal limit

We discuss only an example, the charged stringy black hole, which represents a non-vacuum solution of Einstein-Maxwell dilaton gravity in the string frame [41, 42]:

$$ds^2 = r^2 d\Omega^2 + \frac{dr^2}{(1-a/r)(1-b/r)} - \left(\frac{1-a/r}{1-b/r}\right) dt^2 \tag{30}$$

where $a > b > 0$. The horizon radius is $r = a$.

For this example, the extremal limit as defined by global structure is $b \to a$. The Killing surface gravity $\kappa_\infty \cong 1/2a$ does not vanish in this limit. Garfinkle et al. [42] noted this as puzzling, since extremal black holes are expected to be zero-temperature objects.

Remarkably, the geometrical surface gravity (20)

$$\kappa \cong \frac{a-b}{2a^2} \tag{31}$$

vanishes in the extremal limit. Thus the gravitational dressing effect lowers the temperature to its theoretically expected value.

We conjecture that this is true in general. Indeed, past experience with extremal black holes showed that the horizon of these objects is not only a zero but also a minimum of the expansion $\theta_+ = \partial_+ A/A$ of the radially outgoing null geodesics, θ_+ becoming positive again on crossing the horizon. Thus $\partial_-\theta_+ \cong 0$ should be the appropriate definition of an extremal black hole. Since $\kappa = -e^{-2\varphi}\partial_-\partial_+ r$, this is equivalent to $\kappa = 0$.

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Statefinder analysis of universe models with a viscous cosmic fluid and with a fluid having a non-linear equation of state

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Abstract

In the present article we analyze, by means of the statefinder parameter formalism, some universe models introduced by Brevik and co-workers. We determine constants that earlier were left unspecified, in terms of observable quantities. It is verified that a Big Bang universe model with a fluid having a certain non-linear equation of state behaves in the same way as a model with a viscous fluid [1].

1 Introduction

The statefinder parameters were introduced into cosmology by Alam and co-workers[1] in 2003. They were then applied to flat universe models with dark energy and cold matter, where the dark energy was either of the quintessence type or a Chaplygin gas. The latter models were further investigated by Gorini and co-workers[2]. The formalism was generalized to curved universe models by Evans et al.[3].

The formalism was later applied to several universe models with other properties such as interaction between energy and matter[4-8] or other types of equation of state than that of the quintessence energy, resulting from for example viscosity[9-16]. G. Panotopoulos[17] has applied the statefinder diagnosis to brane universe models.

Iver Brevik and co-workers[21-31] have investigated the future behavior of several universe models filled with cosmic fluids with different equations of state[32-34]. Some of the equations of state may be relevant to fluids with viscosity[35]. They have in particular investigated whether the models arrive at a final so-called Big Rip singularity.

In the present work we want to apply the statefinder formalism to some of the universe models considered by Brevik and co-workers.

[1]This article is **dedicated to 70th aniversary of Professor Iver Brevik**

2 Earlier applications of the statefinder formalism

Alam and co-workers[1] first considered flat universe models with cold matter (dust) and dark energy in the form of quintessence obeying the equation of state

$$p_X = w\rho_X, \tag{1}$$

where p_X is the pressure of the dark energy and ρ_X its density (we use units so that $c = 1$). Here w is a function of time. They defined the statefinder parameters r and s as

$$r = \frac{\dddot{a}}{aH^3} \quad , \quad s = \frac{r-1}{3\,(q-1/2)}, \tag{2}$$

where

$$q = -\frac{\ddot{a}}{aH^2} = -1 - \frac{\dot{H}}{H^2} \tag{3}$$

is the deceleration parameter. In terms of the Hubble parameter and its derivatives with respect to cosmic time the statefinder parameters are given by

$$r = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \quad , \quad s = -\frac{2}{3H}\frac{3H\dot{H}+\ddot{H}}{3H^2+2\dot{H}}. \tag{4}$$

The deceleration parameter and the statefinder parameters can also be expressed in terms of the Hubble parameter and its derivative with respect to the redshift, represented by $x \equiv 1 + z = 1/a$,

$$q = \frac{H'}{H}x - 1, \tag{5}$$

$$r = 1 - 2\frac{H'}{H}x + \left[\frac{H''}{H} + \left(\frac{H'}{H}\right)^2\right]x^2, \tag{6}$$

$$s = \frac{-2H'x/H + \left[H''/H + (H'/H)^2\right]x^2}{3\left[H'x/H - 3/2\right]}. \tag{7}$$

Calculating the deceleration parameter and the statefinder parameters for this class of universe models one finds

$$q = \frac{1}{2}\left(1+3w\right) \quad , \quad r = 1 + \frac{9}{2}w\left(1+w\right)\Omega_X - \frac{3}{2}\frac{\dot{w}}{H}\Omega_X \quad , \quad s = 1 + w - \frac{1}{3}\frac{\dot{w}}{wH}. \tag{8}$$

Where Ω_X is the mass parameter of the dark energy. Hence

$$r = 1 + \frac{9}{2}w\,\Omega_X s \tag{9}$$

It should be noted that the flat ΛCDM universe model, which fits all the cosmological observations, has $w = -1$, $\dot{w} = 0$ and hence $(r, s) = (1, 0)$.
For the type of dark energy called Chaplygin gas, obeying the equation of state

$$p_C = -A/\rho_C^{\alpha}, \tag{10}$$

where A and α are positive constants, the statefinder parameters are found by using the relationships

$$q = \frac{1}{2}\left(1 + 3\frac{p}{\rho}\right) \quad , \quad r = 1 + \frac{9}{2}\frac{\rho + p}{\rho}\frac{\dot{p}}{\dot{\rho}} \quad , \quad s = \frac{\rho + p}{p}\frac{\dot{p}}{\dot{\rho}}, \tag{11}$$

where p and ρ are the total pressure and density, respectively. Hence,

$$r = 1 + \frac{9}{2}\frac{p}{\rho}s. \tag{12}$$

The formalism was generalized to curved universe models by Evans et. Al.[3]. Then the state parameter s is defined by

$$s = \frac{r - \Omega}{3\left(q - \Omega/2\right)}, \tag{13}$$

and eqs.(11) take the form

$$q = \frac{1}{2}\left(1 + 3\frac{p}{\rho}\right)\Omega \quad , \quad r = \left(1 - \frac{3}{2}\frac{\dot{p}}{H\rho}\right)\Omega \quad , \quad s = -\frac{1}{3H}\frac{\dot{p}}{p}. \tag{14}$$

Applied to a universe with only a Chaplygin gas this gives[2]

$$r = 1 - \frac{9}{2\alpha}s\left(1 + s\right). \tag{15}$$

If the source of the dark energy is a scalar field φ with the potential $V(\varphi)$, the equation of state factor w is

$$w = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \tag{16}$$

In this case the statefinder parameters are[3]

$$q = \frac{\Omega}{2} + \frac{\kappa}{2H^2}\left(\frac{1}{2}\dot{\varphi}^2 - V\right) \quad , \quad r = \Omega + \frac{3}{2}\kappa\frac{\dot{\varphi}^2}{H^2} + \kappa\frac{\dot{V}}{H^3} \quad , \quad s = 2\frac{\dot{\varphi}^2 + \frac{2}{3}\frac{\dot{V}}{H}}{\dot{\varphi}^2 - 2V}, \tag{17}$$

where $\kappa = 8\pi G$ is Einstein's constant of gravitation.
W. Zimdahl and D. Pavon[4], and X. Zhang with co-workers[6-9] applied the statefinder formalism to universe models with two interacting fluids. The dark matter component M interacts with the dark energy described by a scalar field φ by

$$\dot{\rho}_M + 3H\rho_M = -Q \quad , \quad \dot{\rho}_\varphi + 3H\rho_\varphi\left(1 + w_\varphi\right) = Q. \tag{18}$$

The deceleration parameter of such a universe model is

$$q = \frac{1}{2}\left(1 + 3w_\varphi\Omega_\varphi\right). \tag{19}$$

Defining effective equations of state for the dark matter and energy by

$$w_M^{eff} = \frac{Q}{3H\rho_M} \quad , \quad w_\varphi^{eff} = -\frac{Q}{3H\rho_\varphi}, \tag{20}$$

the statefinder parameters may be written

$$r = 1 - \frac{3}{2}\left[w'_\phi - 3w_\phi\left(1 + w_\phi^{eff}\right)\right]\Omega_\phi \ , \quad s = 1 - \frac{1}{3}\frac{w'_\phi}{w_\phi} + w_\phi^{eff}, \tag{21}$$

where w'_φ is w_φ differentiated with respect to $u = \ln a = -\ln(1+z)$, where a is the scale factor and $z$the redshift. The relation between r and s is

$$r = 1 + \frac{9}{2}w_\varphi s. \tag{22}$$

M. G. Hu and X. H. Meng[17] have studied flat universe models with a viscous fluid. This type of models shall be investigated in some detail in the next section. They also analyzed a flat universe model with only dark energy obeying the inhomogeneous equation of state,

$$p = w\rho + p_1, \tag{23}$$

where w and p_1 are both constant. Defining the quantities

$$\tilde{\gamma} = -\frac{p_1}{\rho_0} \ , \quad V = p_1\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right), \tag{24}$$

where ρ_0 is the present density, they find for the deceleration parameter and the statefinder parameters,

$$q = \frac{3}{2}V - 1 \ , \quad r = 1 + \frac{9}{2}(\tilde{\gamma} - 1)V \ , \quad s = \frac{(\tilde{\gamma} - 1)V}{V - 1}. \tag{25}$$

The ΛCDM universe model, that is consistent with all present observations and may be considered the standard model of the universe, has $(q, r, s) = (-1, 1, 0)$which is fulfilled for the above model if $\rho = \rho_0$.

3 Viscous dark energy

In this section I will deduce the expressions for the statefinder parameters of flat universe models with dust and viscous dark energy in terms of the Hubble parameter. The dark energy is assumed to obey the usual equation of state, $p = w\rho$, with constant value of w. Friedmann's 1. equation then takes the form

$$H^2 = (\kappa/3)(\rho_m + \rho_x) = (\kappa/3)\rho_{cr}. \tag{26}$$

where ρ_m , ρ_x , ρ_{cr} are the density of the matter, the dark energy and the critical density, respectively. Friedmann's 2. equation is

$$\frac{\ddot{a}}{a} + \frac{H^2}{2} = -\frac{\kappa}{2}\bar{p}_x, \tag{27}$$

where $\kappa = 8\pi G$ and G is Newton's constant of gravitation, and

$$\bar{p}_x = w\rho_x - 3\varsigma H \tag{28}$$

is the effective pressure of the dark energy. We shall assume that the viscosity coefficient ς is constant, and that the dust and the dark energy does not interact. Inserting eq.(28)into eq.(27) we obtain,

$$\frac{\ddot{a}}{a} = -\frac{H^2}{2} - \frac{\kappa}{2} w \rho_x + \frac{3}{2} H \kappa \varsigma. \tag{29}$$

By means of eq.(29) and introducing the mass parameters

$$\Omega_x = \frac{\kappa}{3H^2} \rho_x \ , \ \Omega_m = \frac{\kappa}{3H^2} \rho_m = 1 - \Omega_x, \tag{30}$$

where we have used that the universe is flat, the expression (3) of the deceleration parameter can be written

$$q = \frac{1}{2} \left(1 + 3w\Omega_x - 3\kappa\varsigma / H \right). \tag{31}$$

>From eqs.(3) and (4) we get

$$r = q + 2q^2 - \dot{q}/H. \tag{32}$$

The equation of continuity of the dark matter is

$$\dot{\rho}_m = -3H\rho_m, \tag{33}$$

and of the dark energy

$$\dot{\rho}_x = -3H \left[(1+w) \rho_x - 3H\varsigma \right]. \tag{34}$$

Differentiating eq.(26) and inserting eqs.(33) and (34) we obtain

$$\frac{\dot{H}}{H^2} = -\frac{3}{2} \left(1 + w\Omega_x - \frac{\kappa\xi}{H} \right). \tag{35}$$

Differentiating the first of eqs.(26) and using eqs.(34) and (35) we find

$$\dot{\Omega}_x = 3 \left(1 - \Omega_x \right) \kappa\varsigma. \tag{36}$$

Integration with $\Omega_x(0) = 0$ gives

$$\Omega_x = 1 - e^{-3\kappa\varsigma t} \tag{37}$$

independently of the value of w.
Differentiating the expression (31) and using eqs.(35) and (36) then gives

$$\dot{q} = -\frac{9}{4} \left(1 - 2w + 3w\Omega_x - \frac{\kappa\varsigma}{H} \right) \kappa\varsigma. \tag{38}$$

Inserting the expressions (31) and (38) into eq.(32) finally gives

$$r = 1 + \frac{9}{2} w (1+w) \Omega_x - \frac{9}{4} \left(1 + 2w + w\Omega_x - \frac{\kappa\xi}{H} \right) \frac{\kappa\varsigma}{H}. \tag{39}$$

4 Dark fluid with a non-linear equation of state

In ref. 21 Brevik and co-workers have studied, in their second case, a flat Friedmann-Robertson-Walker universe model dominated by a dark energy with equation of state

$$p = w\rho + A\sqrt{\rho}, \tag{40}$$

where w and A are constants. Lorentz invariant dark energy, which may be represented by a cosmological constant, has $w = -1$ and $A = 0$. For flat universe models with a single fluid Friedmann's 1. equation reduces to

$$H^2 = \frac{\kappa}{3}\rho, \tag{41}$$

and with the equation of state (40) the equation of continuity takes the form

$$\dot{\rho} + \sqrt{3\kappa}\,(1+w)\,\rho^{3/2} + \sqrt{3\kappa}A\rho = 0. \tag{42}$$

Integration of eq.(42) with $\rho\,(0) = (A/w)^2$ gives

$$\rho = \frac{A^2}{\left[e^{\frac{\sqrt{3\kappa}}{2}A\bar{t}} - (1+w)\right]^2}, \tag{43}$$

where the time co-ordinate used by Brevik and co-workers[21] has been called $\bar{t}$. Note that $\rho\,(\bar{t}_S) = \infty$ for $\bar{t}_S = -t_A \ln(1+w)$ where

$$t_A = -\left(2/\sqrt{3\kappa}A\right). \tag{44}$$

The minus is included because a later comparison with observational data will show that $A < 0$. Note also that $\bar{t}_S > 0$ for $w < 0$. I will interpret the singularity as the Big Bang of this universe model. With this interpretation the points of time $\bar{t} < \bar{t}_S$ are without physical significance. Introducing a cosmic time $t = \bar{t} - \bar{t}_S$ with origin at the Big Bang, the expression of the density of the cosmic fluid becomes

$$\rho = \left(\frac{A}{1+w}\right)^2 \frac{1}{\left(1 - e^{-t/t_A}\right)^2}. \tag{45}$$

From eqs.(41), (44) and (45) follow that the Hubble parameter is

$$H = \frac{H_A}{1 - e^{-t/t_A}} \quad , \quad H_A = \frac{2}{3\,(1+w)\,t_A}. \tag{46}$$

For $t >> t_A$ the behavior of the expansion approaches that of DeSitter space with a constant Hubble parameter H_A containing a cosmic fluid with density $\rho_A = [A/(1+w)]^2$. At the initial singularity the Hubble parameter is infinitely great. Normalizing the scale parameter so that it has the value $a\,(t_0) = 1$ at the present time t_0 we find

$$a = \left(\frac{e^{t/t_A} - 1}{e^{t_0/t_A} - 1}\right)^{\frac{2}{3(1+w)}}. \tag{47}$$

Using that

$$H_A t_A = \frac{2}{3\,(1+w)}, \tag{48}$$

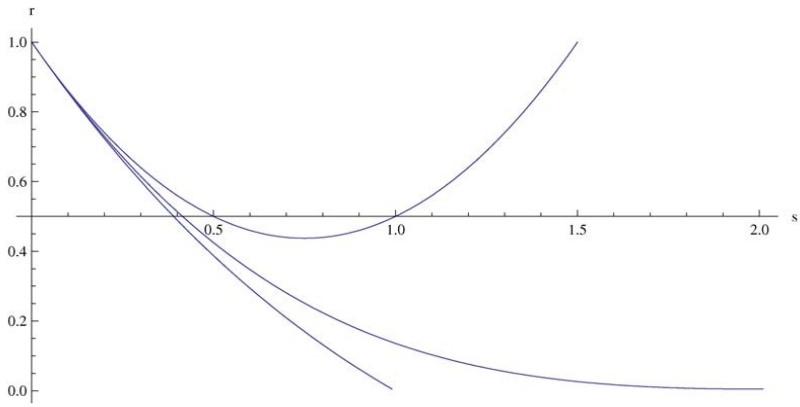

Figure 1: (r, s)-diagram for the universe model with statefinder parameters given in eqs.(52) and (53) The curves are plotted for $t \in [0, \infty]$. The lowest curve is for $w = -2/3$, the next Lone for $w = -1/3$, and the upper one for $w = 0$.

the deceleration parameter is

$$q = -1 + \frac{3}{2}(1+w)\, e^{-t/t_A}. \tag{49}$$

Note that for $t << t_A$ the scale factor is to lowest order in t/t_A

$$a(t) \approx \left(\frac{t}{t_A}\right)^{\frac{2}{3(1+w)}}. \tag{50}$$

which is the same behaviour as that of a universe dominated by a perfect fluid obeying the homogeneous equation of state $p = w\rho$. The late time behavior is that of a de Sitter universe with accelerated expansion independently of the value of w. For $w > -1/3$ there is a transition from decelerated to accelerated expansion at the point of time

$$t_2 = t_A \ln\left[\frac{3}{2}(1+w)\right]. \tag{51}$$

This behavior reflects the fact that the first term of eq.(40) dominates initially when the density is large, but the late time behavior is dominated by the second term in eq.(40). Using eqs.(4) we find that the statefinder parameters for this universe model are

$$r = 1 - \frac{9}{4}\left(1-w^2\right) e^{-t/t_A} + \frac{9}{4}(1+w)^2 e^{-2t/t_A}, \tag{52}$$

$$s = \frac{3}{2}(1+w)\, \frac{1 - w - (1+w)\, e^{-t/t_A}}{e^{t/t_A} - 1 - w}. \tag{53}$$

These expressions are plottet in a (r, s)-diagram in Figure 1.

The present values of the Hubble parameter, $H_0 = H(t_0)$, and the deceleration parameter, $q_0 = q(t_0)$ are determined by observations, which show that $H_0 t_0 \approx 1$ and $q_0 \approx -0.6$. This will be used to determine the quantities H_A and t_A. From eqs.(46) and (49) we find that H_A is determined by the equation

$$\frac{H_0}{H_A} + \frac{1+q_0}{\ln\left(1+\frac{H_A}{H_0}\right)} = 1. \tag{54}$$

Inserting $q_0 \approx -0.6$ and solving the equation numerically, we find $H_A = 0,8H_0$. Hence,

$$t_A = -\frac{t_0}{\ln\left(1-\frac{H_A}{H_0}\right)} \approx 0.6t_0. \tag{55}$$

With $t_0 = 13,7 \cdot 10^9$ years we get $t_A \approx 8.2 \cdot 10^9$ years, so that $H_A t_A \approx 0.8H_0 \cdot 0.6t_0 \approx 0.5$. According to eq.(48) the equation of state factor w then is

$$w = \frac{2}{3H_A t_A} - 1 \approx \frac{1}{3}. \tag{56}$$

This universe model thus contains a fluid behaving somewhat like a combination of electromagnetic radiation and dark energy with an equation of state with negative pressure (note that $A < 0$).
The value $s = 0$ of the ΛCDM- model takes place at the point of time

$$t_3 = t_A \ln\frac{1+w}{1-w}. \tag{57}$$

A positive value of t_3 requires $w > 0$, which is compatible with the value of w obtained from the present values of the Hubble parameter and the deceleration parameter.

5 Viscous cosmic fluid

I. Brevik and O. Gorbunova[24] have recently investigated some universe models dominated by a viscous cosmic fluid with an effective pressure given by eq.(28). They were particularly interested in the late time behaviour of the models and whether they would enter a so-called Big Rip. Therefore they investigated models with $w < -1$. I will consider flat universe models dominated by a viscous fluid with $w > -1$. Then $\Omega_X = 1$, and the expressions (31) and (39) for the deceleration parameter and the statefinder parameter r take the form

$$q = \frac{1}{2} \cdot (1 + 3w - 3\kappa\varsigma/H), \tag{58}$$

$$r = 1 + \frac{9}{2}w(1+w) - \frac{9}{4}\left(1 + 3w - \frac{\kappa\varsigma}{H}\right)\frac{\kappa\varsigma}{H}. \tag{59}$$

The expression (60) may be factorized as

$$r = 1 - \frac{9}{4}\left(1 + w - \frac{\kappa\varsigma}{H}\right)\left(2w - \frac{\kappa\varsigma}{H}\right). \tag{60}$$

For such universe models the statefinder parameter s is

$$s = \frac{3}{2} \cdot \frac{(1 + w - \kappa\varsigma/H)(2w - \kappa\varsigma/H)}{w - \kappa\varsigma/H}. \tag{61}$$

The equation of motion of the cosmic expansion for the present class of models may be written

$$\dot{H} + \frac{3}{2}\left(1+w\right)H^2 - \frac{3}{2}\kappa\varsigma H = 0. \tag{62}$$

For later comparison Brevik and Gorbunova first considered a universe model dominated by a perfect fluid with no viscosity, and wrote the solution of eq.(63) for that case as[23]

$$a\left(t\right) = a_0\left[1 + \frac{3}{2}H_0\left(1+w\right)\left(t-t_0\right)\right]^{-2/3\alpha}. \tag{63}$$

In this paper I will consider Big Bang Universe models with $a\left(0\right)=0$. This demands that

$$H_0 t_0 = \frac{2}{3\left(1+w\right)}. \tag{64}$$

Normalizing the scale factor to unity at the present time, we then obtain

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}. \tag{65}$$

The general solution of eq.(63) with viscosity was written by Brevik as[25]

$$a = a_0\left[1 + \frac{3}{2}\left(1+w\right)H_0 t_\varsigma\left(e^{(t-t_0)/t_\varsigma} - 1\right)\right]^{\frac{2}{3(1+w)}} \quad , \quad t_\varsigma = \left(\frac{3}{2}\kappa\varsigma\right)^{-1}. \tag{66}$$

Demanding again $a\left(0\right)=0$ requires

$$H_0 t_\varsigma = \frac{2}{3\left(1+w\right)\left(1-e^{-t_0/t_\varsigma}\right)}. \tag{67}$$

With the normalization $a\left(t_0\right)=1$ the scale factor is

$$a = \left(\frac{e^{t/t_\varsigma}-1}{e^{t_0/t_\varsigma}-1}\right)^{\frac{2}{3(1+w)}}. \tag{68}$$

The corresponding Hubble parameter is

$$H = H_0\frac{1-e^{-t_0/t_\varsigma}}{1-e^{-t/t_\varsigma}}. \tag{69}$$

Inserting eq.(70) in eqs.(60) and (62) and using eq.(68) give the expressions (52) and (53) for the statefinder parameter with t_A replaced by t_ς.

We see that the expansion behaviour of this model, dominated by a single viscous fluid, is given by identical expressions to those of the model considered in section 4, dominated by a fluid with a non-linear equation of state. This is, however, very natural. The viscous fluid has an effective pressure given by eq.(28). The first Friedmann equation still has the form (41). Hence the effective pressure is

$$\bar{p} = p - 3\kappa\varsigma H = w\rho - \varsigma\kappa^{3/2}\sqrt{3\rho}, \tag{70}$$

and the equation of continuity is

$$\dot{\rho} + \sqrt{3\kappa}\left(1+w\right)\rho^{3/2} - 3\kappa\varsigma\rho = 0, \tag{71}$$

which has the same form as eq.(42).
The initial value of the Hubble parameter is $H(0) = \infty$. When $t \to \infty$ the Hubble-parameter approaches the value.

$$H_\infty = H_0 \left(1 - e^{-t_0/t_\varsigma}\right) = \frac{2}{3(1+w)\,t_\varsigma} = \frac{\kappa\varsigma}{1+w}. \tag{72}$$

The time t_ς depends upon the strength of the viscosity. From the expression of t_ς given in eq.(68) follows that a small viscosity implies a large value of t_ς. Brevik[25] has estimated the value of ς. At $t = 1000s$ after the Big Bang he finds that $t_\varsigma \approx 10^{21}y$. Hence, we have that $e^{t/t_\varsigma} \approx 1$ for the whole history of the universe up to now. Eq.(68) then gives to lowest order in t/t_ς,

$$w \approx \frac{2}{3H_0 t_0} - 1 \approx -\frac{1}{3}. \tag{73}$$

For these models the observationally favoured values of the statefinder parameters, $(r,s) = (1,0)$ take place at the points of time t_{41} and t_{42} given by

$$H(t_{41}) = \frac{\kappa\varsigma}{1+w} \quad , \quad H(t_{42}) = \frac{\kappa\varsigma}{2w}. \tag{74}$$

For $w < 0$ only t_{41} corresponds to an expanding universe. Comparing with eq.(73) we see that $H(t_{41}) = H_\infty$. Hence, the statefinder parameters of these universe models approach the favoured value in the infinite future.

6 Conclusion

Brevik and co-workers have investigated several classes of universe models with different types of dark energy. Two of these classes, one with dark energy having an inhomogeneous equation of state, and one with viscous dark energy, have been considered in this paper, and it has been demonstrated that they are closely related.
While Brevik and co-workers focused upon the late-time behaviour of these models with special emphasis upon the question whether they would enter a so-called Big Rip singularity, I have focused upon these universe models as Big Bang models and their behaviour up to the present time.
The statefinder parameters of the models have been calculated, and some restrictions upon the dark energy has been obtained by demanding that they should pass through an era where the values of the statefinder parameter are not too far from the values $(r,s) = (1,0)$ of the ΛCDM-model that are favoured by observations.

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Casimir densities for wedge-shaped boundaries

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Abstract

The vacuum expectation values of the field squared and the energy-momentum tensor are investigated for a scalar field with Dirichlet boundary conditions and for the electromagnetic field inside a wedge with a coaxial cylindrical boundary. In the case of the electromagnetic field perfectly conducting boundary conditions are assumed on the bounding surfaces. By using the Abel-Plana-type formula for the series over the zeros of the Bessel function, the vacuum expectation values are presented in the form of the sum of two terms. The first one corresponds to the geometry without a cylindrical boundary and the second one is induced by the presence of the cylindrical shell. The additional vacuum forces acting on the wedge sides due the presence of the cylindrical boundary are evaluated and it is shown that these forces are attractive for both scalar and electromagnetic fields [2].

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1 Introduction

The Casimir effect has important implications on all scales, from cosmological to subnuclear, and has become in recent decades an increasingly popular topic in quantum field theory. Since the original work by Casimir [1] many theoretical and experimental works have been done on this problem (see, e.g., [2, 3] and references therein). In particular, a great deal of attention received the investigations of quantum effects for cylindrical boundaries. In addition to traditional problems of quantum electrodynamics under the presence of material boundaries, the Casimir effect for cylindrical geometries can also be important to the flux tube models of confinement [4] and for determining the structure of the vacuum state in interacting field theories [5]. The calculation of the vacuum energy of electromagnetic field with boundary

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conditions defined on a cylinder turned out to be technically a more involved problem than the analogous one for a sphere. First the Casimir energy of an infinite perfectly conducting cylindrical shell has been calculated in Ref. [6] by introducing ultraviolet cutoff and later the corresponding result was derived by zeta function technique [7] (for a recent discussion of the Casimir energy and self-stresses in the more general case of a dielectric-diamagnetic cylinder see [8] and references therein). The local characteristics of the corresponding electromagnetic vacuum such as energy density and vacuum stresses are considered in [9] for the interior and exterior regions of a conducting cylindrical shell, and in [10] for the region between two coaxial shells (see also [11]). The vacuum forces acting on the boundaries in the geometry of two cylinders are also considered in Refs. [12]. The scalar Casimir densities for a single and two coaxial cylindrical shells with Robin boundary conditions are investigated in Refs. [13, 14]. Less symmetric configuration of two eccentric perfectly conducting cylinders is considered in [12]. Vacuum energy for a perfectly conducting cylinder of elliptical section is evaluated in Ref. [15] by the mode summation method, using the ellipticity as a perturbation parameter. The Casimir forces acting on two parallel plates inside a conducting cylindrical shell are investigated in Ref. [16]. The Casimir effect in more complicated geometries with cylindrical boundaries is considered in [17].

Aside from their own theoretical and experimental interest, the problems with this type of boundaries are useful for testing the validity of various approximations used to deal with more complicated geometries. From this point of view the wedge with a coaxial cylindrical boundary is an interesting system, since the geometry is nontrivial and it includes two dynamical parameters, radius of the cylindrical shell and opening angle of the wedge. This geometry is also interesting from the point of view of general analysis for surface divergences in the expectation values of local physical observables for boundaries with discontinuities. The nonsmoothness of the boundary generates additional contributions to the heat kernel coefficients (see, for instance, the discussion in [18] and references therein). In the present paper we review the results of the investigations for the vacuum expectation values of the field squared and the energy-momentum tensor for the scalar and electromagnetic fields in the geometry of a wedge with a coaxial cylindrical boundary. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field. Some most relevant investigations to the present paper are contained in Refs. [2, 19, 20, 21, 22, 23, 24], where the geometry of a wedge without a cylindrical boundary is considered for a conformally coupled scalar and electromagnetic fields in a four dimensional spacetime. The Casimir effect in open geometries with edges is investigated in [25]. The total Casimir energy of a semi-circular infinite cylindrical shell with perfectly conducting walls is considered in [26] by using the zeta function technique. The Casimir energy for the wedge-arc geometry in two dimensions is discussed in [27]. For a scalar field with an arbitrary curvature coupling parameter the Wightman function, the vacuum expectation values of the field squared and the energy-momentum tensor in the geometry of a wedge with an arbitrary opening angle and with a cylindrical boundary are evaluated in [28, 29]. Note that, unlike the case of conformally coupled fields, for a general coupling the vacuum energy-momentum tensor is angle-dependent and diverges on the wedge sides. The corresponding problem for the electromagnetic field, assuming that all boundaries are perfectly conducting, is investigated in [30]. The scalar Casimir densities in the geometry of a wedge with two cylindrical boundaries are discussed in [31]. The closely related problem of the vacuum densities induced by a cylindrical boundary in the geometry of a cosmic string is investigated in Refs. [32] for scalar, electromagnetic and fermionic fields.

We have organized the paper as follows. The next section is devoted to the evaluation of the Wightman function for a scalar field with a general curvature coupling inside a wedge with a cylindrical boundary. By using the formula for the Wightman function, in section 3 we evaluate the vacuum expectation values of the field squared and the energy-momentum tensor inside a wedge without a cylindrical boundary. The vacuum densities for a wedge with the cylindrical shell are considered in section 4. Formulae for the shell contributions are derived and the corresponding surface divergences are investigated. The vacuum expectation values of the electric and magnetic field squared inside a wedge with a cylindrical boundary are investigated in section 5, assuming that all boundaries are perfectly conducting. The corresponding expectation values for the electromagnetic energy-momentum tensor are considered in section 6. The results are summarized in section 7.

2 Wightman function for a scalar field

Consider a free scalar field $\varphi(x)$ inside a wedge with the opening angle ϕ_0 and with a cylindrical boundary of radius a (see figure 1). We will use cylindrical coordinates $(x^1, x^2, \ldots, x^D) = (r, \phi, z_1, \ldots, z_N)$, $N = D-2$, where D is the number of spatial dimensions. The field equation has the form

$$\left(\nabla^i\nabla_i + m^2 + \xi R\right)\varphi(x) = 0, \tag{1}$$

where R is the scalar curvature for the background spacetime and ξ is the curvature coupling parameter. The special cases $\xi = 0$ and $\xi = \xi_c = (D-1)/4D$ correspond to minimally and conformally coupled scalars respectively.

In this section we evaluate the positive frequency Wightman function $\langle 0|\varphi(x)\varphi(x')|0\rangle$ assuming that the field obeys Dirichlet boundary condition on the bounding surfaces:

$$\varphi|_{\phi=0} = \varphi|_{\phi=\phi_0} = \varphi|_{r=a} = 0. \tag{2}$$

The vacuum expectation value (VEV) of the energy-momentum tensor is expressed in terms of the Wightman function as

$$\langle 0|T_{ik}(x)|0\rangle = \lim_{x'\to x}\nabla_i\nabla'_k\langle 0|\varphi(x)\varphi(x')|0\rangle + \left[\left(\xi - \frac{1}{4}\right)g_{ik}\nabla^l\nabla_l - \xi\nabla_i\nabla_k\right]\langle 0|\varphi^2(x)|0\rangle. \tag{3}$$

In addition, the response of a particle detector in an arbitrary state of motion is determined by this function. In (3) we have assumed that the background spacetime is flat and the term with the Ricci tensor is omitted. The Wightman function is presented as the mode sum

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_\alpha \varphi_\alpha(x)\varphi_\alpha^*(x'), \tag{4}$$

where $\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}$ is a complete orthonormal set of solutions to the field equation, satisfying the boundary conditions, α is a set of the corresponding quantum numbers.

2.1 Interior region

In the region $0 \leqslant r \leqslant a$ (region I in figure 1), the eigenfunctions satisfying the boundary conditions (2) on the wedge sides $\phi = 0, \phi_0$ have the form

$$\begin{aligned} \varphi_\alpha(x) &= \beta_\alpha J_{qn}(\gamma r)\sin(qn\phi)\exp\left(i\mathbf{k}\mathbf{r}_\parallel - i\omega t\right), & (5)\\ \omega &= \sqrt{\gamma^2 + k_m^2},\ k_m^2 = |\mathbf{k}|^2 + m^2, \quad q = \pi/\phi_0, & (6)\end{aligned}$$

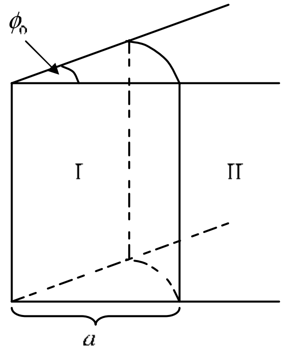

Figure 1: Geometry of a wedge with the opening angle ϕ_0 and cylindrical boundary of radius a.

where $\alpha = (n, \gamma, \mathbf{k})$, $-\infty < k_j < \infty$, $n = 1, 2, \cdots$, $\mathbf{k} = (k_1, \ldots, k_N)$, $\mathbf{r}_\| = (z_1, \ldots, z_N)$, and $J_l(z)$ is the Bessel function. The normalization coefficient β_α is determined from the standard Klein-Gordon scalar product with the integration over the region inside the wedge and is equal to

$$\beta_\alpha^2 = \frac{2}{(2\pi)^N \omega \phi_0 a^2 J'^2_{qn}(\gamma a)}. \tag{7}$$

The eigenvalues for the quantum number γ are quantized by the boundary condition (2) on the cylindrical surface $r = a$. From this condition it follows that

$$\gamma = \lambda_{n,j}/a, \quad j = 1, 2, \cdots , \tag{8}$$

where $\lambda_{n,j}$ are the positive zeros of the Bessel function, $J_{qn}(\lambda_{n,j}) = 0$, arranged in ascending order, $\lambda_{n,j} < \lambda_{n,j+1}$.

Substituting the eigenfunctions (5) into mode sum formula (4) with the set of quantum numbers $\alpha = (n, j, \mathbf{k})$, for the positive frequency Wightman function one finds

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \int d^N\mathbf{k}\, e^{i\mathbf{k}\Delta\mathbf{r}_\|} \sum_{n=1}^{\infty} \sin(qn\phi)\sin(qn\phi') \sum_{j=1}^{\infty} \beta_\alpha^2 J_{qn}(\gamma r) J_{qn}(\gamma r') e^{-i\omega\Delta t}, \tag{9}$$

where $\gamma = \lambda_{n,j}/a$, and $\Delta\mathbf{r}_\| = \mathbf{r}_\| - \mathbf{r}'_\|$, $\Delta t = t - t'$. In order to obtain an alternative form for the Wightman function we apply to the sum over j a variant of the generalized Abel-Plana summation formula [33]

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{2f(\lambda_{n,j})}{\lambda_{n,j} J'^2_{qn}(\lambda_{n,j})} &= \int_0^\infty f(z)dz + \frac{\pi}{4}\mathrm{Res}_{z=0}\left[f(z)\frac{Y_{qn}(z)}{J_{qn}(z)}\right] - \\ &+ \frac{i}{\pi}\int_0^\infty dz\, \frac{K_{qn}(z)}{I_{qn}(z)}[e^{-qn\pi i} f(ze^{\pi i/2}) - e^{qn\pi i} f(ze^{-\pi i/2})], \end{aligned} \tag{10}$$

where $Y_l(z)$ is the Neumann function, and $I_l(z)$, $K_l(z)$ are the modified Bessel functions. The corresponding conditions for the formula (10) to be valid are satisfied if $r + r' + |t - t'| < 2a$. In particular, this is the case in the coincidence limit $t = t'$ for the region under consideration, $r, r' < a$. Formula (10) allows to present the Wightman function in the form

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \langle 0_w|\varphi(x)\varphi(x')|0_w\rangle + \langle\varphi(x)\varphi(x')\rangle_{\mathrm{cyl}}, \tag{11}$$

where

$$\langle 0_w|\varphi(x)\varphi(x')|0_w\rangle = \frac{1}{\phi_0}\int\frac{d^N\mathbf{k}}{(2\pi)^N}e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\int_0^\infty dz\frac{ze^{-i\Delta t\sqrt{z^2+k_m^2}}}{\sqrt{z^2+k_m^2}} \times\sum_{n=1}^\infty\sin(qn\phi)\sin(qn\phi')J_{qn}(zr)J_{qn}(zr'), \quad (12)$$

and

$$\langle\varphi(x)\varphi(x')\rangle_{\rm cyl} = -\frac{2}{\pi\phi_0}\int\frac{d^N\mathbf{k}}{(2\pi)^N}e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\int_{k_m}^\infty dz\frac{z\cosh(\Delta t\sqrt{z^2-k_m^2})}{\sqrt{z^2-k_m^2}} \times\sum_{n=1}^\infty\sin(qn\phi)\sin(qn\phi')I_{qn}(zr)I_{qn}(zr')\frac{K_{qn}(za)}{I_{qn}(za)}. \quad (13)$$

In the limit $a\to\infty$ for fixed r,r', the term $\langle\varphi(x)\varphi(x')\rangle_{\rm cyl}$ vanishes whereas the part (12) does not depend on a. Hence, the term $\langle 0_w|\varphi(x)\varphi(x')|0_w\rangle$ is the Wightman function for the wedge without a cylindrical boundary with the corresponding vacuum state $|0_w\rangle$. Consequently, the term $\langle\varphi(x)\varphi(x')\rangle_{\rm cyl}$ is induced by the presence of the cylindrical boundary. For points away the cylindrical surface this part is finite in the coincidence limit and the renormalization is needed only for the part coming from the term (12).

2.2 Exterior region

In the region outside the cylindrical shell (region II in figure 1): $r>a$, $0\leqslant\phi\leqslant\phi_0$, the eigenfunctions satisfying boundary conditions (2) are obtained from (5) by the replacement

$$J_{qn}(\gamma r)\to g_{qn}(\gamma r,\gamma a)\equiv J_{qn}(\gamma r)Y_{qn}(\gamma a)-J_{qn}(\gamma a)Y_{qn}(\gamma r). \quad (14)$$

Now the spectrum for the quantum number γ is continuous and

$$\beta_\alpha^2=\frac{(2\pi)^{2-D}\gamma}{\phi_0\omega\left[J_{qn}^2(\gamma a)+Y_{qn}^2(\gamma a)\right]}. \quad (15)$$

Substituting the corresponding eigenfunctions into the mode sum formula (4), the positive frequency Whightman function in the exterior region is presented in the form

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \frac{1}{\phi_0}\int\frac{d^N\mathbf{k}}{(2\pi)^N}e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\sum_{n=1}^\infty\sin(qn\phi)\sin(qn\phi') \times\int_0^\infty d\gamma\frac{\gamma g_{qn}(\gamma r,\gamma a)g_{qn}(\gamma r',\gamma a)}{J_{qn}^2(\gamma a)+Y_{qn}^2(\gamma a)}\frac{e^{-i\Delta t\sqrt{\gamma^2+k_m^2}}}{\sqrt{\gamma^2+k_m^2}}. \quad (16)$$

To find the part in the Wightman function induced by the presence of the cylindrical shell, we subtract from (16) the corresponding function for the wedge without a cylindrical shell, given by (12). This allows to present the Wightman function in the form (11) with the cylindrical shell induced part

$$\langle\varphi(x)\varphi(x')\rangle_{\rm cyl} = -\frac{2}{\pi\phi_0}\int\frac{d^N\mathbf{k}}{(2\pi)^N}e^{i\mathbf{k}\Delta\mathbf{r}_\parallel}\int_k^\infty dz\frac{z\cosh(\Delta t\sqrt{z^2-k^2})}{\sqrt{z^2-k^2}} \times\sum_{n=1}^\infty\sin(qn\phi)\sin(qn\phi')K_{qn}(zr)K_{qn}(zr')\frac{I_{qn}(za)}{K_{qn}(za)}. \quad (17)$$

As we see, the expressions for the Wightman functions in the interior and exterior regions are related by the interchange $I_{qn} \rightleftarrows K_{qn}$ of the modified Bessel functions.

3 VEVs inside a wedge without a cylindrical boundary

In this section we consider the geometry of a wedge without a cylindrical boundary. For integer values of q, after the explicit summation over n, the Wightman function is presented in the form

$$\langle 0_w|\varphi(x)\varphi(x')|0_w\rangle = \frac{m^{(D-1)/2}}{(2\pi)^{(D+1)/2}}\sum_{j=1}^{2}(-1)^{j+1}\sum_{l=0}^{q-1}\frac{K_{(D-1)/2}(m\sqrt{u_l^{(j)2}+|\Delta\mathbf{r}_\parallel|^2-(\Delta t)^2})}{\left[u_l^{(j)2}+|\Delta\mathbf{r}_\parallel|^2-(\Delta t)^2\right]^{(D-1)/4}}, \tag{18}$$

where $u_l^{(j)} = \{r^2+r'^2-2rr'\cos[2\pi l/q+\phi+(-1)^j\phi']\}^{1/2}$. Note that the Wightman function in the Minkowski spacetime coincides with the term $j=1$, $l=0$ in formula (18).

Taking the coincidence limit $x'\to x$, for the difference of the VEVs of the field squared,

$$\langle\varphi^2\rangle_{\rm ren}^{(w)} = \langle 0_w|\varphi^2(x)|0_w\rangle - \langle 0_M|\varphi^2(x)|0_M\rangle, \tag{19}$$

where $|0_M\rangle$ is the amplitude for the vacuum state in the Minkowski spacetime without boundaries, we find

$$\langle\varphi^2\rangle_{\rm ren}^{(w)} = \frac{m^{D-1}}{(2\pi)^{(D+1)/2}}\sum_{j=1}^{2}(-1)^{j+1}{\sum_{l=0}^{q-1}}'\frac{K_{(D-1)/2}(2mr\sin\phi_l^{(j)})}{(2mr\sin\phi_l^{(j)})^{(D-1)/2}}. \tag{20}$$

In this formula, the prime means that the term $j=1$, $l=0$ has to be omitted, and we use the notation

$$\phi_l^{(j)} = \pi l/q + (1+(-1)^j)\phi/2. \tag{21}$$

For a massless field, from (20) we find

$$\langle\varphi^2\rangle_{\rm ren}^{(w)} = \frac{\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{\frac{D+1}{2}}r^{D-1}}\sum_{j=1}^{2}{\sum_{l=0}^{q-1}}'\frac{(-1)^{j+1}}{\sin^{D-1}\phi_l^{(j)}}, \tag{22}$$

Note that the terms in this formula with $j=2$, $l=0$ and $j=2$, $l=q-1$ are the corresponding VEVs for the geometry of a single plate located at $\phi=0$ and $\phi=\phi_0$, respectively. In the case $D=3$ for the renormalized VEV of the field square one finds [20]

$$\langle\varphi^2\rangle_{\rm ren}^{(w)} = \frac{q^2-1-3q^2\csc^2(q\phi)}{48\pi^2r^2}. \tag{23}$$

Near the wedge boundaries $\phi=\phi_m$, $m=0,1$ ($\phi_1=0$) the main contribution in (22) comes from the terms $j=2$, $l=0$ and $l=q-1$ for $m=0$ and $m=1$ respectively, and the renormalized VEV of the field squared diverges with the leading behaviour $\langle\varphi^2\rangle_{\rm ren}^{(w)} \propto |\phi-\phi_m|^{1-D}$. The surface divergences in the VEVs of the local physical observables are well known in quantum field theory with boundaries and result from the idealization of the boundaries as perfectly smooth surfaces which are perfect reflectors at all frequencies. These divergences are investigated in detail for various types of fields and general shape of smooth boundary [20, 34]. Near the smooth boundary the leading divergence in the field squared varies as $(D-1)$th power of the distance from the boundary. It seems plausible that such effects as surface roughness, or the microstructure of the boundary on small scales (the atomic nature of matter for the case of the electromagnetic field [35]) can introduce a physical cutoff needed to produce finite values of surface quantities.

Now we turn to the VEVs of the energy-momentum tensor. By making use of formula (3), for the non-zero components one obtains (no summation over i)

$$\langle T_i^i\rangle_{\rm ren}^{(w)} = -\frac{\Gamma\left(\frac{D+1}{2}\right)}{2^{D+2}\pi^{\frac{D+1}{2}}r^{D+1}}\sum_{j=1}^{2}{\sum_{l=0}^{q-1}}{}' \frac{(-1)^{j+1}f_{jl}^{(i)}}{\sin^{D+1}\phi_l^{(j)}},\ \langle T_2^1\rangle_{\rm ren}^{(w)} = \frac{D(\xi_c-\xi)\Gamma\left(\frac{D+1}{2}\right)}{2^D\pi^{\frac{D+1}{2}}r^D}\sum_{l=0}^{q-1}\frac{\cos\phi_l^{(2)}}{\sin^D\phi_l^{(2)}}, \tag{24}$$

where $i=0,1,\ldots,D$, and we use the following notations

$$f_{jl}^{(i)} = 1+(4\xi-1)\left[(D-1)\delta_{j1}\sin^2\phi_l^{(j)}+D\delta_{j2}\right],\quad i=0,3,\ldots,D, \tag{25}$$

$$f_{jl}^{(1)} = f_{jl}^{(0)}-4D(\xi-\xi_c)\sin^2\phi_l^{(j)},\ f_{jl}^{(2)} = D\left[4\sin^2\phi_l^{(j)}(\xi-\xi_c\delta_{j2})-\delta_{j1}\right]. \tag{26}$$

In the case $\phi_0 = \pi/2$ and for minimally and conformally coupled scalar fields, it can be checked that from formulae (24), after the transformation from cylindrical coordinates to the cartesian ones, as a special case we obtain the result derived in [36]. For a conformally coupled scalar field $f_{2l}^{(i)} = 0$ and from (24) one finds

$$\langle T_i^k\rangle_{\rm ren}^{(w)} = -\frac{\Gamma\left(\frac{D+1}{2}\right)}{2^{D+2}\pi^{\frac{D+1}{2}}r^{D+1}}\sum_{l=1}^{q-1}\frac{D-(D-1)\sin^2(\pi l/q)}{D\sin^{D+1}(\pi l/q)}\mathrm{diag}(1,1,-D,1,\ldots,1). \tag{27}$$

In this case the vacuum energy-momentum tensor does not depend on the angular coordinate. For a non-conformally coupled field the VEVs (24) diverge on the boundaries $\phi=\phi_m$ and for points away from the edge $r=0$, these divergences are the same as those for the geometry of a single plate.

In the most important case $D=3$, for the components of the renormalized energy-momentum tensor we find

$$\begin{aligned}\langle T_0^0\rangle_{\rm ren}^{(w)} &= \langle T_3^3\rangle_{\rm ren}^{(w)} = \frac{1}{32\pi^2r^4}\left\{\frac{1-q^4}{45}+\frac{8}{3}\left(1-q^2\right)(\xi-\xi_c)\right.\\ &\left.+12\frac{(\xi-\xi_c)q^2}{\sin^2(q\phi)}\left[\frac{q^2}{\sin^2(q\phi)}-\frac{2}{3}q^2+\frac{2}{3}\right]\right\},\end{aligned} \tag{28}$$

$$\begin{aligned}\langle T_1^1\rangle_{\rm ren}^{(w)} &= \frac{1}{32\pi^2r^4}\left\{\frac{1-q^4}{45}-\frac{4}{3}(1-q^2)(\xi-\xi_c)\right.\\ &\left.+12\frac{(\xi-\xi_c)q^2}{\sin^2(q\phi)}\left[\frac{q^2}{\sin^2(q\phi)}-\frac{2}{3}q^2-\frac{1}{3}\right]\right\},\end{aligned} \tag{29}$$

$$\langle T_2^1\rangle_{\rm ren}^{(w)} = -\frac{3(\xi-\xi_c)}{8\pi^2r^3}\frac{q^3\cos(q\phi)}{\sin^3(q\phi)}, \tag{30}$$

$$\langle T_2^2\rangle_{\rm ren}^{(w)} = \frac{1}{8\pi^2r^4}\left[\frac{q^4-1}{60}+(\xi-\xi_c)\left(1-q^2+\frac{3q^2}{\sin^2(q\phi)}\right)\right]. \tag{31}$$

Though we have derived these formulae for integer values of the parameter q, by the analytic continuation they are valid for non-integer values of this parameter as well. For a conformally coupled scalar field we obtain the result previously derived in the literature [19, 20]. The corresponding vacuum forces acting on the wedge sides are determined by the effective pressure $-\langle T_2^2\rangle_{\rm ren}^{(w)}$. These forces are attractive for the wedge with $q>1$ and are repulsive for $q<1$.

4 Field squared and the energy-momentum tensor

We now turn to the geometry of a wedge with additional cylindrical boundary of radius a. Taking the coincidence limit $x' \to x$ in formula (11) for the Wightman function and integrating over $\mathbf{k}$, the VEV of the field squared is presented as the sum of two terms:

$$\langle 0|\varphi^2|0\rangle = \langle 0_w|\varphi^2|0_w\rangle + \langle \varphi^2\rangle_{\rm cyl}, \tag{32}$$

where the part induced by the cylindrical boundary is given by the formula

$$\langle \varphi^2\rangle_{\rm cyl} = -\frac{2^{3-D}\pi^{\frac{1-D}{2}}}{\Gamma\left(\frac{D-1}{2}\right)\phi_0}\sum_{n=1}^{\infty}\sin^2(qn\phi)\int_m^{\infty} dz\, z\left(z^2-m^2\right)^{\frac{D-3}{2}}\frac{K_{qn}(az)}{I_{qn}(az)}I_{qn}^2(rz). \tag{33}$$

Note that this part vanishes at the wedge sides $\phi = \phi_m$, $0 \leqslant r < a$. Near the edge $r = 0$ the main contribution into $\langle \varphi^2\rangle_{\rm cyl}$ comes from the term $n = 1$ and $\langle \varphi^2\rangle_{\rm cyl}$ behaves like r^{2q}. The part $\langle \varphi^2\rangle_{\rm cyl}$ diverges on the cylindrical surface $r = a$. Near this surface the main contribution into (33) comes from large values n and for $|\phi - \phi_m| \gg 1 - r/a$ the leading behavior is the same as that for a cylindrical surface of radius a.

Similarly, the VEV of the energy-momentum tensor for the situation when the cylindrical boundary is present is written in the form

$$\langle 0|T_{ik}|0\rangle = \langle 0_w|T_{ik}|0_w\rangle + \langle T_{ik}\rangle_{\rm cyl}, \tag{34}$$

where $\langle T_{ik}\rangle_{\rm cyl}$ is induced by the cylindrical boundary. This term is obtained from the corresponding part in the Wightman function, $\langle \varphi(x)\varphi(x')\rangle_{\rm cyl}$, by using formula (3). For points away from the cylindrical surface this limit gives a finite result. For the corresponding components of the energy-momentum tensor one obtains (no summation over i)

$$\begin{aligned}
\langle T_i^i\rangle_{\rm cyl} &= \frac{(4\pi)^{-\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)\phi_0}\sum_{n=1}^{\infty}\int_m^{\infty} dz\, z^3\left(z^2-m^2\right)^{\frac{D-3}{2}}\frac{K_{qn}(az)}{I_{qn}(az)} \\
&\quad \times\left\{a_{i,qn}^{(+)}[I_{qn}(rz)] - a_{i,qn}^{(-)}[I_{qn}(rz)]\cos(2qn\phi)\right\}, \qquad (35)\\
\langle T_2^1\rangle_{\rm cyl} &= \frac{2(4\pi)^{-\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)\phi_0}\sum_{n=1}^{\infty} qn\sin(2qn\phi)\int_m^{\infty} dz\, z^2(z^2-m^2)^{\frac{D-3}{2}}\frac{K_{qn}(az)}{I_{qn}(az)} \\
&\quad \times I_{qn}(rz)\left[\frac{2\xi}{rz}I_{qn}(rz) + (1-4\xi)I'_{qn}(rz)\right], \qquad (36)
\end{aligned}$$

with the notations

$$\begin{aligned}
a_{i,l}^{(\pm)}[g(y)] &= (4\xi-1)\left[g'^2(y) + \left(1 \pm l^2/y^2\right)g^2(y)\right] + 2g^2(y)\frac{1-m^2r^2/y^2}{D-1}, \\
a_{1,l}^{(\pm)}[g(y)] &= g'^2(y) + (4\xi/y)g(y)g'(y) - g^2(y)\left\{1 \pm [1-4\xi(1\mp 1)]\,l^2/y^2\right\}, \\
a_{2,l}^{(\pm)}[g(y)] &= (4\xi-1)\left[g'^2(y) + g^2(y)\right] - (4\xi/y)g(y)g'(y) + g^2(y)\,(4\xi \pm 1)\,l^2/y^2,
\end{aligned} \tag{37}$$

for a given function $g(y)$, $i = 0, 3, \ldots, D$. In accordance with the problem symmetry, the expressions for the diagonal components are invariant under the replacement $\phi \to \phi_0 - \phi$, and the off-diagonal component $\langle T_2^1\rangle_{\rm cyl}$ changes the sign under this replacement. Note that the latter vanishes on the wedge sides $\phi = \phi_m$, $0 \leqslant r < a$ and for $\phi = \phi_0/2$. On the wedge sides for the diagonal components of the energy-momentum tensor we obtain (no summation over i)

$$\langle T_i^i\rangle_{\mathrm{cyl},\phi=\phi_m} = \frac{2^{2-D}\pi^{\frac{5-D}{2}}A_i}{\Gamma\left(\frac{D-1}{2}\right)r^2\phi_0^3}\sum_{n=1}^{\infty}n^2\int_m^{\infty}dz\,z\left(z^2-m^2\right)^{\frac{D-3}{2}}\frac{K_{qn}(az)}{I_{qn}(az)}I_{qn}^2(rz), \tag{38}$$

where $A_i = 4\xi - 1$, $i = 0, 1, 3, \ldots, D$, $A_2 = 1$. In particular, the additional vacuum effective pressure in the direction perpendicular to the wedge sides, $p_a = -\langle T_2^2\rangle_{\mathrm{cyl},\phi=\phi_m}$, does not depend on the curvature coupling parameter and is negative for all values $0 < r < a$. This means that the vacuum forces acting on the wedge sides due to the presence of the cylindrical boundary are attractive. The corresponding vacuum stresses in the directions parallel to the wedge sides are isotropic and the energy density is negative for both minimally and conformally coupled scalars.

For $0 < r < a$ the cylindrical parts (35) and (36) are finite for all values $0 \leqslant \phi \leqslant \phi_0$, including the wedge sides. The divergences on these sides are included in the first term on the right-hand side of (34) corresponding to the case without cylindrical boundary. Near the edge $r = 0$ the main contribution into the boundary parts comes from the summand with $n = 1$ and one has $\langle T_i^i\rangle_{\mathrm{cyl}} \propto r^{2q-2}$, $\langle T_2^1\rangle_{\mathrm{cyl}} \propto r^{2q-1}$. The boundary part $\left\langle T_i^k\right\rangle_{\mathrm{cyl}}$ diverges on the cylindrical surface $r = a$. Expanding over $a - r$, on the wedge sides for the diagonal components one finds

$$\langle T_i^i\rangle_{\mathrm{cyl},\phi=\phi_m} \approx \frac{A_i\Gamma\left(\frac{D+1}{2}\right)}{2(4\pi)^{\frac{D+1}{2}}(a-r)^{D+1}}, \quad r \to a, \tag{39}$$

where the coefficients A_i are defined in the paragraph after formula (38). It can be seen that for the off-diagonal component to the leading order one has $\langle T_2^1\rangle_a \propto (a-r)^{-D}$. For angles $0 < \phi < \phi_0$, and for $|\phi - \phi_m| \gg 1 - r/a$, the leading divergence coincides with the corresponding one for a cylindrical surface of the radius a.

Taking the coincidence limit of the arguments, from formula (17) we obtain the VEV of the field squared in the region $r > a$:

$$\langle\varphi^2\rangle_{\mathrm{cyl}} = -\frac{2^{3-D}\pi^{\frac{1-D}{2}}}{\Gamma\left(\frac{D-1}{2}\right)\phi_0}\sum_{n=1}^{\infty}\sin^2(qn\phi)\int_m^{\infty}dz\,z\left(z^2-m^2\right)^{\frac{D-3}{2}}\frac{I_{qn}(az)}{K_{qn}(az)}K_{qn}^2(rz). \tag{40}$$

As for the interior region, the VEV (40) diverges on the cylindrical surface. For large distances from the cylindrical surface, $r \gg a$, and for a massless field the main contribution comes from the $n = 1$ term and to the leading order one finds $\langle\varphi^2\rangle_{\mathrm{cyl}} \propto (a/r)^{D-1+2q}$. For a massive field and for $mr \gg 1$ the part $\langle\varphi^2\rangle_{\mathrm{cyl}}$ is exponentially suppressed.

For the part in the vacuum energy-momentum tensor induced by the cylindrical surface in the region $r > a$, from (3), (17), (40) one has the following formulae

$$\begin{aligned}
\langle T_i^i\rangle_{\mathrm{cyl}} &= \frac{(4\pi)^{-\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)\phi_0}\sum_{n=1}^{\infty}\int_m^{\infty}dz\,z^3\left(z^2-m^2\right)^{\frac{D-3}{2}}\frac{I_{qn}(az)}{K_{qn}(az)}\\
&\quad\times\left\{a_{i,qn}^{(+)}[K_{qn}(rz)] - a_{i,qn}^{(-)}[K_{qn}(rz)]\cos(2qn\phi)\right\},
\end{aligned} \tag{41}$$

$$\begin{aligned}
\langle T_2^1\rangle_{\mathrm{cyl}} &= \frac{2(4\pi)^{-\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)\phi_0}\sum_{n=1}^{\infty}qn\sin(2qn\phi)\int_m^{\infty}dz\;z^2(z^2-m^2)^{\frac{D-3}{2}}\frac{I_{qn}(az)}{K_{qn}(az)}\\
&\quad\times K_{qn}(rz)\left[\frac{2\xi}{rz}K_{qn}(rz) + (1-4\xi)K_{qn}'(rz)\right],
\end{aligned} \tag{42}$$

with the functions $a_{i,qn}^{(\pm)}[g(y)]$ defined by (37). In the way similar to that used above for the VEV of the field square, it can be seen that at large distances from the cylindrical surface, $r \gg a$, the main contribution comes from the term with $n = 1$ and for a massless field the components of the induced energy-momentum tensor behave as $\langle T_i^i \rangle_{\rm cyl} \propto (a/r)^{D+1+2q}$, $\langle T_2^1 \rangle_{\rm cyl} \propto (a/r)^{D+2q}$. As for the interior region, the vacuum forces acting on the wedge sides due to the presence of the cylindrical shell are attractive and the corresponding energy density is negative for both minimally and conformally coupled scalars.

In the limit $\phi_0 \to 0$, $r, a \to \infty$, assuming that $a - r$ and $a\phi_0 \equiv b$ are fixed, from the results given above we obtain the vacuum densities for the geometry of two parallel plates separated by a distance b, perpendicularly intersected by the third plate. The vacuum expectation values of the energy-momentum tensor for this geometry of boundaries are investigated in [36] for special cases of minimally and conformally coupled massless scalar fields.

5 VEVs for the electromagnetic fields

5.1 Interior region

In this section we consider a wedge with a coaxial cylindrical boundary assuming that all boundaries are perfectly conducting. For this geometry there are two different types of the eigenfunctions corresponding to the transverse magnetic (TM, $\lambda = 0$) and transverse electric (TE, $\lambda = 1$) waves. In the Coulomb gauge, the vector potentials for these modes are given by the formulae

$$\mathbf{A}_\alpha = \beta_\alpha \left\{ \begin{array}{cc} (1/i\omega) \left(\gamma^2 \mathbf{e}_3 + ik\nabla_t \right) J_{qn}(\gamma r) \sin(qn\phi) \exp\left[i\left(kz - \omega t \right) \right], & \lambda = 0 \\ -\mathbf{e}_3 \times \nabla_t \left\{ J_{qn}(\gamma r) \cos(qn\phi) \exp\left[i\left(kz - \omega t \right) \right] \right\}, & \lambda = 1 \end{array} \right. , \tag{43}$$

where $\mathbf{e}_3$ is the unit vector along the axis of the wedge, ∇_t is the part of the nabla operator transverse to this axis, and $\omega^2 = \gamma^2 + k^2$. In Eq. (43), $n = 1, 2, \ldots$ for $\lambda = 0$ and $n = 0, 1, 2, \ldots$ for $\lambda = 1$. From the normalization condition one finds

$$\beta_\alpha^2 = \frac{4qT_{qn}(\gamma a)}{\pi\omega a\gamma}\delta_n, \; \delta_n = \left\{ \begin{array}{cc} 1/2, & n = 0 \\ 1, & n \neq 0 \end{array} \right. , \tag{44}$$

where we have introduced the notation $T_\nu(x) = x\left[J_\nu'^2(x) + (1 - \nu^2/x^2) J_\nu^2(x) \right]^{-1}$. Eigenfunctions (43) satisfy the standard boundary conditions on the wedge sides. From the boundary conditions on the cylindrical shell it follows that the eigenvalues for γ are roots of the equation

$$J_{qn}^{(\lambda)}(\gamma a) = 0, \quad \lambda = 0, 1, \tag{45}$$

where $J_\nu^{(0)}(x) = J_\nu(x)$ and $J_\nu^{(1)}(x) = J_\nu'(x)$. We will denote the corresponding eigenmodes by $\gamma a = \lambda_{n,j}^{(\lambda)}$, $j = 1, 2, \ldots$.

First we consider the VEVs of the squares of the electric and magnetic fields inside the shell. Substituting the eigenfunctions (43) into the corresponding mode-sum formula, we find

$$\langle 0|F^2|0\rangle = \frac{4q}{\pi a^3} \sum_{m=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \sum_{\lambda=0,1} \sum_{n=1}^{\infty} \frac{\lambda_{n,j}^{(\lambda)3} T_{qm}(\lambda_{n,j}^{(\lambda)})}{\sqrt{\lambda_{n,j}^{(\lambda)2} + k^2 a^2}} g^{(\eta_{F\lambda})}[\Phi_{qn}^{(\lambda)}(\phi), J_{qn}(\lambda_{n,j}^{(\lambda)} r/a)], \tag{46}$$

where $F = E, B$ with $\eta_{E\lambda} = \lambda$, $\eta_{B\lambda} = 1 - \lambda$, and the prime in the summation over n means that the term $n = 0$ should be halved. In formula (46) we have introduced the notations

$$\begin{aligned} g^{(0)}[\Phi(\phi), f(x)] &= (k^2 r^2/x^2) \left[\Phi^2(\phi) f'^2(x) + \Phi'^2(\phi) f^2(x)/x^2 \right] + \Phi^2(\phi) f^2(x), \\ g^{(1)}[\Phi(\phi), f(x)] &= (1 + k^2 r^2/x^2) \left[\Phi^2(\phi) f'^2(x) + \Phi'^2(\phi) f^2(x)/x^2 \right], \end{aligned} \tag{47}$$

and

$$\Phi_\nu^{(0)}(\phi) = \sin(\nu\phi),\ \Phi_\nu^{(1)}(\phi) = \cos(\nu\phi). \tag{48}$$

The expressions (46) corresponding to the electric and magnetic fields are divergent. They may be regularized introducing a cutoff function $\psi_\mu(\omega)$ with the cutting parameter μ which makes the divergent expressions finite and satisfies the condition $\psi_\mu(\omega) \to 1$ for $\mu \to 0$. After the renormalization the cutoff function is removed by taking the limit $\mu \to 0$.

In order to further simplify the VEVs, we apply to the series over n the summation formula (10) for the modes with $\lambda = 0$ and the similar formula from [33] for the modes with $\lambda = 1$. As it can be seen, for points away from the shell the contribution to the VEVs coming from the second integral terms on the right-hand sides of these formulae are finite in the limit $\mu \to 0$ and, hence, the cutoff function in these terms can be safely removed. As a result the VEVs are written in the form

$$\langle 0|F^2|0\rangle = \langle 0_w|F^2|0_w\rangle + \left\langle F^2\right\rangle_{\rm cyl}, \tag{49}$$

where

$$\begin{aligned}\langle 0_w|F^2|0_w\rangle &= \frac{q}{\pi}\sum_{n=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \int_0^\infty d\gamma\, \frac{\gamma^3\psi_\mu(\omega)}{\sqrt{\gamma^2+k^2}} \left\{ \left(1+\frac{2k^2}{\gamma^2}\right) \left[J_{qn}'^2(\gamma r) + \frac{q^2n^2}{\gamma^2 r^2} J_{qn}^2(\gamma r)\right]\right. \\ &\left. + J_{qn}^2(\gamma r) - (-1)^{\eta_{F1}} \cos(2qn\phi) \left[J_{qn}'^2(\gamma r) - \left(1+\frac{q^2n^2}{\gamma^2r^2}\right) J_{qn}^2(\gamma r)\right]\right\}, \end{aligned} \tag{50}$$

and

$$\langle F^2\rangle_{\rm cyl} = \frac{2q}{\pi}\sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1} \int_0^\infty dx\, x^3 \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} G^{(\eta_{F\lambda})}[\Phi_{qn}^{(\lambda)}(\phi), I_{qn}(xr)]. \tag{51}$$

In formula (51) we have introduced the notations

$$\begin{aligned} G^{(0)}[\Phi(\phi), f(x)] &= \Phi^2(\phi) f'^2(x) + \Phi'^2(\phi) f^2(x)/x^2 + 2\Phi^2(\phi) f^2(x), \\ G^{(1)}[\Phi(\phi), f(x)] &= -\Phi^2(\phi) f'^2(x) - \Phi'^2(\phi) f^2(x)/x^2. \end{aligned} \tag{52}$$

The second term on the right-hand side of Eq. (49) vanishes in the limit $a \to \infty$ and the first one does not depend on a. Thus, we can conclude that the term $\langle 0_w|F^2|0_w\rangle$ corresponds to the part in the VEVs when the cylindrical shell is absent.

First, let us concentrate on the part corresponding to the wedge without a cylindrical shell. In (50) the part which does not depend on the angular coordinate ϕ is the same as in the corresponding problem of the cosmic string geometry with the angle deficit $2\pi - \phi_0$ (see [32]), which we will denote by $\langle 0_{\rm s}|F^2|0_{\rm s}\rangle$. For this part we have

$$\langle 0_{\rm s}|F^2|0_{\rm s}\rangle = \langle 0_{\rm M}|F^2|0_{\rm M}\rangle - \frac{(q^2-1)(q^2+11)}{180\pi r^4}, \tag{53}$$

where $\langle 0_{\rm M}|F^2|0_{\rm M}\rangle$ is the VEV in the Minkowski spacetime without boundaries and in the last expression we have removed the cutoff. To evaluate the part in (50) which depends on ϕ, we firstly consider the case when the parameter q is an integer. In this case, the summation over n can be done explicitly and the integrals are evaluated by introducing polar coordinates in the (k,γ)-plane. As a result, for the renormalised VEVs of the field squared in the geometry of a wedge without a cylindrical boundary we find

$$\langle F^2\rangle_{\rm ren}^{(w)} = -\frac{(q^2-1)(q^2+11)}{180\pi r^4} - \frac{(-1)^{\eta_{F1}} q^2}{2\pi r^4 \sin^2(q\phi)} \left[1 - q^2 + \frac{3q^2}{2\sin^2(q\phi)}\right], \tag{54}$$

with $\eta_{E1} = 1$ and $\eta_{B1} = 0$. Though we have derived this formula for integer values of the parameter q, by the analytic continuation it is valid for non-integer values of this parameter as well. The expression on the right of formula (54) is invariant under the replacement $\phi \to \phi_0 - \phi$ and, as we could expect, the VEVs are symmetric with respect to the half-plane $\phi = \phi_0/2$. Formula (54) for $F = E$ was derived in Ref. [22] within the framework of Schwinger's source theory.

Now, we turn to the investigation of the parts in the VEVs of the field squared induced by the cylindrical boundary and given by formula (51). These parts are symmetric with respect to the half-plane $\phi = \phi_0/2$. The expression in the right-hand side of (51) is finite for $0 < r < a$ including the points on the wedge sides, and diverges on the shell. To find the leading term in the corresponding asymptotic expansion, we note that near the shell the main contribution comes from large values of n. By using the uniform asymptotic expansions of the modified Bessel functions for large values of the order, up to the leading order, for the points $a - r \ll a|\sin\phi|, a|\sin(\phi_0 - \phi)|$ we find $\langle F^2\rangle_{\rm cyl} \approx -3(-1)^{\eta_{F1}}/[4\pi(a-r)^4]$. These surface divergences originate in the unphysical nature of perfect conductor boundary conditions. In reality the expectation values will attain a limiting value on the conductor surface, which will depend on the molecular details of the conductor. From the formulae given above it follows that the main contribution to $\langle F^2\rangle_{\rm cyl}$ are due to the frequencies $\omega \lesssim (a-r)^{-1}$. Hence, we expect that formula (51) is valid for real conductors up to distances r for which $(a-r)^{-1} \ll \omega_0$, with ω_0 being the characteristic frequency, such that for $\omega > \omega_0$ the conditions for perfect conductivity fail.

Near the edge $r = 0$, assuming that $r/a \ll 1$, the asymptotic behavior of the part induced in the VEVs of the field squared by the cylindrical shell depends on the parameter q. For $q > 1 + \eta_{F1}$, the dominant contribution comes from the lowest mode $n = 0$ and to the leading order one has $\langle F^2\rangle_{\rm cyl} \propto r^{2\eta_{F1}}$. In this case the quantity $\langle B^2\rangle_{\rm cyl}$ takes a finite limiting value on the edge $r = 0$, whereas $\langle E^2\rangle_{\rm cyl}$ vanishes as r^2. For $q < 1 + \eta_{F1}$ the main contribution comes from the mode with $n = 1$ and the shell-induced parts diverge on the edge $r = 0$ with $\langle F^2\rangle_{\rm cyl} \propto r^{2(q-1)}$. In accordance with (54), near the edge $r = 0$ the total VEV is dominated by the part coming from the wedge without the cylindrical shell.

5.2 Exterior region

In the exterior region (region II in figure 1), the corresponding eigenfunctions for the vector potential are obtained from formulae (43) by the replacement

$$J_{qn}(\gamma r) \to g_{qn}^{(\lambda)}(\gamma a, \gamma r) = J_{qn}(\gamma r) Y_{qn}^{(\lambda)}(\gamma a) - Y_{qn}(\gamma r) J_{qn}^{(\lambda)}(\gamma a), \tag{55}$$

where, as before, $\lambda = 0, 1$ correspond to the waves of the electric and magnetic types, respectively. The eigenvalues for γ are continuous and

$$\beta_\alpha^{-2} = (8\pi/q)\delta_n \gamma\omega \left[J_{qn}^{(\lambda)2}(\gamma a) + Y_{qn}^{(\lambda)2}(\gamma a)\right]. \tag{56}$$

Substituting the eigenfunctions into the corresponding mode-sum formula, for the VEV of the field squared one finds

$$\langle 0|F^2|0\rangle = \frac{2q}{\pi} \sum_{n=0}^{\infty}{}' \int_{-\infty}^{+\infty} dk \int_0^\infty d\gamma \sum_{\lambda=0,1} \frac{\gamma^3}{\sqrt{k^2+\gamma^2}} \frac{g^{(\eta_{F\lambda})}[\Phi_{qn}^{(\lambda)}(\phi), g_{qn}^{(\lambda)}(\gamma a, \gamma r)]}{J_{qn}^{(\lambda)2}(\gamma a) + Y_{qn}^{(\lambda)2}(\gamma a)}, \tag{57}$$

where the functions $g^{(\eta_{F\lambda})}[\Phi(\phi), f(x)]$ are defined by relations (47) with $f(x) = g_{qn}^{(\lambda)}(\gamma a, x)$. To extract from this VEV the part induced by the cylindrical shell, we subtract from the right-hand side the corresponding expression for the wedge without the cylindrical boundary. As a result, the VEV of the field squared is written in the form (49), where the part induced by the cylindrical shell is given by the formula

$$\langle F^2\rangle_{\rm cyl} = \frac{2q}{\pi}\sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^{\infty} dx\, x^3 \frac{I_{qn}^{(\lambda)}(xa)}{K_{qn}^{(\lambda)}(xa)} G^{(\eta_{F\lambda})}[\Phi_{qn}^{(\lambda)}(\phi), K_{qn}(xr)]. \tag{58}$$

In this formula the functions $G^{(\eta_{F\lambda})}[\Phi(\phi), f(x)]$ are defined by expressions (52). Comparing this result with formula (51), we see that the expressions for the shell-induced parts in the interior and exterior regions are related by the interchange $I_{qn} \rightleftarrows K_{qn}$.

The VEV (58) diverges on the cylindrical shell with the leading term being the same as that for the interior region. At large distances from the cylindrical shell we introduce a new integration variable $y = xr$ and expand the integrand over a/r. For $q > 1$ the main contribution comes from the lowest mode $n = 0$ and up to the leading order we have

$$\langle E^2\rangle_{\rm cyl} \approx \frac{4q\,(a/r)^2}{5\pi r^4},\ \langle B^2\rangle_{\rm cyl} \approx -\frac{28q\,(a/r)^2}{15\pi r^4}. \tag{59}$$

For $q < 1$ the dominant contribution into the VEVs at large distances is due to the mode $n = 1$ with the leading term

$$\langle F^2\rangle_{\rm cyl} \approx -\frac{4q^2(q+1)}{\pi r^4}\left(\frac{a}{r}\right)^{2q}\left[\frac{\cos(2q\phi)}{2q+3} + (-1)^{\eta_{F1}}\frac{q+1}{2q+1}\right]. \tag{60}$$

For the case $q = 1$ the contributions of the modes $n = 0$ and $n = 1$ are of the same order and the corresponding leading terms are obtained by summing these contributions. The latter are given by the right-hand sides of formulae (59) and (60). As we see, at large distances the part induced by the cylindrical shell is suppressed with respect to the part corresponding to the wedge without the shell by the factor $(a/r)^{2\beta}$ with $\beta = \min(1, q)$.

6 Energy-momentum tensor for the electromagnetic field

Now let us consider the VEV of the energy-momentum tensor in the region inside the cylindrical shell. Substituting the eigenfunctions (43) into the corresponding mode-sum formula, for the non-zero components we obtain (no summation over i)

$$\begin{aligned}
\langle 0|T_i^i|0\rangle &= \frac{q}{2\pi^2 a^3}\sum_{n=0}^{\infty}{}'\int_{-\infty}^{+\infty} dk \sum_{\lambda=0,1}\sum_{j=1}^{\infty}\frac{\lambda_{n,j}^{(\lambda)3}T_{qn}(\lambda_{n,j}^{(\lambda)})}{\sqrt{\lambda_{n,j}^{(\lambda)2}+k^2a^2}} f^{(i)}[\Phi_{qn}^{(\lambda)}(\phi), J_{qn}(\lambda_{n,j}^{(\lambda)}r/a)], &(61)\\
\langle 0|T_2^1|0\rangle &= \frac{-q^2}{4\pi^2 a}\frac{\partial}{\partial r}\sum_{n=0}^{\infty}{}' n\sin(2qn\phi)\int_{-\infty}^{+\infty} dk\sum_{\lambda=0,1}(-1)^{\lambda}\\
&\quad\times\sum_{j=1}^{\infty}\frac{\lambda_{n,j}^{(\lambda)}T_{qn}(\lambda_{n,j}^{(\lambda)})}{\sqrt{\lambda_{n,j}^{(\lambda)2}+k^2a^2}} J_{qn}^2(\lambda_{n,j}^{(\lambda)}r/a), &(62)
\end{aligned}$$

where $i = 0, 1, 2, 3$, and we have introduced the notations

$$\begin{aligned}
f^{(j)}[\Phi(\phi), f(x)] &= (-1)^i\left(2k^2/\gamma^2+1\right)\left[\Phi^2(\phi)f'^2(x) + \Phi'^2(\phi)f^2(x)/y^2\right] + \Phi^2(\phi)f^2(x),\\
f^{(l)}[\Phi(\phi), f(x)] &= (-1)^l\Phi^2(\phi)f'^2(x) - \left[\Phi^2(\phi) + (-1)^l\Phi'^2(\phi)/x^2\right]f^2(x), &(63)
\end{aligned}$$

with $j = 0, 3$ and $l = 1, 2$. As in the case of the field squared, in formulae (61) and (62) we introduce a cutoff function and apply formula (10) for the summation over n. This enables us to present the vacuum energy-momentum tensor in the form of the sum

$$\langle 0|T_i^k|0\rangle = \langle 0_w|T_i^k|0_w\rangle + \langle T_i^k\rangle_{\rm cyl}, \tag{64}$$

where $\langle 0_w|T_i^k|0_w\rangle$ is the part corresponding to the geometry of a wedge without a cylindrical boundary and $\langle T_i^k\rangle_{\rm cyl}$ is induced by the cylindrical shell. The latter may be written in the form (no summation over i)

$$\langle T_i^i\rangle_{\rm cyl} = \frac{q}{2\pi^2}\sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^\infty dx x^3 \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} F^{(i)}[\Phi_{qn}^{(\lambda)}(\phi), I_{qn}(xr)], \tag{65}$$

$$\langle T_2^1\rangle_{\rm cyl} = \frac{q^2}{4\pi^2}\frac{\partial}{\partial r}\sum_{n=0}^{\infty}{}' n\sin(2qn\phi)\sum_{\lambda=0,1}(-1)^\lambda \int_0^\infty dx x \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} I_{qn}^2(xr), \tag{66}$$

with the notations

$$\begin{aligned} F^{(i)}[\Phi(\phi), f(y)] &= \Phi^2(\phi) f^2(y),\ i = 0, 3, \\ F^{(i)}[\Phi(\phi), f(y)] &= -(-1)^i \Phi^2(\phi) f'^2(y) - \left[\Phi^2(\phi) - (-1)^i \Phi'^2(\phi)/y^2\right] f^2(y),\ i = 1, 2. \end{aligned} \tag{67}$$

The diagonal components are symmetric with respect to the half-plane $\phi = \phi_0/2$, whereas the off-diagonal component is an odd function under the replacement $\phi \to \phi_0 - \phi$. As it can be easily checked, the tensor $\langle T_i^k\rangle_{\rm cyl}$ is traceless and satisfies the covariant continuity equation. The off-diagonal component $\langle T_2^1\rangle_{\rm cyl}$ vanishes at the wedge sides and for these points the VEV of the energy-momentum tensor is diagonal. The vacuum energy density induced by the cylindrical shell in the interior region is always negative.

The renormalized VEV of the energy density for the geometry without the cylindrical shell is obtained by using the corresponding formulae for the field squared. Other components are found from the tracelessness condition and the continuity equation and one has [2, 19, 20]

$$\langle T_i^k\rangle_{\rm ren}^{(w)} = -\frac{(q^2-1)(q^2+11)}{720\pi^2 r^4}{\rm diag}(1, 1, -3, 1). \tag{68}$$

Formula (68) coincides with the corresponding result for the geometry of the cosmic string with the angle deficit $2\pi - \phi_0$ and in the corresponding formula $q = 2\pi/\phi_0$.

The normal force acting on the wedge sides is determined by the component $\langle T_2^2\rangle_{\rm ren}$ of the vacuum energy-momentum tensor evaluated for $\phi = 0$ and $\phi = \phi_0$. On the base of formula (64) for the corresponding effective pressure one has

$$p_2 = -\langle T_2^2\rangle_{\rm ren}|_{\phi=0,\phi_0} = p_{2w} + p_{2\rm cyl}, \tag{69}$$

where $p_{2w} = -\langle T_2^2\rangle_{\rm ren}^{(w)}$ is the normal force acting per unit surface of the wedge for the case without a cylindrical boundary and the additional term

$$p_{2\rm cyl} = -\langle T_2^2\rangle_{\rm cyl}|_{\phi=0,\phi_0} = -\frac{q}{\pi^2}\sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^\infty dx x^3 \frac{K_{qn}^{(\lambda)}(xa)}{I_{qn}^{(\lambda)}(xa)} F_{qn}^{(\lambda)}[I_{qn}(xr)], \tag{70}$$

with the notations

$$F_\nu^{(0)}[f(y)] = \nu^2 f^2(y)/y^2,\ F_\nu^{(1)}[f(y)] = -f'^2(y) - f^2(y), \tag{71}$$

is induced by the cylindrical shell. Note that the normal force on the wedge sides is the sum of the corresponding forces for Dirichlet and Neumann scalars corresponding to the terms with $\lambda = 0$ and $\lambda = 1$ respectively. The finiteness of the normal stress on the wedge sides is a consequence of the fact that for a single perfectly conducting plane boundary this stress vanishes. Note that this result can be directly obtained from the symmetry of the corresponding problem with combination of the continuity equation for the energy-momentum tensor. It also survives for more realistic models of the plane boundary (see, for instance, [37]) though the corresponding energy density and parallel stresses no longer vanish. So we expect that the obtained formula for the normal force acting on the wedge sides will correctly approximate the corresponding results of more realistic models in the perfectly conducting limit. The corresponding vacuum forces are attractive for $q > 1$ and repulsive for $q < 1$. In particular, the equilibrium position corresponding to the geometry of a single plate ($q = 1$) is unstable. As regards to the part induced by the cylindrical shell, from (70) it follows that $p_{2\mathrm{cyl}} < 0$ and, hence, the corresponding forces are always attractive. In figure 2 we have plotted the vacuum pressure on the wedge sides induced by the cylindrical boundary versus r/a for Dirichlet scalar (left panel) and for the electromagnetic field (right panel). The full (dashed) curves correspond to the wedge with $\phi_0 = \pi/2$ ($\phi_0 = 3\pi/2$).

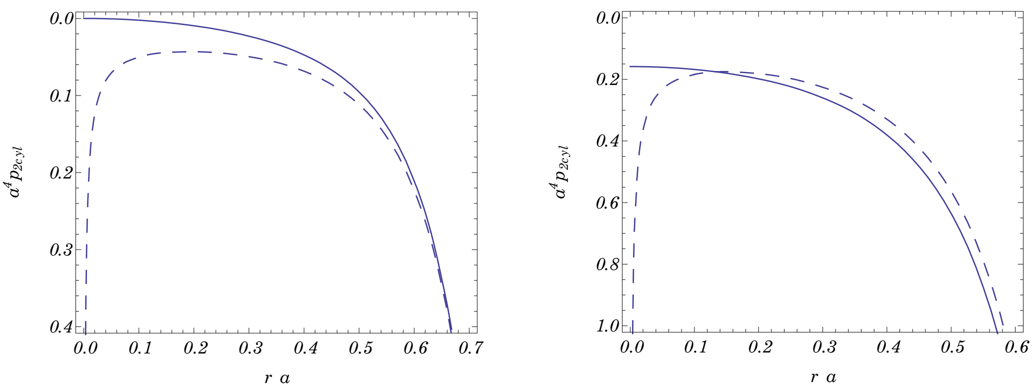

Figure 2: The effective azimuthal pressure induced by the cylindrical shell on the wedge sides, $a^4 p_{2\mathrm{cyl}}$, as a function of r/a for Dirichlet scalar (left panel) and for the electromagnetic field (right panel). The full (dashed) curves correspond to $q = 2$ ($q = 2/3$).

Now, let us discuss the behavior of the boundary-induced part in the VEV of the energy-momentum tensor in the asymptotic regions of the parameters. Near the cylindrical shell the main contribution comes from large values of n and for the points $a - r \ll a|\sin\phi|, a|\sin(\phi_0 - \phi)|$ the leading terms are the same as those for a cylindrical shell when the wedge is absent. For points near the edges ($r = a, \phi = 0, \phi_0$) the leading terms in the corresponding asymptotic expansions are the same as for the geometry of a wedge with the opening angle $\phi_0 = \pi/2$. Near the edge, $r \to 0$, for the components (no summation over i) $\langle T_i^i \rangle_{\mathrm{cyl}}$, $i = 0, 3$, the main contribution comes from the mode $n = 0$ and we find $\langle T_i^i \rangle_{\mathrm{cyl}} \approx -0.0590q/a^4$, $i = 0, 3$. For the components (no summation over i) $\langle T_i^i \rangle_{\mathrm{cyl}}$, $i = 1, 2$, when $q > 1$ the main contribution again comes form $n = 0$ term and one has $\langle T_i^i \rangle_{\mathrm{cyl}} \approx -\langle T_0^0 \rangle_{\mathrm{cyl}}$, $i = 1, 2$. For $q < 1$ the main contribution into the components $\langle T_i^i \rangle_{\mathrm{cyl}}$, $i = 1, 2$, comes from the term $n = 1$ and we have (no summation over i) $\langle T_i^i \rangle_{\mathrm{cyl}} \propto r^{2(q-1)}$, $i = 1, 2$. In this case the radial and azimuthal stresses induced by the cylindrical shell diverge on the edge $r=0$. For the off-diagonal component the main contribution comes from the $n=1$ mode and $\langle T_2^1\rangle_{\rm cyl}$ behaves like r^{2q-1}.

Now we turn to the VEVs of the energy-momentum tensor in the exterior region. Subtracting from these VEVs the corresponding expression for the wedge without the cylindrical boundary, analogously to the case of the field square, it can be seen that the VEVs are presented in the form (64), with the parts induced by the cylindrical shell given by the formulae (no summation over i)

$$\langle T_i^i\rangle_{\rm cyl} = \frac{q}{2\pi^2}\sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^\infty dx x^3 \frac{I_{qn}^{(\lambda)}(xa)}{K_{qn}^{(\lambda)}(xa)} F^{(i)}[\Phi_{qn}^{(\lambda)}(\phi), K_{qn}(xr)], \tag{72}$$

$$\langle T_2^1\rangle_{\rm cyl} = \frac{q^2}{4\pi^2}\frac{\partial}{\partial r}\sum_{m=0}^{\infty}{}' n\sin(2qn\phi)\sum_{\lambda=0,1}(-1)^\lambda\int_0^\infty dx x \frac{I_{qn}^{(\lambda)}(xa)}{K_{qn}^{(\lambda)}(xa)} K_{qn}^2(xr). \tag{73}$$

Here the functions $F^{(i)}\left[\Phi(\phi), f(y)\right]$ are defined by expressions (67). It can be seen that the vacuum energy density induced by the cylindrical shell in the exterior region is positive.

In the way similar to that for the interior region, the force acting on the wedge sides is presented in the form of the sum (69), where for the part due to the presence of the cylindrical shell we have

$$p_{2\rm cyl} = -\langle T_2^2\rangle_{\rm cyl}|_{\phi=0,\phi_0} = -\frac{q}{\pi^2}\sum_{n=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^\infty dx x^3 \frac{I_{qn}^{(\lambda)}(xa)}{K_{qn}^{(\lambda)}(xa)} F_{qn}^{(\lambda)}[K_{qn}(xr)]. \tag{74}$$

In this formula, the function $F_\nu^{(\lambda)}\left[f(y)\right]$ is defined by relations (71) and the corresponding forces are always attractive.

The leading divergence in the boundary induced part (72) on the cylindrical surface is given by the same formulae as for the interior region. For large distances from the shell and for $q>1$ the main contribution into the VEVs of the diagonal components comes from the $n=0$, $\lambda=1$ term and one has (no summation over i)

$$\langle T_i^i\rangle_{\rm cyl} \approx -\frac{qc_i\,(a/r)^2}{15\pi^2 r^4},\ c_0=c_3=2,\ c_1=1,\ c_2=-5. \tag{75}$$

In the case $q<1$ the main contribution into the VEVs of the diagonal components at large distances from the cylindrical shell comes from the $n=1$ mode. The leading terms in the corresponding asymptotic expansions are given by the formulae

$$\langle T_i^i\rangle_{\rm cyl} \approx -q^2(q+1)c_i(q)\frac{\cos(2q\phi)}{\pi^2 r^4}\left(\frac{a}{r}\right)^{2q}, \tag{76}$$

with the notations

$$c_0(q)=c_3(q)=\frac{1}{2q+3},\ c_1(q)=\frac{2q^2+q+1}{(2q+1)(2q+3)},\ c_2(q)=-\frac{q+1}{2q+1}. \tag{77}$$

In the case $q=1$ the asymptotic terms are determined by the sum of the contributions coming from the modes $n=0$ and $n=1$. The latter are given by formulae (75), (76). For the off-diagonal component, for all values q the main contribution at large distances comes from the $n=1$ mode and $\langle T_2^1\rangle_{\rm cyl} \propto (a/r)^{2q} r^{-3}$.

7 Conclusion

We have investigated the polarization of the scalar and electromagnetic vacua by a wedge with coaxial cylindrical boundary, assuming Dirichlet boundary conditions in the case of a scalar field and perfectly conducting boundary conditions for the electromagnetic field. The application of the Abel-Plana-type summation formula for the series over the zeros of the Bessel function and its derivative allowed us to extract from the VEVs the parts due to the wedge without a cylindrical boundary and to present the additional parts induced by this boundary in terms of exponentially convergent integrals. The vacuum densities for the geometry of a wedge without a cylindrical boundary are considered in section 3. We have derived formulae for the renormalized VEVs of the field squared and the energy-momentum tensor, formulae (22), (24). For a conformally coupled scalar the energy-momentum tensor is diagonal and does not depend on the angular variable ϕ. The corresponding vacuum forces acting on the wedge sides are attractive for $\phi_0 < \pi$ and are repulsive for $\phi_0 > \pi$.

For a scalar field the parts in the Wightman function induced by the cylindrical boundary are given by formulae (13) and (17) for the interior and exterior regions respectively. The corresponding VEVs for the field squared and the energy-momentum tensor are investigated in section 4. The field squared is given by formula (33) and vanishes on the wedge sides $\phi = \phi_m$ for all points away from the cylindrical surface. The energy-momentum tensor induced by the cylindrical surface is non-diagonal and the corresponding components are determined by formulae (35), (36). The off-diagonal component vanishes on the wedge sides. The additional vacuum forces acting on the wedge sides due to the presence of the cylindrical surface are determined by the $^2_2$-component of the corresponding stress and are attractive for all values ϕ_0. On the wedge sides the corresponding vacuum stresses in the directions parallel to the wedge sides are isotropic and the energy density is negative for both minimally and conformally coupled scalars. The formulae in the exterior region differ from the corresponding formulae for the interior region by the interchange $I_{qn}(z) \leftrightarrows K_{qn}(z)$. For large distances from the cylindrical surface, $r \gg a$, the VEVs behave as $(a/r)^{D-1+2q}$ for the field squared and as $(a/r)^{D+1+2q}$ for the diagonal components of the energy-momentum tensor.

In the second part of the paper we have evaluated the VEVs of the field squared and the energy-momentum tensor for the electromagnetic field. For the wedge without the cylindrical shell the VEVs of the field squared are presented in the form (54). The first term on the right of this formula corresponds to the VEVs for the geometry of a cosmic string with the angle deficit $2\pi - \phi_0$. The parts induced by the cylindrical shell are presented in the form (51) for the interior region and in the form (58) for the exterior region. We have discussed these general formulae in various asymptotic regions of the parameters. In section 6 we consider the VEV of the energy-momentum tensor. For the geometry of a wedge without the cylindrical boundary the vacuum energy-momentum tensor does not depend on the angle ϕ and is the same as in the geometry of the cosmic string. The corresponding vacuum forces acting on the wedge sides are attractive for $\phi_0 < \pi$ and repulsive for $\phi_0 > \pi$. In particular, the equilibrium position corresponding to the geometry of a single plate is unstable. For the region inside the shell the part in the VEV of the energy-momentum tensor induced by the presence of the cylindrical shell is non-diagonal and the corresponding components are given by formulae (65), (66) for the interior region and by (72), (73) for the exterior region. The vacuum energy density induced by the cylindrical shell is negative in the interior region and is positive in the exterior region. For a wedge with $\phi_0 < \pi$ the part in the vacuum energy-momentum tensor induced by the shell is finite on the edge $r = 0$. For $\phi_0 > \pi$ the shell-induced parts in the energy density and the axial stress remain finite, whereas the radial and azimuthal stresses diverge as $r^{2(\pi/\phi_0-1)}$. The corresponding off-diagonal component behaves like $r^{2\pi/\phi_0-1}$ for all values ϕ_0. For points near the edges ($r = a, \phi = 0, \phi_0$), the leading terms in the corresponding asymptotic expansions are the same as for the geometry of a wedge with the opening angle $\phi_0 = \pi/2$. The presence of the shell leads to additional forces acting on the wedge sides. The corresponding effective azimuthal pressures are given by formulae (70), (74) and these forces are always attractive.

Acknowledgments

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From multiple scattering to van der Waals interactions: exact results for eccentric cylinders

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Abstract

In this paper, dedicated to the career of Iver Brevik, we review the derivation of the retarded van der Waals or Casimir-Polder interaction between polarizable molecules from the general multiple scattering formulation of Casimir interactions between bodies. We then apply this van der Waals potential to examine the interaction between tenuous cylindrical bodies, including eccentric cylinders, and revisit the vanishing self-energy of a tenuous dielectric cylinder. In each case, closed-form expressions are obtained [4].

1 Introduction

Since the earliest calculations of fluctuation forces between bodies [1], that is, Casimir or quantum vacuum forces, multiple scattering methods have been employed. Rather belatedly, it has been realized that such methods could be used to obtain accurate numerical results in many cases [2, 3, 4, 5]. These results allow us to transcend the limitations of the proximity force theorem (PFT) [6, 7], and so make better comparison with experiment, which typically involve curved surfaces. (For a review of the experimental situation, see Ref. [8].)

These improvements in technique were inspired in part by the development of the numerical Monte-Carlo worldline method of Gies and Klingmüller [9, 10, 11, 12] but the difficulty with this latter method lies in the statistical limitations of Monte Carlo methods and in the complexity of incorporating electromagnetic boundary conditions. Optical path approximations have been studied extensively for many years, with considerable success [13, 14, 15, 16]. However, there always remain uncertainties because of unknown errors in excluding diffractive effects. Direct numerical methods [17, 18], based on finite-difference engineering techniques,

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may have promise, but the requisite precision of 3-dimensional calculations may prove challenging [19].

The multiple scattering formalism, which is in principle exact, dates back at least into the 1950s [20, 21]. Particularly noteworthy is the seminal work of Balian and Duplantier [22]. (For more complete references see Ref. [23].) This technique, which has been brought to a high state of perfection by Emig et al. [5], has concentrated on numerical results for the Casimir forces between conducting and dielectric bodies such as spheres and cylinders. Recently, we have noticed that the multiple-scattering method can yield exact, closed-form results for bodies that are weakly coupled to the quantum field [24, 23]. This allows an exact assessment of the range of applicability of the PFT. The calculations there, however, as those in recent extensions of our methodology [25], have been restricted to scalar fields with δ-function potentials, so-called semitransparent bodies. (These are closely related to plasma shell models [3, 26, 27, 28].) The technique was recently extended to dielectric bodies [29], characterized by a permittivity ε. Strong coupling would mean a perfect metal, $\varepsilon \to \infty$, while weak coupling means that ε is close to unity.

In this paper we will give details of the formalism, and show how in weak coupling (dilute dielectrics) we recover the sum of Casimir-Polder or retarded van der Waals forces between atoms. Exact results have been found in the past in such summations, for example for the self-energy of a dilute dielectric sphere [30] or a dilute dielectric cylinder [31]. Thus it is not surprising that exact results for the interaction of different dilute bodies can be obtained. It is only surprising that such results were not found much earlier. (We note that the additive approximation has been widely used in the past, for example, see Ref. [32], but here the method is exact. Also, there are many exact computations for non-retarded London forces betwen bodies, e.g., Ref. [33], but these results can only apply to very tiny objects on the nanometer scale.) In our previous letter [29] we considered the force and torque on a slab of finite extent above an infinite plane, and the force between spheres and parallel cylinders outside each other. Here we will examine further cylindrical geometries, such as concentric cylinders, and eccentric circular cylinders with parallel axes, in cases where the dielectric materials do not overlap. We will prove that the results can be obtained by analytic continuation of the energies found earlier for non-contained bodies. Finally, we will re-examine the self-energy of a dielectic cylinder [31].

2 Green's dyadic formalism

For electromagnetism, we can start from the formalism of Schwinger [34], which is based on the electric Green's dyadic $\mathbf{\Gamma}$. This object can be identified as the one-loop vacuum expectation value of the correlation function of electric fields,

$$\mathbf{\Gamma}(\mathbf{r},t;\mathbf{r}',t') = i\langle \mathrm{T}\{\mathbf{E}(\mathbf{r},t)\mathbf{E}(\mathbf{r}',t')\}\rangle. \tag{1}$$

Alternatively, we regard the Green's dyadic as the propagator between a polarization source $\mathbf{P}$ and a phenomenological field $\mathbf{E}$ (where $x^\mu = (\mathbf{r},t)$):

$$\mathbf{E}(x) = \int (dx')\mathbf{\Gamma}(x,x)\cdot\mathbf{P}(x'). \tag{2}$$

We will only be contemplating static geometries, so it is convenient to consider a specific frequency ω, as introduced through a Fourier transform,

$$\mathbf{\Gamma}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\mathbf{\Gamma}(\mathbf{r},\mathbf{r}';\omega), \tag{3}$$

in terms of which the Maxwell equations in a region where the permittivity $\varepsilon(\omega)$ and the permeability $\mu(\omega)$ are constant in space read

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{\Gamma} &= i\omega\boldsymbol{\Phi}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}=0, && \text{(4a)}\\ \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{\Phi} &= -i\omega\varepsilon\boldsymbol{\Gamma}', \quad \boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}'=0, && \text{(4b)} \end{aligned}$$

where we have introduced $\boldsymbol{\Gamma}'=\boldsymbol{\Gamma}+\mathbf{1}/\varepsilon$, where the unit dyadic includes a spatial δ function. The two Green's dyadics given here satisfy the following inhomogenous Helmholtz equations,

$$\begin{aligned} (\nabla^2+\omega^2\varepsilon\mu)\boldsymbol{\Gamma}' &= -\frac{1}{\varepsilon}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{1}), && \text{(5a)}\\ (\nabla^2+\omega^2\varepsilon\mu)\boldsymbol{\Phi} &= i\omega\mu\boldsymbol{\nabla}\times\mathbf{1}. && \text{(5b)} \end{aligned}$$

In the following, it will prove more convenient to use, instead of Eq. (5a),

$$\left(\frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\varepsilon\right)\boldsymbol{\Gamma}=\mathbf{1}. \tag{6}$$

In the presence of a polarization source, the action is, in symbolic form,

$$W=\frac{1}{2}\int\mathbf{P}\cdot\boldsymbol{\Gamma}\cdot\mathbf{P}, \tag{7}$$

so if we consider the interaction between bodies characterized by particular values of ε and μ, the change in the action due to moving those bodies is

$$\delta W=\frac{1}{2}\int\mathbf{P}\cdot\delta\boldsymbol{\Gamma}\cdot\mathbf{P}=-\frac{1}{2}\int\mathbf{E}\cdot\delta\boldsymbol{\Gamma}^{-1}\cdot\mathbf{E}, \tag{8}$$

where the symbolic inverse dyadic, in the sense of Eq. (6), is

$$\boldsymbol{\Gamma}^{-1}=\frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\varepsilon, \tag{9}$$

that is,

$$\delta\boldsymbol{\Gamma}\cdot\boldsymbol{\Gamma}^{-1}=-\boldsymbol{\Gamma}\cdot\delta\boldsymbol{\Gamma}^{-1}. \tag{10}$$

By comparing with the iterated source term in the vacuum-to-vacuum persistence amplitude $\exp iW$, we see that an infinitesimal variation of the bodies results in an effective source product,

$$\mathbf{P}(\mathbf{r})\mathbf{P}(\mathbf{r}')\Big|_{\text{eff}}=i\delta\boldsymbol{\Gamma}^{-1}, \tag{11}$$

from which we deduce from Eq. (7) that

$$\delta W=\frac{i}{2}\mathrm{Tr}\,\boldsymbol{\Gamma}\cdot\delta\boldsymbol{\Gamma}^{-1}=-\frac{i}{2}\mathrm{Tr}\,\delta\boldsymbol{\Gamma}\cdot\boldsymbol{\Gamma}^{-1}=-\frac{i}{2}\delta\mathrm{Tr}\,\ln\boldsymbol{\Gamma}, \tag{12}$$

where the trace includes integration over space-time coordinates. We conclude, by ignoring an integration constant,

$$W=-\frac{i}{2}\mathrm{Tr}\,\ln\boldsymbol{\Gamma}. \tag{13}$$

This is in precise analogy to the expression for scalar fields. Another derivation of this result is given in the Appendix. Incidentally, note that the first equality in Eq. (12) implies for dielectric bodies ($\mu=1$)

$$\delta W=-\frac{i}{2}\int\frac{d\omega}{2\pi}\int(d\mathbf{r})\,\delta\varepsilon(\mathbf{r},\omega)\Gamma_{kk}(\mathbf{r},\mathbf{r}';\omega), \tag{14}$$

which is the starting point for the derivation of the Lifshitz formula [35] in Ref. [34].

3 Rederivation of Casimir-Polder formula

Henceforth, let us consider pure dielectrics, that is, set $\mu = 1$. The free Green's dyadic, in the absence of dielectric bodies, satisfies the equation

$$\left[\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -1\right]\boldsymbol{\Gamma}_0 = \mathbf{1}, \tag{15}$$

so the equation satisfied by the full Green's dyadic is

$$(\boldsymbol{\Gamma}_0^{-1} - V)\boldsymbol{\Gamma} = \mathbf{1}, \tag{16}$$

where $V = \varepsilon - 1$ within the body. From this we deduce immediately that

$$\boldsymbol{\Gamma} = (1 - \boldsymbol{\Gamma}_0 V)^{-1}\boldsymbol{\Gamma}_0. \tag{17}$$

From the trace-log formula (13) we see that the energy for a static situation ($W = -\int dt\, E$) relative to the free-space background is

$$E = \frac{i}{2}\mathrm{Tr}\,\ln\boldsymbol{\Gamma}_0^{-1}\cdot\boldsymbol{\Gamma} = -\frac{i}{2}\mathrm{Tr}\,\ln(1 - \boldsymbol{\Gamma}_0 V). \tag{18}$$

The trace here is only over spatial coordinates. We will now consider the interaction between two bodies, with non-overlapping potentials, $V = V_1 + V_2$, where $V_a = \varepsilon_a - 1$ is confined to the interior of body a, $a = 1, 2$. Although it is straightforward to proceed to write the interaction between the bodies in terms of scattering operators, for our limited purposes here, we will simply treat the potentials as weak, and retain only the first, bilinear term in the interaction:

$$E_{12} = \frac{i}{2}\mathrm{Tr}\,\boldsymbol{\Gamma}_0 V_1 \boldsymbol{\Gamma}_0 V_2. \tag{19}$$

Here, as may be verified by direct calculation [36],

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}') = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}G_0(\mathbf{r}-\mathbf{r}') - \mathbf{1} = (\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\zeta^2)G_0(\mathbf{r}-\mathbf{r}'), \tag{20}$$

where the scalar Helmholtz Green's function which satisfies causal or Feynman boundary conditions is

$$G_0(\mathbf{r}-\mathbf{r}') = \frac{e^{-|\zeta|R}}{4\pi R}, \quad R = |\mathbf{r}-\mathbf{r}'|, \tag{21}$$

the Fourier transform of the Euclidean Green's function, which obeys the differential equation

$$(-\nabla^2 + \zeta^2)G_0(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{22}$$

and $\zeta = -i\omega$.

Thus the interaction between the two potentials is given by

$$E_{12} = -\frac{1}{2}\int\frac{d\zeta}{2\pi}\int(d\mathbf{r})(d\mathbf{r}')\left[(\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\right]^2 V_1(\mathbf{r})V_2(\mathbf{r}'). \tag{23}$$

The derivatives occurring here may be easily worked out:

$$(\boldsymbol{\Gamma}_0)_{ij} = (\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta|R|}}{4\pi R} = \left[-\delta_{ij}(1 + |\zeta|R + \zeta^2R^2) + \frac{R_iR_j}{R^2}(3 + 3|\zeta|R| + \zeta^2R^2)\right]\frac{e^{-|\zeta|R}}{4\pi R^3}, \tag{24}$$

and then contracting two such factors together gives

$$(\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta|R}}{4\pi R}(\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta|R}}{4\pi R} = (6+12t+10t^2+4t^3+2t^4)\frac{e^{-2t}}{(4\pi R^3)^2}, \tag{25}$$

where $t = |\zeta|R$. Inserting this into Eq. (23), we obtain for the integral over ζ

$$-\frac{1}{64\pi^3 R^7}\int_0^\infty du\, e^{-u}\left(6+6u+\frac{5}{2}u^2+\frac{1}{2}u^3+\frac{1}{8}u^4\right) = -\frac{23}{64\pi^3 R^7}, \tag{26}$$

or

$$E_{12} = -\frac{23}{(4\pi)^3}\int (d\mathbf{r})(d\mathbf{r}')\frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^7}, \tag{27}$$

which is the famous Casimir-Polder potential [37]. This formula is valid for bodies, which are presumed to be composed of material filling nonoverlapping volumes v_1 and v_2, respectively, characterized by dielectric constants ε_1 and ε_2, both nearly unity. We emphasize that this formula is exact in the limit $\varepsilon_{1,2} \to 1$, as discussed in Ref. [32].

4 Energy of cylinder parallel to a plane

In Ref. [29] we derived the energy of two uniform dilute cylinders, of radius a and b respectively, the parallel axes of which are separated by a distance R, $R > a + b$. In terms of the constant

$$N = \frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1). \tag{28}$$

the energy of interaction per unit length is

$$\mathfrak{E}_{\text{cyl}-\text{cyl}} = -\frac{32\pi N}{3}\frac{a^2b^2}{R^6}\frac{1-\frac{1}{2}\left(\frac{a^2+b^2}{R^2}\right)-\frac{1}{2}\left(\frac{a^2-b^2}{R^2}\right)^2}{\left[\left(1-\left(\frac{a+b}{R}\right)^2\right)\left(1-\left(\frac{a-b}{R}\right)^2\right)\right]^{5/2}}. \tag{29}$$

If we take R and b to infinity, such that $Z = R - b$ is held fixed, we describe a cylinder of radius a parallel to a dielectric plane, where Z is the distance between the axis of the cylinder and the plane. That limit gives the simple result

$$\mathfrak{E}_{\text{cyl}-\text{pl}} = -\frac{N\pi a^2}{Z^4}\frac{1}{(1-a^2/Z^2)^{5/2}}. \tag{30}$$

This is to be compared to the corresponding result for a sphere of radius a a distance Z above a plane:

$$E_{\text{sph}-\text{pl}} = -N\frac{v}{Z^4}\frac{1}{(1-a^2/Z^2)^2}, \tag{31}$$

where v is the volume of the sphere.

As an illustration of how the calculation is done, let us rederive this result directly from Eq. (27). We see immediately that the energy between an infinite halfspace (of permittivity ε_1) and a parallel slab (of permittivity ε_2) of area A and thickness dz separated by a distance z is

$$\frac{dE}{A} = -N\frac{dz}{z^4}, \tag{32}$$

so the energy per length between the cylinder and the plane is

$$\mathfrak{E}_{\text{cyl}-\text{pl}} = -2Na^2\int_{-1}^{1} d\cos\theta\frac{\sin\theta}{(Z+a\cos\theta)^4} = -\frac{N\pi a^2 Z}{(Z^2-a^2)^{5/2}}, \tag{33}$$

which is the result (30).

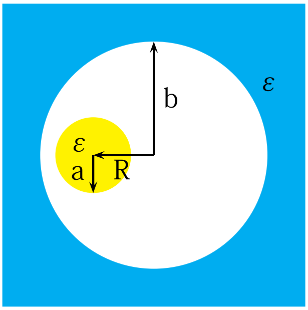

Figure 1: Dielectric ε_2 hollowed out by a cylindrical cavity which contains an offset parallel dielectric cylinder ε_1.

5 Eccentric cylinders

As a second illustration, consider two coaxial cylinders, of radii a and b, $a < b$. The inner cylinder is filled with material of permittivity ε_1, while the outer cylinder is the inner boundary of a region with permittivity ε_2 extending out to infinity. An easy calculation from the van der Waals interaction (27)

$$\begin{aligned} \mathfrak{E}_{\text{co}-\text{cyl}} &= -\frac{64N}{3}\int_0^a d\rho\,\rho\int_b^\infty d\rho'\,\rho'\int_0^{2\pi}\frac{d\theta}{(\rho^2+\rho'^2-2\rho\rho'\cos\theta)^3} \\ &= -\frac{32\pi N}{3}\int_0^{a^2}dx\int_{b^2}^\infty dy\frac{x^2+y^2+4xy}{(y-x)^5} \\ &= -\frac{16N\pi a^2b^2}{3(b^2-a^2)^3}. \end{aligned} \tag{34}$$

This reduces to the dilute Lifshitz formula for the interaction between parallel plates if we take the limit $b\to\infty$, $a\to\infty$, with $b-a=d$ held fixed:

$$\mathfrak{E}_{\text{co}-\text{cyl}} \to -\frac{2N\pi b}{3d^3}, \quad \text{or} \quad \frac{E}{A} = -\frac{N}{3d^3}. \tag{35}$$

Note that the result (34) may be obtained by analytically continuing the energy between two externally separated cylinders, given by Eq. (29). We simply take R to zero there, and choose the sign of the square root so that the energy is negative. That suggests that the same thing can be done to obtain the energy of interaction between two parallel cylinders, one inside the other, but whose axes are displaced by an offset R, with $R+a<b$, as shown in Fig. 1:

$$\mathfrak{E}_{\text{ecc}-\text{cyl}} = -\frac{16\pi N}{3}\frac{a^2}{b^4}\frac{(1-a^2/b^2)^2+(1+a^2/b^2)R^2/b^2-2R^4/b^4}{[(1-a^2/b^2)^2+R^4/b^4-2(1+a^2/b^2)R^2/b^2]^{5/2}}. \tag{36}$$

We can verify this is true by carrying out the integral from the Casimir-Polder formula (27)

$$\begin{aligned} \mathfrak{E}_{\text{ecc}-\text{cyl}} &= -\frac{32N}{3\pi}\int_0^a \rho\,d\rho\int_b^\infty \rho'\,d\rho'\int_0^{2\pi}d\theta\int_0^{2\pi}d\theta' \\ &\quad\times\left[\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')+R^2-2R(\rho\cos\theta-\rho'\cos\theta')\right]^{-3}. \end{aligned} \tag{37}$$

The angular integrals can be done, but the remaining integrals are rather complicated. Therefore, let us simply expand immediately in the quantities $x = \rho/\rho'$ and $y = R/\rho'$, which are both less than one. Then we can carry out the four integrals term by term. In this way we find

$$\mathfrak{E}_{\text{ecc}-\text{cyl}} = -\frac{16\pi N a^2}{3b^4} \sum_{n,m=0}^{\infty} \left(\frac{a^2}{b^2}\right)^n \left(\frac{R^2}{b^2}\right)^m \frac{(m+1)^2}{2} \binom{n+m+1}{m+1} \binom{n+m+2}{m+1}, \tag{38}$$

which is exactly the series expansion of Eq. (36) for small a/b and R/b.

By differentiating this energy with respect to the offset R, we obtain the force of interaction between the inner cylinder and the outer one, $\mathfrak{F} = -\partial\mathfrak{E}/\partial R$. Evidently, that force is zero for coaxial cylinders, since that is a point of unstable equilibrium. For small R, $\mathfrak{F}$ grows linearly with R with a positive coefficient. The inner cylinder is attracted to the closest point of the opposite cylinder. Similar considerations for conducting cylinders were given in Refs. [38, 39], with the idea that the cylindrical geometry might prove to be a useful proving ground for Casimir experiments.

6 Self-energy of dilute cylinder

This is a rederivation of the result found by dimensional continuation in Ref. [31]. The summation of the Casimir-Polder forces between the molecules in a single cylinder of radius a is given by (in N $\varepsilon_1 = \varepsilon_2$)

$$\begin{aligned}\mathfrak{E}_{\text{cyl}} &= -\frac{32N}{3}\int_0^a d\rho\,\rho \int_0^a d\rho'\,\rho' \int_0^{2\pi} \frac{d\theta}{(\rho^2+\rho'^2-2\rho\rho'\cos\theta)^3} \\ &= -\frac{16N}{3}\int_0^{a^2} \frac{dx}{x^2} \int_0^1 du \left(\frac{1}{u^3} - \frac{6}{u^4} + \frac{6}{u^5}\right).\end{aligned} \tag{39}$$

This is, of course, terribly divergent. We can regulate it by analytic continuation: replace the highest power of $(\rho^2 - \rho'^2)^{-1} = (xu)^{-1}$, 5, by β, and regard β as less than 1. Then,

$$\begin{aligned}\mathfrak{E}_{\text{cyl}} &= -\frac{16N}{3}\int_0^{a^2} dx\, x^{3-\beta} \int_0^1 du\left(u^{2-\beta} - 6u^{1-\beta} + 6u^{-\beta}\right) \\ &= -\frac{16N}{3}(a^2)^{4-\beta}\frac{(5-\beta)}{(1-\beta)(2-\beta)(3-\beta)}.\end{aligned} \tag{40}$$

Now if we analytically continue to $\beta = 5$ we get an vanishing self-energy to order $(\varepsilon - 1)^2$. This result was first discovered by Romeo (private communication), verified in Ref. [31], and confirmed later by full Casimir calculations [40, 41].

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8 Derivation of Green's dyadic formalism from canonical theory

Here we begin by sketching the development of the Green's dyadic equation from canonical quantum electrodynamics. For simplicity of the discussion, we will consider a medium without dispersion, so that ε and μ are constant. First we must state the canonical equal-time commutation relations. We will require (only transverse fields are relevant)

$$[E_i(\mathbf{r},t), E_j(\mathbf{r}',t)] = 0. \tag{41}$$

In a medium, it is the electric displacement field which is canonically conjugate to the vector potential, so we have the equal-time commutation relation (Coulomb gauge)

$$[\mathbf{A}(\mathbf{r},t), \partial_0\mathbf{A}(\mathbf{r}',t)] = \frac{i}{\varepsilon}\left(\mathbf{1} - \frac{\boldsymbol{\nabla}\boldsymbol{\nabla}}{\nabla^2}\right)\delta(\mathbf{r}-\mathbf{r}'). \tag{42}$$

Now in view of Eq. (1), and the Maxwell equations

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \tag{43a}$$

$$\boldsymbol{\nabla}\times\frac{1}{\mu}\mathbf{B} = \frac{\partial}{\partial t}\varepsilon\mathbf{E}, \tag{43b}$$

we deduce from Eq. (1)

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Gamma}' + \varepsilon\mu\frac{\partial^2}{\partial t^2}\boldsymbol{\Gamma}' = i\varepsilon\mu\delta(t-t')\langle[\dot{\mathbf{E}}(\mathbf{r},t), \mathbf{E}(\mathbf{r}',t)]\rangle. \tag{44}$$

But according to Maxwell's equations and Eq. (42)

$$[\dot{\mathbf{E}}(\mathbf{r},t), \mathbf{E}(\mathbf{r}',t)] = \frac{1}{\varepsilon\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times[\mathbf{A}(\mathbf{r},t), -\partial_0\mathbf{A}(\mathbf{r}',t)] = -\frac{i}{\varepsilon^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}\delta(\mathbf{r}-\mathbf{r}'). \tag{45}$$

If we now insert this into Eq. (44) we obtain for the Fourier transform of the Green's dyadic (3)

$$(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -\varepsilon\mu\omega^2)\boldsymbol{\Gamma}' = \frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}, \tag{46}$$

which is indeed the equation satisfied by the solenoidal Green's dyadic, Eq. (5a).

It is equally easy to derive the trace-log formula. We have the variational statement, for infinitesimal changes in the permittivity and the permeability [42],

$$\delta E = -\frac{1}{2}\int(d\mathbf{r})\langle\delta\varepsilon E^2 + \delta\mu H^2\rangle. \tag{47}$$

Given Eqs. (1) and (43a) we can write this in terms of the coincident-point limit of Green's dyadic,

$$\delta E = \frac{i}{2}\int(d\mathbf{r})\left[\delta\varepsilon - \delta\left(\frac{1}{\mu}\right)\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\right]\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\bigg|_{\mathbf{r}'\to\mathbf{r}} = -\frac{i}{2}\mathrm{Tr}\,\delta\boldsymbol{\Gamma}^{-1}\cdot\boldsymbol{\Gamma}, \tag{48}$$

according to Eq. (9), which involves an integration by parts, and coincides with the first equality in Eq. (12).

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Casimir force for electrolytes

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Abstract

The Casimir force between a pair of parallell plates filled with ionic particles is considered. We use a statistical mechanical approach and consider the classical high temperature limit. In this limit the ideal metal result with no transverse electric (TE) zero frequency mode is recovered. This result has also been obtained by Jancovici and Šamaj earlier. Our derivation differs mainly from the latter in the way the Casimir force is evaluated from the correlation function. By our approach the result is easily extended to electrolytes more generally. Also we show that when the plates are at contact the Casimir force is in accordance with the bulk pressure as follows from the virial theorem of classical statistical mechanics [2].

1 Introduction

It is a pleasure to contribute this work to a festshrift volume for Professor Iver Brevik. We have had an extensive collaboration through many years on problems connected to the Casimir effect. In our works we have fruitfully utilized methods from different fields of research. In particular we have explored the statistical mechnanical aspects of the Casimir problem. The present contribution is a work that continues in the statistical mechanical direction.

A pair of metallic or dielectric plates attract each other. This is the well known Casimir effect, and it is commonly regarded to be due to fluctuations of the quantum electrodynamic field in vacuum. However, Høye and Brevik considered this in a different way by regarding the problem as a statistical mechanical one of interacting fluctuating dipole moments of polarizable particles. In this way the Casimir force between a pair of polarizable point particles was recovered [1]. To do so the path integral formulation of quantized particle systems was utilized [2]. Before that this method was fruitfully utilized for a polarizable fluid [3]. With this approach the role of the electromagnetic field is to mediate the pair interaction between polarizable particles. Later this type of evaluation was generalized to a pair of parallell plates, and the well known Lifshitz result was recovered [4]. Similar evaluations were performed for other situations [5, 6].

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The statistical mechanical approach opens new perspectives for evaluations of the Casimir force. Instead of focusing upon the quantization of the electromagnetic field itself one can regard the problem as one of polarizable particles interacting via the electromagnetic field. It is found that these two viewpoints are equivalent [1, 4, 6, 7].

Metals are materials that have electrons that can be regarded as free. When deriving the Lifshitz formula they are regarded as dielectric media that have infinite dielectric constant for zero frequency. Jancovici and Šamaj realized that it should be possible to evaluate the Casimir force for metals by regarding an electron plasma. Thus they considered parallell plates filled with charged particles at low density in a neutralizing background [8]. Further they considered the classical case, i.e. the high temperature limit. In this situation the Debye-Hückel theory of electrolytes is fully applicable. Then they use the Ornstein-Zernike equation (OZ) equation, and utilize its equivalence with the differential equation for the screened Coulomb potential to obtain the pair correlation function. This function is used to obtain the local ionic density at the surfaces of the plates. The difference between local and bulk densities is attributed to the Casimir force in accordance with the ideal gas law. The result obtained coincides with a result for ideal metals in the high temperature limit. The latter has been a dispute of controversy [9]. The ionic plasma result coincides with the one where there is no transverse electric mode at zero frequency. This is also in accordance with Maxwell's equations of electromagnetism.

The ionic plasma has also been extended to the quantum mechanical case by use of the path integral formalism from a statistical mechanical viewpoint, and it has been shown that magnetic interactions do not contribute in the classical high temperature limit [13].

In the present work we reconsider the ionic plasma in the classical limit. We arrive at the the same pair correlation function as in Ref. [8]. But we use a different approach to obtain the Casimir force. As we see it, our method better utilizes the methods of classical statistical mechanics especially for possible further developments. Thus we use the correlation function to directly evaluate the average force between pairs of particles in the two plates and then integrate to obtain the total force. This is the method used in Refs. [1, 4]. In this way the result of Ref. [8] is recovered. A noteable feature of this comparison is that it demonstrates that the modification of the density profile at the surface is a perturbing effect that can be neglected to leading order by our approach.

With our approach the evaluations are extended in a straightforward way to electrolytes of more arbitrary density. To do so known properties of the direct correlation function is utilized. The main change with this extension is that the large distance inverse shielding length is modified while the Casimir force remains unchanged for large separations.

An additional result of our approach is that it is shown that when the plates are at contact the Casimir pressure more generally is nothing but the contribution to the bulk pressure (with opposite sign) that follows from the virial theorem of classical statistical mechnanics.

2 General expressions

Consider a pair of harmonic oscillators with static polarizability α. They interact via a potential $\psi s_1 s_2$ where s_1 and s_2 are fluctuating polarizations. This interaction creates a shift in the free energy of the system. This is easily evaluated to be [1]

$$-\beta F = -\frac{1}{2}\ln[1-(\alpha\psi)^2] = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha\psi)^{2n} \tag{1}$$

with $\beta = 1/(k_B T)$ where T is temperature and k_B is Boltzmanns constant. The last sum is the expansion performed in Ref. [4] where the two particles were replaced with two plane parallell plates. In the latter case the terms can be interpreted as the sum of graph contributions due to the mutual interaction ψ. The α will represent correlations within each plate separately while each ψ gives a link between the plates while $2n$ is the symmetry factor of the graphs that form closed rings. With plates the endpoints of each link ψ should be integrated over the plates. In the quantum mechanical case there is also a sum over Matzubara frequencies upon which α and ψ may depend.

The parallell plates are separated by a distance a. Due to the interaction there will be an attractive force K between the plates. This force is found from [4]

$$K = -\frac{\partial F}{\partial a} = \frac{1}{\beta}\frac{\alpha\psi\alpha}{1-(\alpha\psi)^2}\frac{\partial\psi}{\partial a}. \tag{2}$$

The fraction in the middle of this expression represents the graph expansion of the pair correlation function with the endpoints in separate plates. These graphs form chains where each ψ forms a link between the plates. Thus we can write

$$K = \rho^2 \int h(\mathbf{r}_2, \mathbf{r}_1)\, \psi'_z(\mathbf{r}_2 - \mathbf{r}_1)\, d\mathbf{r}_1 d\mathbf{r}_2 \tag{3}$$

where ρ is number density, $h(\mathbf{r}_2, \mathbf{r}_1)$ is the pair correlation function, and $\psi'_z(\mathbf{r}_2 - \mathbf{r}_1) = \partial\psi/\partial a$ with the z-direction normal to the plates. For polarizable particles integral (3) will also contain integrations with respect to polarizations [4].

For infinite plates integral (3) diverges, so as usual we will consider the force f per unit area which then will be

$$f = \frac{\rho^2}{(2\pi)^2} \int\limits_{z_1<0, z_2>0} \hat{h}(k_\perp, z_2, z_1)\hat{\psi}'_z(k_\perp, z_2 - z_1)\, dk_x dk_y dz_1 dz_2 \tag{4}$$

where the hat denotes Fourier transform with respect to the x- and y-coordinates. (Here we have used $\int fg\, dxdy = \int \hat{f}\hat{g}\, dk_x dk_y/(2\pi)^2$ and translational symmetry along the xy-plane.

Now we can introduce

$$q^2 = k_\perp^2 = k_x^2 + k_y^2, \quad \text{with} \quad dk_x dk_y = 2\pi q\, dq. \tag{5}$$

Further with $z_2 = u_2 + a$ and $z_1 = -u_1$ we then get

$$f = \frac{\rho^2}{2\pi} \int\limits_{u_1, u_2>0} \hat{h}(q, z_2, z_1)\hat{\psi}'_z(q, z_2 - z_1) q\, dq dz_1 dz_2 \tag{6}$$

An interesting feature of result (6) or (4) is that it is fully consistent with the virial theorem in statistical mechanics. This means that when the plates are at contact for $a = 0$ the Casimir force equals the contribution to the pressure from the virial integral with pair interaction ψ. With $a = 0$ translational symmetry is also present in the z-direction, so we have

$$f = \rho^2 \int\limits_{z_1<0, z_2>0} h(|\mathbf{r}_2 - \mathbf{r}_1|)\psi'_z(|\mathbf{r}_2 - \mathbf{r}_1|)\, dxdydz_1dz_2. \tag{7}$$

With new variable $z = z_2 - z_1$ one can first integrate with respect to z_2 which then will be confined to the region $0 \le z_2 \le z$. Thus with $\int_0^z dz_2 = z$ we obtain ($\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$)

$$f = \rho^2 \int\limits_{z>0} h(r)\psi'_z(r)\, d\mathbf{r} = \frac{\rho^2}{6} \int h(r)\mathbf{r}\nabla\psi(r)\, d\mathbf{r} \tag{8}$$

where symmetry with respect to the x-, y-, and z-directions and with respect to positive and negative z is used. (It may be noted that the above is correct if the average of ψ is zero. Otherwise the pair distribution function $1 + h$ should be used. But for neutral plates as for dielectric plates with dipolar interaction this average will be zero.)

3 Pair correlation function

To obtain the correlation function we use the Ornstein-Zernike (OZ) equation

$$h(\mathbf{r}_2, \mathbf{r}_1) = c(\mathbf{r}_2, \mathbf{r}_1) + \int c(\mathbf{r}_2, \mathbf{r}')\rho(\mathbf{r}')h(\mathbf{r}', \mathbf{r}_1)\, d\mathbf{r}' \tag{9}$$

which here has been extended to non-homogeneous fluids. The $c(\mathbf{r})$ is the direct correlation function. For week long-range forces [14] or to leading order the $c(\mathbf{r})$ is related to the interaction in a simple way

$$c(\mathbf{r}_2, \mathbf{r}_1) = -\beta\psi. \tag{10}$$

For plate separations beyond interparticle distances the ψ will be small anyway. For a plasma at low density we can write for all r

$$c(\mathbf{r}_2, \mathbf{r}_1) = c(r) = -\beta\frac{q_c^2}{r}, \quad (\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1) \tag{11}$$

where q_c is the ionic charge assuming one component for simplicity. (Here Gaussian units are used.) To keep the system neutral a uniform background is assumed. As noted in Ref. [8] the OZ-equation is now equivalent to Maxwells equation of electrostatics. The similar situation was utilized in Ref. [4] for dipolar interactions.

Since ψ is the electrostatic potential from a charge one has

$$\nabla^2 c(r) = 4\pi\beta q^2\delta(\mathbf{r}). \tag{12}$$

With this Eq. (9) can be rewritten as

$$\nabla^2\Phi - 4\pi\beta q_c^2\rho(\mathbf{r})\Phi = -4\pi\delta(\mathbf{r} - \mathbf{r}_0), \quad h(\mathbf{r}, \mathbf{r}_0) = -\beta q_c^2\Phi, \tag{13}$$

where $\mathbf{r}_2$ and $\mathbf{r}_1$ have been replaced by $\mathbf{r}$ and $\mathbf{r}_0$ respectively. In the present case with parallell plates the number density is

$$\rho(\mathbf{r}) = \begin{cases} \rho, & z < 0 \\ 0, & 0 < z < a \\ \rho, & a < z \end{cases} \tag{14}$$

with equal densities $\rho =$ const. on both plates. By Fourier transform in the x- and y-directions Eq. (11) becomes

$$\left(\frac{\partial^2}{\partial z^2} - k_\perp^2 - \kappa_z^2\right)\hat{\Phi} = -4\pi\delta(z - z_0) \tag{15}$$

where the hat denotes Fourier transform and with $\kappa = 4\pi\beta q_c^2\rho$

$$\kappa_z^2 = \kappa \begin{cases} 1, & z < 0 \\ 0, & 0 < z < a \\ 1, & a < z, \end{cases} \tag{16}$$

The κ is the inverse Debye-Hückel shielding length in the media. Solution of Eq. (13) can be written in the form

$$\hat{\Phi} = 2\pi e^{q_\kappa z_0} \begin{cases} \frac{1}{q_\kappa} e^{-q_\kappa z} + B e^{q_\kappa z}, & z_0 < z < 0 \\ C e^{-qz} + C_1 e^{qz}, & 0 < z < a \\ D e^{-q_\kappa z}, & a < z \end{cases} \tag{17}$$

where $q = k_\perp$, $q_\kappa = \sqrt{k_\perp^2 + \kappa^2}$. (For $z < z_0$ the solution is the first line of Eq. (15) where the resulting exponent of first exponential has changed sign.)

With continuous $\hat{\Phi}$ and $\partial\hat{\Phi}/\partial z$ as conditions, one finds for the coefficient of interest

$$D = \frac{4q e^{(q_\kappa - q)a}}{(q_\kappa + q)^2 (1 - A e^{-2qa})}, \qquad A = \left(\frac{q_\kappa - q}{q_\kappa + q}\right)^2 = \frac{\kappa^4}{(q_k + q)^4}. \tag{18}$$

With this the pair correlation function for $z_0 < 0$ and $z > a$ is

$$\hat{h}(k_\perp, z, z_0) = -2\pi\beta q_c^2 D e^{-q_\kappa(z - z_0)}. \tag{19}$$

4 Casimir force

Besides $\hat{h}$ the $\hat{\psi}_z'$ is needed to obtain the Casimir force f. In accordance with Eq. (10) the ionic pair interaction is $\psi = q_c^2/r$. Its full Fourier transform is $\tilde{\psi} = q_c^2/k^2$ which is consistent with Eq. (10). With $k^2 = k_\perp^2 + k_z^2$ this can be transformed backwards to obtain ($q = k_\perp$)

$$\hat{\psi}(k_\perp, z - z_0) = -2\pi\beta q_c^2 e^{-q_\kappa(z - z_0)}. \tag{20}$$

This is consistent with solution (15) for Φ. The derivative of (20) with respect to z is now together with expression (19) inserted in Eq. (6) to first obtain ($z - z_0 \to z_2 - z_1 = u_1 + u_2 + a$)

$$\begin{aligned} f &= \frac{\rho^2}{2\pi} \int_0^\infty (-2\pi\beta q_c^2) D (-2\pi q_c^2) \int_0^\infty \int_0^\infty e^{-(q_\kappa + q)(u_1 + u_2 + a)}\, du_1 du_2\, q\, dq \\ &= \frac{\kappa^4}{8\pi\beta} \int_0^\infty \frac{D e^{-(q_\kappa + q)a}}{(q_\kappa + q)^2} q\, dq = \frac{1}{2\pi\beta} \int_0^\infty \frac{A e^{-2qa}}{1 - A e^{2qa}} q^2\, dq. \end{aligned} \tag{21}$$

First one can note that this result is precisely result (3.44) in Ref. [8]. This is seen by some rearrangement of the latter result with the substitutions $\kappa_0 \to \kappa$, $k \to q/\kappa$, and $d \to a$ for dimensionality $\nu = 3$.

Expression (16) and result (21 may be simplified further with new variable of integration

$$q = \kappa \sinh t, \quad dq = \kappa \cosh t\, dt.$$

With this we have $q_\kappa + q = \kappa(\cosh t + \sinh t) = \kappa e^t$, $q_\kappa - q = \kappa e^{-t}$, and $A = e^{-4t}$ by which the Casimir force becomes

$$f = \frac{\kappa^3}{2\pi\beta} \int_0^\infty \frac{e^{-g(t)}}{1 - e^{-g(t)}} \sinh^2 t \cosh t\, dt. \tag{22}$$

where $g(t) = 4t + 2\kappa a \sinh t$.

For large separation a only small values of t will contribute, and one can put

$$g(t) = (2\kappa a + 4)t \quad \text{and} \quad \sinh^2 t \cosh t = t^2.$$

With this and expansion of the denominator the force becomes

$$f = \frac{\kappa^3}{2\pi\beta}\frac{2\zeta(3)}{(2\kappa a + 4)^3} = \frac{k_B T\zeta(3)}{8\pi a^3(1 + 2/(\kappa a))^3} = \frac{k_B T\zeta(3)}{8\pi a^3}\left(1 - \frac{6}{\kappa a} + \cdots\right). \tag{23}$$

The $\zeta(3)$ is the Riemann ζ-function, $\zeta(p) = \sum_{n=1}^{\infty} 1/n^p$.

As noted earlier [8, 10, 11] this is the ideal metal result for high temperatures when the transverse electric mode is absent. Also one sees that for large a the effective separation between the plates is increasesd by twice the Debye shielding length, i. e. $a \to a + 2/\kappa$. Thus for semiconductors the influence of free ions vanishes due to the increase of effective separation for decreasing ionic density. The small conductivity of semiconductors has been an issue of some controversy [15]. It has been argued that small concentrations of free ions in semiconductors should be neglected [16]. However, result (20) suggests that lack of influence for small ionic concentration is due to increased effective separation for vanishing κ.

When the plates are in contact, $a = 0$, the integral (19) can be evaluated exlicitly. With $1 - e^{-4t} = 4e^{-2t}\sinh t \cosh t$ one finds

$$f = \frac{\kappa^3}{8\pi\beta}\int_0^\infty e^{-2t}\sinh t\, dt = \frac{\kappa^3}{24\pi\beta}. \tag{24}$$

For an ionic system at low density this is precisely the contribution to the pressure (with opposite sign) from the ionic interaction (beyond the ideal gas pressure) in accordance with the virial integral (8).

5 Electrolytes in general

For higher densities and lower temperatures the direct correlation function c given by Eq. (10) will be modified. However, the crucial point is that for large $r \to \infty$ this expression is still valid while for small r there will be changes. On the scale of plate separation this change will be a term that can be regarded as a δ-function in $\mathbf{r}$-space such that

$$c(\mathbf{r}_2, \mathbf{r}_1) = c_0(r) + \tau\delta(\mathbf{r}_2 - \mathbf{r}_1) \tag{25}$$

where $c_0(r) = -\beta q_c^2/r$ and τ is a constant that will depend upon the local density. When the local density varies the OZ-equation (9) can be regarded as a matrix equation. Multiplying it from both left and right with ρ and adding ρ on both sides of it the equation after some rearrangement becomes

$$(1 - \rho c)\rho(1 + h\rho) = \rho. \tag{26}$$

Insertion of expression (25) then yields

$$(1 - \rho\tau - \rho c_0)\rho(1 + h\rho) = \rho.$$

$$\rho(1 + h\rho) = \frac{\rho_e}{1 - \rho_e c_0}, \quad \rho_e = \frac{\rho}{1 - \rho\tau}. \tag{27}$$

Thus the only change in the resulting pair corrrelation function $\rho h\rho$ is that ρ is replaced by an effective density ρ_e on the right hand side. In this way only the inverse shielding length is affected by which we get $\kappa^2 = 4\pi\beta\rho_e q_c^2$. But for large plate separations the Casimir force (23) does not depend upon κ by which the ideal metal result is generally valid for large separations for any electrolyte.

6 Summary

The Casimir force between a pair of parallell plates filled with ionic particles has been evaluated in the classical high temperature limit. To do so methods of classical statistical mechanics have been used. The pair correlation function is evaluated from which the average force between pairs of particles in different plates is found. When the plates are at contact the magnitude of the force equals the contribution to the pressure from the virial theorem. This latter result makes the force consistent with bulk pressure. The force found is the same as the one found earlier in Ref. [8] for charged particles at low density. There the force was evaluated on basis of the difference between surface and bulk densities. By the present approach it thus follows that this difference in densities can be neglected to leading order.

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Shear viscosity, relaxation and collision times in spherically symmetric spacetimes

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Abstract

We interpret as shear viscosity the anisotropic pressure that emerges in inhomogeneous spherically symmetric spacetimes described by the Lemaître–Tolman–Bondi (LTB) metric in a comoving frame. By assuming that local isotropic pressure and energy density satisfy a generic ideal gas equation of state, we reduce the field equations to a set of evolution equations based on auxiliary quasi–local variables. We examine the transport equation of shear viscosity from Extended Irreversible Thermodynamics and use a numerical solution of the evolution equations to obtain the relaxation times for the full and "truncated" versions. Considering a gas of cold dark matter WIMPS after its decoupling from the cosmic fluid, we show that the relaxation times for the general equation are qualitatively analogous to collision times, while the truncated version is inadequate to describe transient phenomena of transition to equilibrium [2].

1 Introduction.

It is a well known fact that dissipative effects in the context of General Relativity must comply with causality and stability requirements [1, 2, 3, 4]. Also, there is an evident theoretical connection between dissipative phenomena and anisotropy or inhomogeneity of self–gravitating sources. This emerges from the fact that heat flux and shear viscosity couple with the 4–acceleration, shear and spacelike gradients in their corresponding evolution (or transport) equations. Since bulk viscosity is the only dissipative stress compatible with global isotropy and homogeneity, most articles on dissipative cosmological sources deal with the effects of this stress in a Friedman–Lemaître–Robertson–Walker (FLRW) context [5]. However, the literature contains also a large number of studies of dissipative cosmological sources under anisotropic and inhomogeneous conditions, using Bianchi or Kantowski–Sachs models [6], involving heat flux [7] or shear viscosity with the Lemaitre–Tolman–Bondi metric [8, 9, 10] (see also [11] for inhomogeneous spacetimes with dissipative sources).

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Besides mathematical simplicity, the main justification for preferring a FLRW framework, or linear perturbations on a FLRW background, in cosmological studies is the conjecture (supported by observations) that the universe is approximately FLRW at a large "homogeneity" scale of 150-300 Mpc [12]. Thermal dissipation might play a minor role in these scales, as observations seem to reveal that cosmic dynamics is presently dominated by (apparently) non–thermal sources (cold dark matter and dark energy [12]). While dissipative phenomena of a thermal nature are relevant for understanding early universe interactions, inhomogeneity can be safely assumed to be very small in these conditions. Dissipative phenomena also arise in self–gravitating (and inhomogeneous) sources at local scales, either stellar or galactic [1]. In some cases (inter–stelar or inter–galactic clouds of ionized gas), characteristic velocities and energies are basically non–relativistic, but in other cases (gas accretion to compact objects or AGN's, jets, photon or neutrino transport) we can have non–trivial relativistic and ultra-relativistic effects in conditions of non–linear inhomogeneity [13]. However, as long as we ignore the fundamental physics of dark matter and dark energy, we can still try to probe theoretically the possibility of some forms of thermal dissipation in these sources and/or their interactions at the cosmic scale.

Fully general inhomogeneity requires numerical codes of high complexity, hence we offer in this article a compromise by looking at dissipative phenomena in spherically symmetric sources, which in spite of their obviously idealized nature, are still useful to examine non–linear phenomena that cannot be studied in a FLRW framework or with linear perturbations. By considering "LTB spacetimes" that generalize to nonzero pressure the well known LTB dust solutions [14], we obtain a class of spacetimes that can be fully described by autonomous first order evolution equations that can be well handled by simple numerical methods. These models are quite general and readily allow for an inhomogeneous generalization of a large number of known FLRW solutions. The reader can consult [14] for an extended and comprehensive discussion on these spacetimes and their physical and geometric properties.

The plan of the article is as follows. We describe in section 2 the basic features of LTB spacetimes [11, 14], in which the anisotropic pressure is considered as a shear viscous stress [8, 9, 10]. Assuming a conserved particle current and the entropy current associated with Extended Irreversible Thermodynamics, we derive in section 3 the full causal transport equation for shear viscosity [1, 2, 3, 4]. In section 4, we provide the "fluid flow" evolution equations for LTB spacetimes [15], equivalent to the field equations, in terms of suitably defined quasi–local variables [14]. In this description, the local thermodynamical state variables are gauge invariant "exact" perturbations of their quasi–local equivalents. In section 5 we specialize the evolution equations for a local equation of state corresponding to a generic ideal gas that covers the cases of (i) a classical ideal gas and (ii) the coupled mixture of a non–relativistic gas and radiation (the "radiative gas" [1, 3, 9, 10]). In section 6 we specialize the evolution and transport equations for the ideal gas, as a model of a gas of non–relativistic WIMPS after their decoupling from the cosmic fluid [1, 13, 16], when particle numbers are conserved. We evaluate the relaxation times for the full transport equation and for its "truncated" version (the Maxwell–Cattaneo equation). In section 7 we compare numerically these times with mean collision times, showing that they are qualitatively analogous in the relaxation time scale. These numerical examples also show that the truncated equation is inadequate to describe the transient phenomena of transition to equilibrium for gas of WIMPS. This result is analogous to that obtained for the decoupling of matter and radiation in the radiative gas [10]. We summarize the results obtained in section 8.

2 LTB spacetimes in the "fluid flow" description.

Spherically symmetric inhomogeneous dust sources are usually described by the well known Lemaître–Tolman–Bondi metric [11, 14]

$$ds^2 = -c^2 dt^2 + \frac{R'^2}{1-K} dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right). \tag{1}$$

where $R = R(ct, r)$, $R' = \partial R/\partial r$ and $K = K(r)$. A large class of spherically symmetric spacetimes follow at once by considering the most general source for (1) in a comoving frame ($u^a = \delta^a_0$), which is the energy–momentum tensor

$$T^{ab} = \mu\, u^a u^b + p\, h^{ab} + \Pi^{ab}, \tag{2}$$

where μ and p are the matter–energy density and the isotropic pressure, $h^{ab} = u^a u^b + g^{ab}$ is the induced metric of hypersurfaces T orthogonal to u^a, and Π^{ab} is the symmetric traceless tensor of anisotropic pressure. We will call "LTB' spacetimes" to all solutions of Einstein's equations for (1) and (2).

Besides the scalars μ and p, and the tensor Π^{ab}, the remaining basic covariant objects of LTB spacetimes are:

$$\begin{aligned}
\Theta &= \tilde{\nabla}_a u^a = \frac{2\dot{R}}{R} + \frac{\dot{R}'}{R'}, && \text{Expansion scalar,} && (3)\\
{}^3\mathcal{R} &= \frac{2(KR)'}{R^2 R'}, && \text{Ricci scalar of the } T, && (4)\\
\sigma_{ab} &= \tilde{\nabla}_{(a} u_{b)} - \frac{\Theta}{3} h_{ab}, && \text{Shear tensor,} && (5)\\
E^{ab} &= u_c u_d C^{abcd}, && \text{Electric Weyl tensor,} && (6)
\end{aligned}$$

where $\dot{R} = u^a \nabla_a R$, $\tilde{\nabla}_a = h_a^b \nabla_b$, and C^{abcd} is the Weyl tensor.

For spherically symmetric spacetimes, the symmetric traceless tensors σ^{ab}, Π^{ab} and E^{ab} can be expressed in terms of single scalar functions as

$$\sigma^{ab} = \Sigma\, \Xi^{ab}, \qquad \Pi^{ab} = \mathcal{P}\, \Xi^{ab}, \qquad E^{ab} = \mathcal{E}\, \Xi^{ab}, \tag{7}$$

where $\Xi^{ab} = h^{ab} - 3\eta^a \eta^b$ and $\eta^a = \sqrt{h^{rr}}\delta^a_r$ is the unit vector orthogonal to u^a and to the 2–spheres orbits of SO(3) parametrized by (θ, ϕ).

The field equations $G^{ab} = \kappa T^{ab}$ (with $\kappa = 8\pi G/c^4$) for (1) and (2) are numparts

$$\begin{aligned}
\kappa\, \mu\, R^2 R' &= \left[R(\dot{R}^2 + K) \right]', && (8)\\
\kappa\, p\, R^2 R' &= -\frac{1}{3} \left[R(\dot{R}^2 + K) + 2R^2 \ddot{R} \right]', && (9)\\
\kappa\, \mathcal{P} \frac{R'}{R} &= -\frac{1}{6} \left[\frac{\dot{R}^2 + K}{R^2} + \frac{2\ddot{Y}}{Y} \right]', && (10)
\end{aligned}$$

endnumpartsFrom (3), (8)–(10) and (7) we obtain the expressions for $\mathcal{E}$ and Σ in terms of metric functions

$$\Sigma = \frac{1}{3} \left[\frac{\dot{R}}{R} - \frac{\dot{R}'}{R'} \right], \qquad \mathcal{E} = -\frac{\kappa}{2} \mathcal{P} - \frac{\kappa}{6} \mu + \frac{\dot{R}^2 + K}{2R^2}. \tag{11}$$

The energy–momentum balance equations $\nabla_b T^{ab} = 0$ for (2) are numparts

$$\dot\mu = -(\mu+p)\,\Theta - \sigma_{ab}\Pi^{ab} = -(\mu+p)\,\Theta - 6\,\Sigma\,\mathcal{P}, \tag{12}$$

$$\tilde\nabla_a p = -\tilde\nabla_b \Pi^b{}_a, \qquad \Rightarrow \quad p' - 2\,\mathcal{P}' = 6\,\mathcal{P}\,\frac{R'}{R}, \tag{13}$$

endnumpartsso that pressure gradients are effectively supported by the anisotropic pressure.

Bearing in mind (7) and the remaining previous equations, all covariant objects (scalars and proper tensors) in LTB spacetimes can be fully characterized by the following set of local covariant scalars:

$$\{\mu,\ p,\ \mathcal{P},\ \Theta,\ \Sigma,\ \mathcal{E},\ {}^3\mathcal{R}\}. \tag{14}$$

Given the covariant "1+3" time slicing afforded by u^a, the evolution of these scalars can be completely determined by the following set of "fluid flow" scalar evolution equations [15] numparts

$$\dot\Theta = -\frac{\Theta^2}{3} - \frac{\kappa}{2}\,(\,\mu + 3p\,) - 6\,\Sigma^2, \tag{15}$$

$$\dot\mu = -\,(\mu+p)\,\Theta - 6\,\Sigma\,\mathcal{P}, \tag{16}$$

$$\dot\Sigma = -\frac{2\Theta}{3}\,\Sigma + \Sigma^2 - \mathcal{E} + \frac{\kappa}{2}\mathcal{P}, \tag{17}$$

$$\dot{\mathcal{E}} = -\frac{\kappa}{2}\dot{\mathcal{P}} - \frac{\kappa}{2}\,(\mu + p - 2\mathcal{P})\,\Sigma - 3\left(\mathcal{E} + \frac{\kappa}{6}\mathcal{P}\right)\left(\frac{\Theta}{3} + \Sigma\right), \tag{18}$$

endnumpartstogether with the spacelike constraints numparts

$$(p - 2\mathcal{P})\,' - 6\,\mathcal{P}\,\frac{R'}{R} = 0, \tag{19}$$

$$\left(\Sigma + \frac{\Theta}{3}\right)' + 3\,\Sigma\,\frac{R'}{R} = 0, \tag{20}$$

$$\frac{\kappa}{6}\left(\mu + \frac{3}{2}\mathcal{P}\right)' + \mathcal{E}\,' + 3\,\mathcal{E}\,\frac{R'}{R} = 0, \tag{21}$$

endnumpartsand the Friedman equation (or "Hamiltonian" constraint)

$$\left(\frac{\Theta}{3}\right)^2 = \frac{\kappa}{3}\,\mu - \frac{{}^3\mathcal{R}}{6} + \Sigma^2, \tag{22}$$

The system (15)–(22) is equivalent to the field plus conservation equations $\nabla_b T^{ab} = 0$ (equations (16) and (19)). However, this system requires an equation of state linking μ, p and $\mathcal{P}$ to become determined, and the time and radial derivatives (in general) do not decouple. Hence, we will consider in section 5 another set of equivalent (but easier to handle) scalar evolution equations.

3 Extended Irreversible Thermodynamics.

In order to arrive to a determined set of evolution equations, we need to prescribe an equation of state that is suitable for a given physical model. If the desired model is a thermal system, it is necessary to consider μ and p as thermodynamical scalars. In particular, a very useful system is the ideal gas associated with the following equilibrium equation of state [1, 8]

$$\mu = mc^2\, n + \frac{p}{\gamma - 1}, \qquad k\, T = \frac{p}{n} \tag{23}$$

where n is the particle number density for a gas whose particles have mass m, T is the temperature, k is Boltzmann's constant and γ is a constant.

For $\gamma = 5/3$, the generic equation of state (23) becomes [1, 8, 9, 10]

$$\mu = mc^2\, n + \frac{3}{2}\, p, \qquad k\, T = \frac{p}{n} \tag{24}$$

which is the equation of state of a non–relativistic limit of the classical ideal gas (Maxwell–Boltzmann gas). Another system that can be described by (23) is a suitable approximation to a mixture a non–relativistic and an ultra–relativistic gas [1, 3, 4, 8, 9, 10]:

$$\begin{aligned} \mu = m_{(\mathrm{nr})} n_{(\mathrm{nr})} c^2 + m_{(\mathrm{ur})} c^2 n_{(\mathrm{ur})} + \frac{3}{2} p_{(\mathrm{nr})} + 3 p_{(\mathrm{ur})}, \\ k\, T_{(\mathrm{nr})} = \frac{p_{(\mathrm{nr})}}{n_{(\mathrm{nr})}}, \quad k\, T_{(\mathrm{ur})} = \frac{p_{(\mathrm{ur})}}{n_{(\mathrm{ur})}}, \end{aligned} \tag{25}$$

where subindices $_{(\mathrm{nr})}$ and $_{(\mathrm{ur})}$ respectively stand for non–relativistic and ultra–relativistic. If we assume that $p_{(\mathrm{nr})} \ll p_{(\mathrm{ur})}$, but the non–relativistic gas provides the major contribution to rest mass $(m_{(\mathrm{nr})} n_{(\mathrm{nr})} \gg m_{(\mathrm{ur})} n_{(\mathrm{ur})})$, then (25) becomes

$$\mu = m_{(\mathrm{nr})} c^2\, n_{(\mathrm{nr})} + 3\, p_{(\mathrm{ur})}, \qquad k\, T_{(\mathrm{ur})} = \frac{p_{(\mathrm{ur})}}{n_{(\mathrm{ur})}}, \tag{26}$$

which is the equation of state (23) with $\gamma = 4/3$. In practice, one uses (26) to describe the so–called "radiative gas", which is a tightly coupled mixture of baryons and photons described as a single "dust plus radiation" fluid. In particular, since we neglect thermal motions of non–relativistic particles, $m_{(\mathrm{nr})} c^2\, n_{(\mathrm{nr})}$ could be the rest mass density of cold or "warm" dark matter and non–relativistic baryons, and so $m_{(\mathrm{nr})}$ could be taken as the mass of a neutralino or another supersymmetric DM particle candidate.

For either form (24) or (26), we will assume particle number conservation

$$n^a = n u^a, \qquad \nabla_a n^a = 0, \qquad \Rightarrow \qquad \dot{n} + n\,\Theta = 0, \tag{27}$$

hence, if we consider a dark or warm DM gas described by (24), we would be necessarily looking at dissipative effects after the "freeze out" era, when thermal equilibrium is no longer kept by particle annihilation [12, 13, 16]. On the other hand, considering the radiative gas model, then (26) with particle conservation is appropriate to describe the photon–electron interaction associated with Thomson or Compton scattering.

Since (2) contains anisotropic pressure, which is not involved in the equation of state (23), it is natural to consider this pressure as a shear viscous stress associated to irreversible processes, whether in the classical ideal gas of WIMPS (24) or in the radiative gas (26). Considering the fact that Extended Irreversible Thermodynamics (EIT) provides the most advanced theory complying with causality and stability [1, 2, 3, 4], we construct an entropy current S^a within the framework of this theory. Since $\dot{u}_a = 0$ for LTB spacetimes and the only dissipative stress is shear viscosity, the entropy current is

$$S^a = S\, n^a = \left[S^{(\mathrm{eq})} - \frac{c\,\tau\, \Pi_{ab}\Pi^{ab}}{2\eta\, n\, T} \right] n^a = \left[S^{(\mathrm{eq})} - \frac{3\, c\,\tau\, \mathcal{P}^2}{\eta\, n\, T} \right] n\, u^a \tag{28}$$

where we used (7), and τ, η are, respectively, the relaxation time and the coefficient of shear viscosity, while the specific entropy, $S^{(\mathrm{eq})}$, is given by the equilibrium Gibbs equation

$$T\mathrm{d}S^{(\mathrm{eq})} = \mathrm{d}\left(\frac{\mu}{n}\right) + p\mathrm{d}\left(\frac{1}{n}\right), \tag{29}$$

so its projection with respect to u^a and the balance equation (12) yield

$$n\,T\,\dot{S}^{(\mathrm{eq})} = -\sigma_{ab}\Pi^{ab} = -6\,\Sigma\,\mathcal{P}. \tag{30}$$

The condition $\nabla_a S^a \geq 0$, together with (30), leads to the transport equation for shear viscosity [1, 2, 3, 4]

$$c\,\tau\,h_a^c h_b^d \dot{\Pi}_{cd} + \Pi_{ab}\left[1 + \eta\,T\,\tilde{\nabla}_c\left(\frac{c\,\tau}{2\,\eta\,T}\,u^c\right)\right] + 2\,\eta\,\sigma_{ab} = 0, \tag{31}$$

where $\epsilon_0 = 0, 1$ is a "switch", so that (31) is the complete transport equation if $\epsilon_0 = 1$, and we get the "truncated" or Maxwell–Cattaneo equation if $\epsilon_0 = 0$. Using (3) and (7), equation (31) becomes the following scalar equation

$$c\,\tau\,\dot{\mathcal{P}} + \mathcal{P} + 2\,\eta\,\Sigma + \frac{\epsilon_0\,c\,\tau\,\mathcal{P}}{2}\left[\frac{\dot{\tau}}{\tau} - \frac{\dot{\eta}}{\eta} - \frac{\dot{T}}{T} + \Theta\right] = 0. \tag{32}$$

To apply EIT to the non–relativistic and radiative gases, we need to substitute the equation of state (23) and utilize the forms of the coefficient of shear viscosity for these gases. From [1, 2, 3, 4], we have

$$\eta = \alpha\,p\,c\,\tau, \qquad \alpha = \begin{cases} 1, & \text{non–relativistic ideal gas} \\ \frac{4}{5}, & \text{radiative gas} \end{cases} \tag{33}$$

Hence, inserting $p = nkT$ and the particle conservation law (27), the transport equation (32) becomes

$$c\,\tau\,\left[\dot{\mathcal{P}} + 2\,\alpha\,p\,\Sigma + \epsilon_0\mathcal{P}\frac{\dot{T}}{T}\right] + \mathcal{P} = 0, \tag{34}$$

which clearly reveals how the difference between the complete and truncated equations can be dynamically significant, as it involves the term $\mathcal{P}\,\dot{T}/T$. The entropy production subjected to the conservation law (27) follows readily from (28) and (30) as

$$\nabla_a S^a = n\dot{S} = 3\,k\,n\,\left[\frac{1}{c\,\tau} + (1-\epsilon_0)\frac{\dot{p}}{p}\right]\,\frac{\mathcal{P}^2}{\alpha\,p^2}, \tag{35}$$

where we used (33) and (34) to eliminate $\dot{\mathcal{P}}$.

In order to examine (28) and (34) we need to solve the field equations, or their equivalent "fluid flow" evolutions equations (15)–(22), which would render the functional forms of the involved thermodynamical scalars. We look at this matter in the following section.

4 Quasi–local evolution equations.

We can obtain an alternative set to the evolution equations (15)–(22) that is completely equivalent, but easier to deal with numerically [14]. This follows from using instead of the local scalars (14), the scalar representation given by quasi–local variables A_* defined by the map

$$\mathcal{J}_* : X(\mathcal{D}) \to X(\mathcal{D}), \qquad A_* = \mathcal{J}_*(A) = \frac{\int_0^r A R^2 R' \mathrm{d}x}{\int_0^r R^2 R' \mathrm{d}x}. \tag{36}$$

where $X(\mathcal{D})$ is the set of all smooth integrable scalar functions A defined in any spherical comoving region $\mathcal{D}$ of the hypersurfaces T orthogonal to u^a, containing a symmetry center marked by $r = 0$. The functions $A_* : \mathcal{D} \to \mathrm{R}$ that are images of $\mathcal{J}_*$ will be denoted by "quasi–local" (QL) scalars. In particular, we will call A_* the QL dual of A. See [14] for a comprehensive discussion of of the map (36).

Applying the map (36) to the scalars Θ and ${}^3\mathcal{R}$ in (3) and (4) we obtain their QL duals

$$\Theta_* = \frac{3\dot{R}}{R}, \qquad {}^3\mathcal{R}_* = \frac{6K}{R^2}. \tag{37}$$

Applying now (36) to μ and p, comparing with (8)–(9), and using (37), these two field equations transform into numparts

$$\left(\frac{\Theta_*}{3}\right)^2 = \frac{\kappa}{3}\mu_* - \frac{{}^3\mathcal{R}_*}{6}, \tag{38}$$

$$\dot{\Theta}_* = -\frac{\Theta_*^2}{3} - \frac{\kappa}{2}\left(\mu_* + 3p_*\right). \tag{39}$$

endnumpartswhich are identical to the FLRW Friedman and Raychaudhuri equations, but among QL scalars. These equations can be further combined to yield identically the FLRW energy balance equation:

$$\dot{\mu}_* = -\left(\mu_* + p_*\right)\Theta_*. \tag{40}$$

so that the QL scalars $\{\mu_*,\, p_*,\, \Theta_*\}$ effectively satisfy FLRW evolution laws.

In order to relate local scalars to their and QL duals, we introduce the following "relative deviations" or "perturbations"

$$\delta^{(A)} \equiv \frac{A - A_*}{A_*}, \quad \Rightarrow \quad A = A_*\left[1 + \delta^{(A)}\right]. \tag{41}$$

Therefore, all scalars A in (14) can be expressed in terms of their duals A_* and perturbations $\delta^{(A)}$:

$$\mu = \mu_*\left[1 + \delta^{(\mu)}\right], \quad p = p_*\left[1 + \delta^{(p)}\right], \quad \Theta = \Theta_*\left[1 + \delta^{(\Theta)}\right], \quad {}^3\mathcal{R} = {}^3\mathcal{R}_*\left[1 + \delta^{({}^3\mathcal{R})}\right], \tag{42}$$

whereas Σ, $\mathcal{P}$ and $\mathcal{E}$ follow as numparts

$$\Sigma = -\frac{1}{3}\left[\Theta - \Theta_*\right] = -\frac{1}{3}\,\Theta_*\,\delta^{(\Theta)}, \tag{43}$$

$$\mathcal{P} = \frac{1}{2}\left[p - p_*\right] = \frac{1}{2}\,p_*\,\delta^{(p)}, \tag{44}$$

$$\mathcal{E} = -\frac{\kappa}{6}\left[\mu - \mu_* + \frac{3}{2}(p - p_*)\right] = -\frac{\kappa}{6}\left[\mu_*\delta^{(\mu)} + \frac{3}{2}p_*\delta^{(p)}\right], \tag{45}$$

endnumpartswhich leads to an alternative QL scalar representation $\{A_*,\, \delta^{(A)}\}$ that it is fully equivalent to the local representation. We derive now the evolution and constraint equations for this representation.

From differentiating both sides of (36) and using (41), we can relate radial gradients of μ_*, p_* and $\mathcal{H}_*$ with their corresponding δ functions by

$$\frac{\Theta_*{}'}{\Theta_*} = \frac{3R'}{R}\,\delta^{(\Theta)}, \qquad \frac{\mu_*{}'}{\mu_*} = \frac{3R'}{R}\delta^{(\mu)}, \qquad \frac{p_*{}'}{p_*} = \frac{3R'}{R}\delta^{(p)}, \tag{46}$$

while (39) and (40) are evolution equations for $\dot\mu_*$ and $\dot\Theta_*$. Hence, the evolution equations for $\delta^{(\mu)}$ and $\delta^{(\Theta)}$ follow from the consistency condition $[A_*']^{\cdot} = [\dot A_*]'$ applied to (39), (40) and (46) for $A = \Theta_*$ and μ_*. The result is the following set of autonomous evolution equations for the QL scalar representation $\{A_*,\, \delta^{(A)}\}$: numparts

$$\dot\mu_* = -\,[\,1+w\,]\;\mu_*\,\Theta_*, \tag{47}$$

$$\dot\Theta_* = -\frac{\Theta_*^2}{3} - \frac{\kappa}{2}\,[\,1+3\,w\,]\;\mu_*, \tag{48}$$

$$\dot\delta^{(\mu)} = \Theta_*\,\left[\left(\delta^{(\mu)} - \delta^{(p)}\right)\,w - \left(1+w+\delta^{(\mu)}\right)\delta^{(\Theta)}\right], \tag{49}$$

$$\dot\delta^{(\theta)} = -\frac{\Theta_*}{3}\,\left(1+\delta^{(\Theta)}\right)\,\delta^{(\Theta)} + \frac{\kappa\mu_*}{6\,(\Theta_*/3)}\left[\delta^{(\Theta)} - \delta^{(\mu)} + 3w\,\left(\delta^{(\Theta)} - \delta^{(p)}\right)\right], \tag{50}$$

endnumpartswhere $w \equiv p_*/\mu_*$.

The constraints associated with these evolution equations are simply the spatial gradients (46), while the Friedman equation (or Hamiltonian constraint) is (38). Notice that (46) follow directly from differentiating the integral definition (36), so by using the QL variables we do not need to solve these constraints in order to integrate (47)–(50).

It is straightforwards to prove that the evolution equations (47)–(50) and the constraints (38) and (46) are wholly equivalent to the evolution equations (15)–(22) of the fluid flow formalism of Ellis, Bruni and Dunsbury [15]. It is also important to mention that the QL representation $\{A_*,\, \delta^{(A)}\}$ leads to a characterization of LTB spacetimes as exact, non–linear, gauge invariant and covariant perturbations over a FLRW formal background defined by the QL scalars A_*, which satisfy FLRW dynamics. See [14] for details.

5 Evolution equations for the generic ideal gas.

In order to integrate the system (47)–(50) we need to prescribe a relation between μ_*, p_* and $\delta^{(\mu)}$, $\delta^{(p)}$. Since we are interested in thermal dissipative phenomena characteristic of a hydrodynamical regime of short range interactions, the physically meaning full equation of state (23) is that relating local variables μ and p, and not QL variables. However, (23) is a linear functional relation, hence its validity as a local relation and the assumption of particle current conservation (27) are sufficient conditions to render (47)–(50) a fully determined system in which the QL variables are basically auxiliary variables (and the physical variables are the local ones).

Assuming the local equation of state (23) and using (41) with $A = \mu,\, n,\, p$ leads to the following conditions on the QL variables numparts

$$\mu_* = m\,c^2\,n_* + \frac{p_*}{\gamma-1}, \tag{51}$$

$$\delta^{(\mu)} = \frac{m\,c^2\,n_*}{\mu_*}\,\delta^{(n)} + \frac{p_*}{(\gamma-1)\,\mu_*}\,\delta^{(p)}, \tag{52}$$

endnumpartsUsing the particle numbers conservation law (27) with $n = n_*(1+\delta^{(n)})$, together with (51)–(52), transforms (47)–(50) into the fully determined system numparts

$$\dot{n}_* = -n_*\,\Theta_*, \tag{53}$$

$$\dot{p}_* = -\gamma\, p_*\,\Theta_*, \tag{54}$$

$$\dot{\Theta}_* = -\frac{\Theta_*^2}{3} - \frac{\kappa}{2}\left[mc^2\,n_* + \gamma_1\,p_*\right], \tag{55}$$

$$\dot{\delta}^{(n)} = -\left(1+\delta^{(n)}\right)\Theta_*\,\delta^{(\Theta)}, \tag{56}$$

$$\dot{\delta}^{(p)} = -\left(\gamma+\delta^{(p)}\right)\Theta_*\,\delta^{(\Theta)}, \tag{57}$$

$$\dot{\delta}^{(\theta)} = -\frac{1}{3}\left(1+\delta^{(\Theta)}\right)\Theta_*\,\delta^{(\Theta)} - \frac{\kappa}{6}\left[mc^2\,n_*\left(\delta^{(n)}-\delta^{(\Theta)}\right) + \gamma_1\,p_*\left(\delta^{(p)}-\delta^{(\Theta)}\right)\right], \tag{58}$$

endnumpartswith $\gamma_1 \equiv (3\gamma-2)/(\gamma-1)$. Once the system (53)–(58) is solved numerically for appropriate initial conditions (see appendices of [14]), we obtain the local variables p, Θ, Σ, $\mathcal{P}$ from (42), (43) and (44), while the temperature T follows readily as

$$k\,T = \frac{p}{n} = \frac{p_*\,\left[1+\delta^{(p)}\right]}{n_*\,\left[1+\delta^{(n)}\right]}, \tag{59}$$

With the help from (42), (43), (44), (53)–(58) and (59), the transport equation (34) reduces to the following two algebraic constraints defining the relaxation times for the full ($\epsilon = 1$) and truncated ($\epsilon = 0$) cases:

$$c\tau = \frac{3\delta^{(p)}\left(1+\delta^{(p)}\right)}{\Theta_*\delta^{(\Theta)}\,\left[4\alpha(\delta^{(p)})^2 + (3+8\alpha)\delta^{(p)} + 3\gamma + 4\alpha\right]}, \qquad \epsilon = 1, \tag{60}$$

$$c\tau = \frac{3\delta^{(p)}}{\Theta_*\,\left[((4\alpha+3)\delta^{(\Theta)} + 3\gamma)\delta^{(p)} + (4\alpha+3\gamma)\delta^{(\Theta)}\right]}, \qquad \epsilon = 0, \tag{61}$$

while the entropy production law (35) leads to

$$\dot{S} = \frac{3k(\delta^{(p)})^2}{4\alpha\,[1+\delta^{(p)}]^2}\left[\frac{1}{c\tau} + \frac{(\epsilon_0-1)\Theta_*[(\gamma+\delta^{(\Theta)})\delta^{(p)} + (1+\delta^{(\Theta)})\gamma]}{1+\delta^{(p)}}\right]. \tag{62}$$

However, substituting τ from either (60) or (61) into (62) we obtain the same expression of $\dot{S}$ for the full and truncated cases:

$$\dot{S} = \frac{k\,\Theta_*\,\delta^{(\Theta)}\,\delta^{(p)}\,[4\alpha\,(\delta^{(p)})^2 + (3+8\alpha)\,\delta^{(p)} + 4\alpha + 3\gamma]}{4\alpha\,[1+\delta^{(p)}]^3}. \tag{63}$$

Dissipative effects associated with shear viscosity for thermal systems associated with (23) can be now examined by using the numerical solution of (53)–(58) to calculate the relaxation time scale given by (60) or (61), as well as the entropy production $n\dot{S}$ from (62).

6 The gas of WIMPS

Dissipative phenomena associated with shear viscosity in spacetimes with LTB metrics have been studied mostly on the radiative gas model [9, 10] (but see [8]). In particular, the

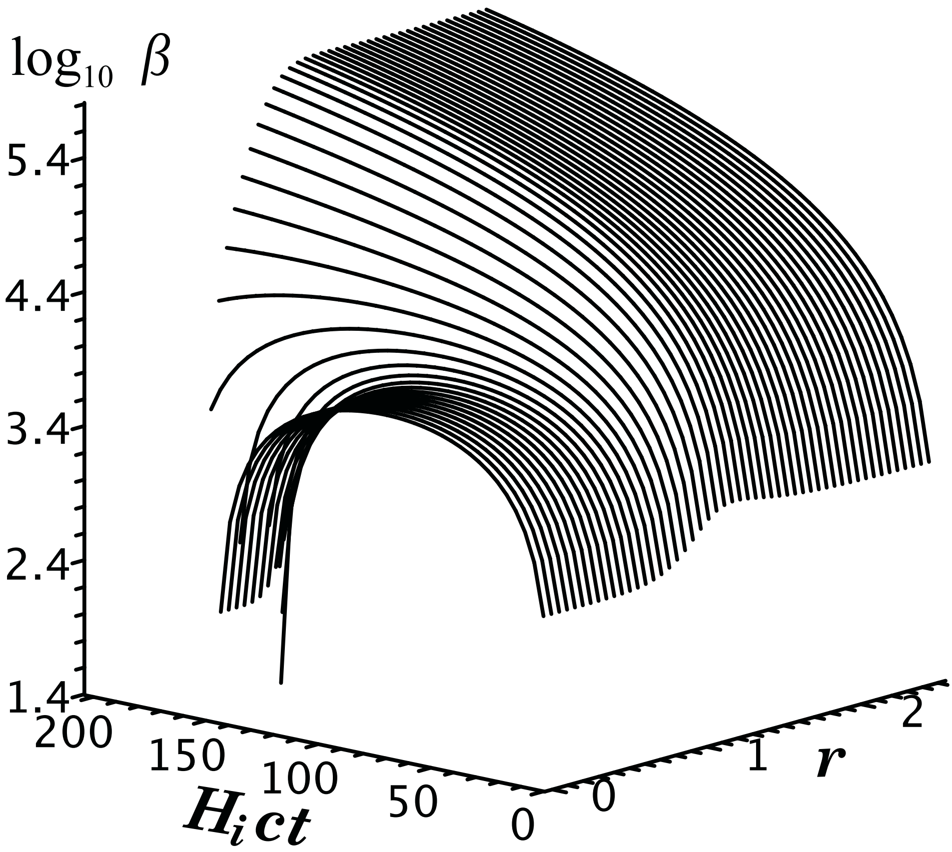
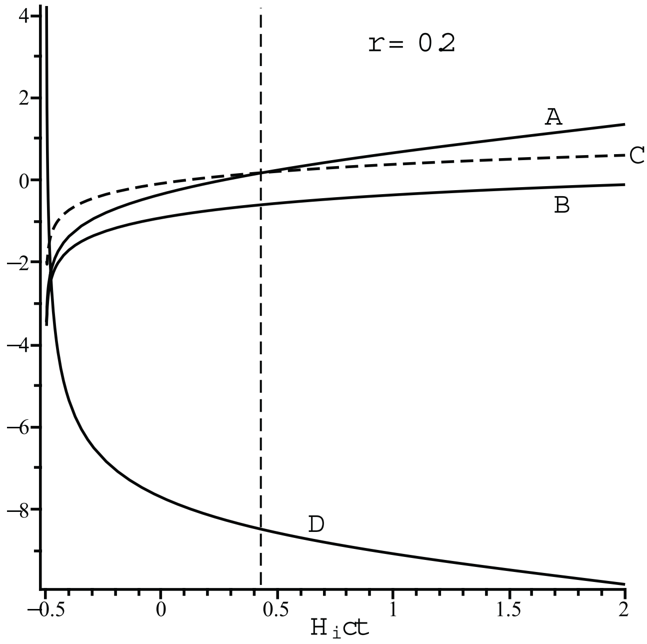
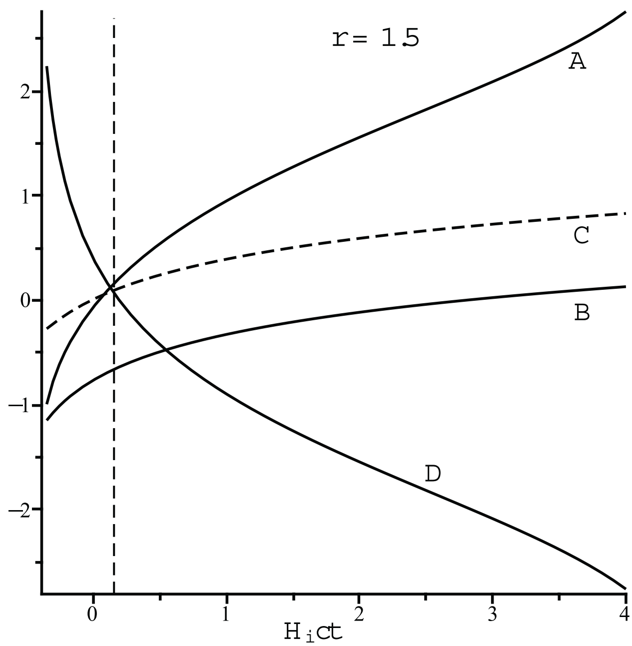
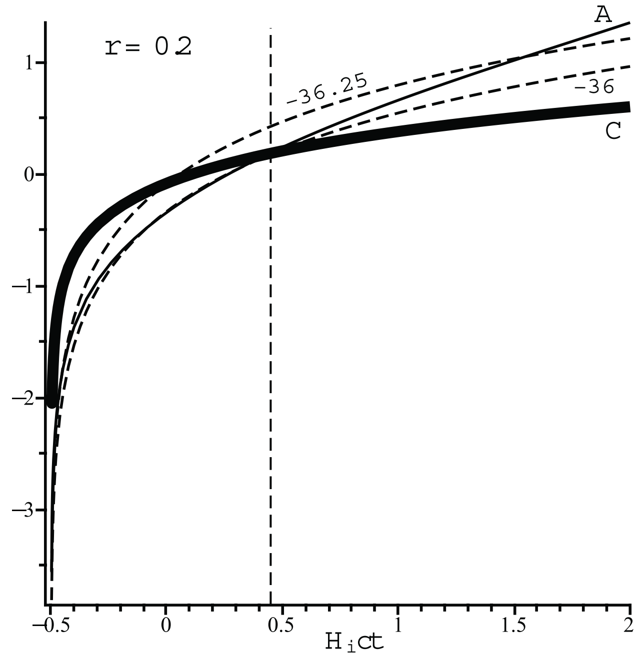
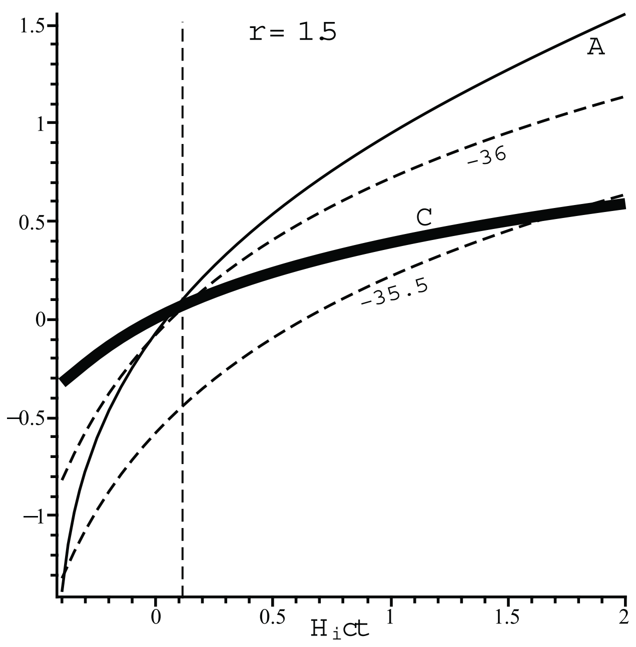


Figure 1: **Coldness parameter** β**.** The figure displays the function $\beta = mc^2/(kT)$ given by (71) for the ideal gas of WIMPS configuration described in section 7. The layers near the center ($r = 0$) bounce and collapse to a black hole where thermal motions dominate rest mass ($\beta \to 0$), though the hydrodynamical regime is no longer valid in this stage.


r= 0.2
A
C
B
D
4
2
0
–2
–4
–6
–8
–0.5
0
0.5
1
1.5
2
H_i ct


r= 1.5
A
C
B
D
2
1
0
–1
–2
0
1
2
3
4
H_i ct


r= 0.2
A
-36.25
-36
C
1
0
–1
–2
–3
–0.5
0
0.5
1
1.5
2
$H_i$ct


r= 1.5
A
-36
C
-35.5
1.5
1
0.5
0
–0.5
–1
0
0.5
1
1.5
2
$H_i$ct

comparison between relaxation and collision times was examined in [10] for this model in the context of the cosmological radiative era. In this article we consider the same issue, but for a gas of non–relativistic cold dark matter particles (WIMPS) after its decoupling or "freeze out" from the cosmic fluid, when thermal equilibrium is no longer maintained by particle annihilation [12, 13, 16]. Since cold dark matter has no effect on cosmic nucleosynthesis, this decoupling must have happened before nucleosynthesis at around $t \sim 200$ sec, and so the gas of WIMPS can be described as an ideal gas in the non–relativistic limit, corresponding to the equation of state (24). Hence, the expressions for the coefficient of shear viscosity, relaxation times and entropy production are (33), (51)–(63), for the the values $\gamma = 5/3$ and $\alpha = 1$.

The relaxation time is a mesoscopic quantity that could be, in principle, obtained by means of collision integrals [2], but cannot be given in terms of an "equation of state" relating macroscopic thermodynamical scalars. Usually, this quantity is taken simply as a mean collision time, or it is assumed to have the same order of magnitude value as these times. However, as shown by the results of [10] in the radiative gas model, there is no reason for this to be the case. Since these two time scales follow from physically distinct concepts, they must be different functions that could exhibit qualitatively analogous behavior and/or could be of the same order of magnitude.

For an ideal gas the mean collision time is given as [1, 13, 16]

$$ct_{\mathrm{col}} = \frac{1}{\sigma\, n} = \frac{1}{\sigma\, n_*\,(1+\delta^{(n)})}, \tag{64}$$

where $\sigma = \sigma(n, T)$ is the collision cross section area, whose precise functional form follows from the specific particle interactions involved in the gas model. For a gas of WIMPS, we can identify a decoupling stage as cosmic times $ct = ct_{\mathrm{D}}$ for which the reaction times compare with the Hubble expansion time t_H

$$n\,\sigma(n, T) \approx c\,t_H \sim \frac{3}{\Theta}, \tag{65}$$

so that for $t < t_{\mathrm{D}}$, before its decoupling from the cosmic fluid, σ is associated with particle pair annihilations and its form follows from theoretical considerations pertinent to supersymmetric cold dark matter candidate particles [13, 16]. Moreover, we will examine dissipative effects in the gas of WIMPS for $t > t_{\mathrm{D}}$, after this freeze out when particle numbers are conserved. The justification for these after freeze out dissipative processes comes from the assumption that there could have been dissipation in the earlier stage $t < t_{\mathrm{D}}$, and so it is reasonable to assume that once particle annihilations stop at (65), there should be a short timed relaxation process characterized by a weak self–interaction associated with very small cross section areas, so that after this process the fluid becomes completely non–collisional. Since this relaxation should be of short duration, we can model these cross section areas empirically by the simple ansatz [16, 13]

$$\sigma \sim 10^{s_0}\,\mathrm{cm}^2, \qquad -40 < s_0 < -34. \tag{66}$$

Hence, we expect τ in (60) and (61) to exhibit a qualitatively analogous behavior as (64) for cross section areas having magnitudes given by (66). In particular, the existence of an interaction that can be associated with shear viscosity requires that these time scales are of lesser magnitude than the Hubble time: $c\tau < 3/\Theta$ and $ct_{\mathrm{col}} < 3/\Theta$, with the relaxation time scale given by the cosmic time ct such that $c\tau \sim 3/\Theta$ and $ct_{\mathrm{col}} \sim 3/\Theta$, and thus, $c\,\tau \sim ct_{\mathrm{col}}$ at these cosmic time values. Hence, when $c\tau > 3/\Theta$ the gas expands in a non–collisional stage. However, for earlier times $c\tau$ and ct_{col} need not be the same function, just have comparable magnitudes. Also, for the relaxation time scale in which $c\tau < 3/\Theta$, we must have necessarily

$\dot{S} > 0$, so that there is entropy production with $\dot{S} \to 0$ as $c\tau$ and $ct_{\rm col}$ overtake $3/\Theta$ and entropy becomes a maximum associated with equilibrium conditions.

In order to test numerically these conditions, we define the following dimensionless variables associated with n_*, p_* and Θ_* in (53)–(58) (notice that the δ functions are already dimensionless):

$$x \equiv \frac{\kappa m\, c^2\, n_*}{3\, H_i^2}, \qquad y \equiv \frac{\kappa p_*}{3H_i^2}, \qquad z \equiv \frac{\Theta}{3\, H_i}, \tag{67}$$

where $H_i \sim 1/(ct_i)$ is taken as the Hubble scale factor for the initial time surface $t_i \sim 200$ sec., and we will consider $mc^2 = 100$ GeV to be the rest mass–energy of the WIMP. In terms of (67), the collision time (64) is given by

$$ct_{\rm col} = \frac{\kappa mc^2}{\sigma\, x\,(1+\delta^{(n)})} = 4.3\times 10^{-26}\times \frac{m\,c^2}{\rm GeV}\times\frac{\rm cm^2}{10^{s_0}}\times\frac{1}{x\,(1+\delta^{(n)})}. \tag{68}$$

We will examine in the following section these different time scales associated with the relaxation scale using the numeric solutions of (53)–(58).

7 Relaxation time scales: numeric results.

In order to set up appropriate initial conditions for x, y and z, we use equations (38) and (51) for $\gamma = 5/3$, leading to

$$z_i^2(r) = x_i(r) + y_i(r) - k_i(r), \qquad k_i(r) = \frac{[^3\mathcal{R}_*]_i}{6\, H_i^2}, \tag{69}$$

where the subindex $_i$ denotes evaluation at $t = t_i$. Initial conditions for a central over–density with small positive curvature that smoothy blends to a cosmic background with small negative spatial curvature can be achieved by choosing $k_i(r)$ as any smooth function for which $k_i(0) = 0.1$ and $k_i(r) \to -0.1$ for $r \to \infty$. Central and asymptotic values for x_i and y_i are given by

$$x_i(0) = 1.5, \qquad x_i(\infty) = 0.9, \qquad\qquad y_i(0) = 0.08, \qquad y_i(\infty) = 0.02. \tag{70}$$

The form of z_i follows from (69) and (70), while the forms for the initial value functions $[\delta^{(n)}]_i$, $[\delta^{(p)}]_i$ and $[\delta^{(\Theta)}]_i$ can be obtained from x_i, y_i, z_i by means of (46) evaluated at $t = t_i$ (see the appendices of [14]).

An important parameter in thermal systems is the "coldness" parameter

$$\beta = \frac{m\,c^2}{k\,T} = \frac{x\,[1+\delta^{(n)}]}{y\,[1+\delta^{(p)}]}, \tag{71}$$

which, with the numeric values of (70), takes initial values $\beta_i(0) \sim 20$ and $\beta_i(\infty) \sim 40$, which are reasonable values for cold dark matter WIMPS that are non–relativistic when they decouple at $t = t_{\rm D}$ [12, 16, 13]. We display in figure 1 the function β that results from the numeric solution of (53)–(58) for the configuration outlined above. As the configuration expands we can see how β increases for all r to clear non–relativistic values $\beta \gg 1$, but layers in the over-density region (around $r = 0$) collapse to a black hole at around $H_i ct \sim 150$, with $\beta \to 0$, indicating dominance of internal energy density over rest mass energy density near the final collapse. However, the hydrodynamical regime is no longer a valid approximation in this stage, as WIMP configurations do not evolve to black holes. In more realistic structure formation scenarios the WIMP gas becomes non–collisional and undergoes non–collisional relaxation phenomena, such as virialization [13], leading to stable bound structures.

The relaxation of a viscous dissipative stress requires that $\dot{S} > 0$ while $c\,\tau < 3/\Theta$, but both τ and $t_{\rm col}$ must overtake $3/\Theta$ as $\dot{S} \to 0$. We test these conditions numerically in figure 2, for two different values of r (at the over–density in the left panel and at the cosmic background in the right panel), and for the relaxation times of the full (60) and truncated (61) transport equations and for $\dot{S}$ given by (63). As shown by this figure, we have $c\,\tau < 3/\Theta$ for all times for the relaxation time (61) of the truncated equation. Therefore, the relaxation time for the truncated (Maxwell–Cattaneo) does not exhibit the appropriate behavior of a relaxation parameter, which means that the full transport equation is needed to provide an adequate description of the transient dissipative phenomena for the ideal gas of WIMPS. The same result was obtained for the radiative gas model in [10]. On the other hand, the relaxation time (60) of the full transport equation exhibits the expected behavior and overtakes the Hubble time $3/\Theta$ as $\dot{S} \to 0$. We show in figure 3 how the relaxation time (60) of the full transport equation (for $mc^2 = 100$ GeV) in the whole relaxation time scale is qualitatively analogous to collision times with cross sections given by (66) with $s_0 \sim -36$, which characterize expected weak interactions for decoupled WIMPS [12, 13, 16].

8 Conclusion

We have examined causal dissipation from shear viscosity in the context of a large class of inhomogeneous spherically symmetric spacetimes described by the LTB metric (1). A generic equation of state was suggested, which contains as particular cases the classical, non–relativistic, ideal gas, as well as the radiative gas in the approximation in which thermal motions of the non–relativistic species are ignored. We obtained a set of evolution equations equivalent to the field and balance equations, whose numeric solutions can be used to compute the relaxation times for the full and truncated transport equations, the rate of change of specific entropy and collision times for suitable cross section areas. We considered the non–relativistic ideal gas as an appropriate equation of state for a gas of cold dark matter WIMPS undergoing a transition to equilibrium soon after their freeze out and decoupling from the cosmic fluid at the outset of cosmic nucleosynthesis. The comparison between relaxation and collision times yielded similar results as those obtained in [10] with the radiative gas model, namely, that only the relaxation time from the full transport equation exhibits the expected behavior of a relaxation parameter, being also qualitatively analogous and of the same order of magnitude as collision times with reasonable cross sections for a gas of WIMPS. This result is shown in figures 2 and 3.

It is evident that the study of shear viscosity without other dissipative fluxes (heat flux and bulk viscosity) is an idealized situation which follows from the constraints of the LTB metric. Although the inhomogeneous conditions provided by this metric are mathematically tractable, they are not trivial and contain enough structure to examine non–linear effects that cannot be studied in a FLRW context or with linear perturbation. Another shortcoming is the use the transport equation itself to define the relaxation times, as it was done in [8, 9, 10], instead of using it as a free parameter to be specified. The resulting expressions (60) and (61) are, evidently, approximations to the actual relaxation times, but this approximation will be reasonable if the obtained quantities behave as a relaxation parameters. As shown in section 7 and in figures 2 and 3, the relaxation time for the full equation does exhibit the expected behavior, and so this approximation is reasonable. Future work along these lines would necessarily require a more general metric framework and more elaborated numerical methods. This work is presently under consideration.

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Finite-time future singularities in modified gravity

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Abstract

We review finite-time future singularities in modified gravity. We reconstruct an explicit model of modified gravity realizing a crossing of the phantom divide and show that the Big Rip singularity appears in the modified gravitational theory. It is also demonstrated that the (finite-time) Big Rip singularity in the modified gravity is transformed into the infinite-time singularity in the corresponding scalar field theory obtained through the conformal transformation. Furthermore, we study several models of modified gravity which produce accelerating cosmologies ending at the finite-time future singularities of all four known types [2].

1 Introduction

Recent observations confirmed that the current expansion of the universe is accelerating. There are two approaches to explain the current accelerated expansion of the universe. One is to introduce some unknown matter, which is called "dark energy" in the framework of general relativity. The other is to modify the gravitational theory, e.g., to study the action described by an arbitrary function of the scalar curvature R. This is called "$F(R)$ gravity", where $F(R)$ is an arbitrary function of the scalar curvature R (for reviews, see [1, 2]).

According to the recent various observational data, there exists the possibility that the effective equation of state (EoS) parameter, which is the ratio of the effective pressure of the universe to the effective energy density of it, evolves from larger than -1 (non-phantom phase) to less than -1 (phantom one, in which superacceleration is realized), namely, crosses -1 (the phantom divide) currently or in near future. Various attempts to realize the crossing of the phantom divide have been made in the framework of general relativity. Recently, a crossing of the phantom divide in modified gravity has also been investigated [1, 3, 4, 5]. Moreover, it is known that modified gravity may lead to the effective phantom/quintessence phase [1], while the phantom/quintessence-dominated universe may end up with finite-time future singularities, which can be categorized into four types [6].

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[2] This article is **dedicated to 70th aniversary of Professor Iver Brevik**

In the present article, we review finite-time future singularities in modified gravity. Following the considerations in Ref. [5], we reconstruct an explicit model of modified gravity realizing a crossing of the phantom divide by using the reconstruction method proposed in Refs. [7, 8]. We show that the Big Rip singularity appears in this modified gravitational theory, whereas that the (finite-time) Big Rip singularity in the modified gravity is transformed into the infinite-time singularity in the corresponding scalar field theory. Next, following the investigations in Ref. [9], we explore several examples of $F(R)$ gravity which predict the accelerating cosmological solutions ending at the finite-time future singularities. It is demonstrated that not only the Big Rip but other three types of the finite-time future singularities may appear.

This article is organized as follows. In Sec. II we explain the reconstruction method of modified gravity [7, 8]. Using this method, we reconstruct an explicit model of modified gravity in which a crossing of the phantom divide can be realized. We also consider the corresponding scalar field theory. In Sec. III we present several models of $F(R)$ gravity which predict accelerating cosmologies ending at the finite-time future singularities by using the reconstruction method. Finally, summary is given in Sec. IV. We use units in which $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by κ^2, so that $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2$, where $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\mathrm{GeV}$ is the Planck mass.

2 Modified gravitational theory realizing a crossing of the phantom divide

We investigate modified gravity in which a crossing of the phantom divide can be realized by using the reconstruction method.

2.1 Reconstruction of modified gravity

First, we review the reconstruction method of modified gravity proposed in Refs. [7, 8] (for the related study of reconstruction in $F(R)$ gravity, see [10]).

The action of $F(R)$ gravity with general matter is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}}\right], \tag{1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_{\mathrm{matter}}$ is the matter Lagrangian.

The action (1) can be rewritten to the following form by using proper functions $P(\phi)$ and $Q(\phi)$ of a scalar field ϕ:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[P(\phi)R + Q(\phi)\right] + \mathcal{L}_{\mathrm{matter}}\right\}. \tag{2}$$

The scalar field ϕ may be regarded as an auxiliary scalar field because ϕ has no kinetic term. Taking the variation of the action (2) with respect to ϕ, we obtain

$$0 = \frac{dP(\phi)}{d\phi}R + \frac{dQ(\phi)}{d\phi}, \tag{3}$$

which may be solved with respect to ϕ as $\phi = \phi(R)$. Substituting $\phi = \phi(R)$ into the action (2), we find that the expression of $F(R)$ in the action of $F(R)$ gravity in Eq. (1) is given by

$$F(R) = P(\phi(R))R + Q(\phi(R)). \tag{4}$$

Taking the variation of the action (2) with respect to the metric $g_{\mu\nu}$, we find that the field equation of modified gravity is given by

$$\frac{1}{2}g_{\mu\nu}\left[P(\phi)R+Q(\phi)\right]-R_{\mu\nu}P(\phi)-g_{\mu\nu}\Box P(\phi)+\nabla_\mu\nabla_\nu P(\phi)+\kappa^2 T^{(\mathrm{matter})}_{\mu\nu}=0\,, \tag{5}$$

where ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian for a scalar field, and $T^{(\mathrm{matter})}_{\mu\nu}$ is the contribution to the matter energy-momentum tensor.

We assume the flat Friedmann-Robertson-Walker (FRW) space-time with the metric $ds^2=-dt^2+a^2(t)d\vec{x}^2$, where $a(t)$ is the scale factor. In this background, the $(\mu,\nu)=(0,0)$ component and the trace part of the $(\mu,\nu)=(i,j)$ component of Eq. (5), where i and j run from 1 to 3, read

$$-6H^2P(\phi(t))-Q(\phi(t))-6H\frac{dP(\phi(t))}{dt}+2\kappa^2\rho=0\,, \tag{6}$$

$$2\frac{d^2P(\phi(t))}{dt^2}+4H\frac{dP(\phi(t))}{dt}+\left(4\dot{H}+6H^2\right)P(\phi(t))+Q(\phi(t))+2\kappa^2p=0\,, \tag{7}$$

respectively, where $H=\dot{a}/a$ is the Hubble parameter. Here, ρ and p are the sum of the energy density and pressure of matters with a constant EoS parameter w_i, respectively, where i denotes some component of the matters.

Eliminating $Q(\phi)$ from Eqs. (6) and (7), we obtain

$$\begin{aligned}&\frac{d^2P(\phi(t))}{dt^2}-H\frac{dP(\phi(t))}{dt}+2\dot{H}P(\phi(t))+\\&\kappa^2\left(\rho+p\right)=0\,.\end{aligned} \tag{8}$$

We note that the scalar field ϕ may be taken as $\phi=t$ because ϕ can be redefined properly.

We now consider that $a(t)$ is described as $a(t)=\bar{a}\exp\left(\tilde{g}(t)\right)$, where $\bar{a}$ is a constant and $\tilde{g}(t)$ is a proper function. In this case, Eq. (9) is reduced to

$$\begin{aligned}&\frac{d^2P(\phi)}{d\phi^2}-\frac{d\tilde{g}(\phi)}{d\phi}\frac{dP(\phi)}{d\phi}+2\frac{d^2\tilde{g}(\phi)}{d\phi^2}P(\phi)\\&\qquad+\kappa^2\sum_i\left(1+w_i\right)\bar{\rho}_i\bar{a}^{-3(1+w_i)}\exp\left[-3\left(1+w_i\right)\tilde{g}(\phi)\right]=0\,,\end{aligned} \tag{9}$$

where $\bar{\rho}_i$ is a constant and we have used $H=d\tilde{g}(\phi)/\left(d\phi\right)$. Moreover, it follows from Eq. (6) that $Q(\phi)$ is given by

$$Q(\phi)=-6\left[\frac{d\tilde{g}(\phi)}{d\phi}\right]^2P(\phi)-6\frac{d\tilde{g}(\phi)}{d\phi}\frac{dP(\phi)}{d\phi}+2\kappa^2\sum_i\bar{\rho}_i\bar{a}^{-3(1+w_i)}\exp\left[-3\left(1+w_i\right)\tilde{g}(\phi)\right]\,. \tag{10}$$

Hence, if we obtain the solution of Eq. (9) with respect to $P(\phi)$, then we can find $Q(\phi)$. Consequently, using Eq. (4), we can reconstruct $F(R)$ gravity for any cosmology expressed by $a(t)=\bar{a}\exp\left(\tilde{g}(t)\right)$. In Refs. [7, 11, 12], specific models unifying the sequence: the early-time acceleration, radiation/matter-dominated stage and dark energy epoch have been constructed.

Next, using the above method, we reconstruct an explicit model in which a crossing of the phantom divide can be realized. We start with Eq. (9) without matter:

$$\frac{d^2P(\phi)}{d\phi^2}-\frac{d\tilde{g}(\phi)}{d\phi}\frac{dP(\phi)}{d\phi}+2\frac{d^2\tilde{g}(\phi)}{d\phi^2}P(\phi)=0\;. \tag{11}$$

By redefining $P(\phi)$ as $P(\phi) = e^{\tilde{g}(\phi)/2}\tilde{p}(\phi)$, Eq. (11) is rewritten to

$$\frac{1}{\tilde{p}(\phi)}\frac{d^2\tilde{p}(\phi)}{d\phi^2} = 25e^{\tilde{g}(\phi)/10}\frac{d^2\left(e^{-\tilde{g}(\phi)/10}\right)}{d\phi^2}\,. \tag{12}$$

We explore the following model: $\tilde{g}(\phi) = -10\ln\left[(\phi/t_0)^{-\gamma} - C(\phi/t_0)^{\gamma+1}\right]$, where C and γ are positive constants, and t_0 is the present time. In this case, Eq. (12) is reduced to $(1/\tilde{p}(\phi))\left[d^2\tilde{p}(\phi)/\left(d\phi^2\right)\right] = 25\gamma(\gamma+1)/\phi^2$, which can be solved as $\tilde{p}(\phi) = \tilde{p}_+\phi^{\beta_+} + \tilde{p}_-\phi^{\beta_-}$. Here, $\tilde{p}_\pm$ are arbitrary constants and $\beta_\pm$ are given by $\beta_\pm = \left[1 \pm \sqrt{1+100\gamma(\gamma+1)}\right]/2$. From the above expression of $\tilde{g}(\phi)$, we find that $\tilde{g}(\phi)$ diverges at finite ϕ when $\phi = t_s \equiv t_0 C^{-1/(2\gamma+1)}$, which tells that there could be the Big Rip singularity at $t = t_s$. We only need to consider the period $0 < t < t_s$ because $\tilde{g}(\phi)$ should be real number. In this case, the Hubble rate $H(t)$ is given by

$$H(t) = \frac{d\tilde{g}(\phi)}{d\phi} = \left(\frac{10}{t_0}\right)\left[\gamma\left(\frac{\phi}{t_0}\right)^{-\gamma-1} + (\gamma+1)C\left(\frac{\phi}{t_0}\right)^{\gamma}\right]\left[\left(\frac{\phi}{t_0}\right)^{-\gamma} - C\left(\frac{\phi}{t_0}\right)^{\gamma+1}\right]^{-1}, \tag{13}$$

where it is taken $\phi = t$.

In the flat FRW background, even for modified gravity described by the action (1), the effective energy-density and pressure of the universe are given by $\rho_{\mathrm{eff}} = 3H^2/\kappa^2$ and $p_{\mathrm{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2$, respectively. The effective EoS parameter $w_{\mathrm{eff}} = p_{\mathrm{eff}}/\rho_{\mathrm{eff}}$ is defined as $w_{\mathrm{eff}} \equiv -1 - 2\dot{H}/\left(3H^2\right)$ [1]. For the case of $H(t)$ in Eq. (13), w_{eff} is expressed as $w_{\mathrm{eff}} = -1 + U(t)$, where

$$\begin{aligned} U(t) \equiv -\frac{2\dot{H}}{3H^2} = & -\frac{1}{15}\left[-\gamma + 4\gamma(\gamma+1)\left(\frac{t}{t_s}\right)^{2\gamma+1} + (\gamma+1)\left(\frac{t}{t_s}\right)^{2(2\gamma+1)}\right]\times \\ & \left[\gamma + (\gamma+1)\left(\frac{t}{t_s}\right)^{2\gamma+1}\right]^{-2}. \end{aligned} \tag{14}$$

For the case of Eq. (13), the scalar curvature $R = 6\left(\dot{H} + 2H^2\right)$ is expressed as

$$\begin{aligned} R = & \; 60\left[\gamma(20\gamma-1) + 44\gamma(\gamma+1)\left(\frac{t}{t_s}\right)^{2\gamma+1} + (\gamma+1)(20\gamma+21)\left(\frac{t}{t_s}\right)^{2(2\gamma+1)}\right]\times \\ & t^{-2}\left[1 - \left(\frac{t}{t_s}\right)^{2\gamma+1}\right]^{-2}. \end{aligned} \tag{15}$$

In deriving Eqs. (15) and (16), we have used $t_s = t_0 C^{-1/(2\gamma+1)}$.

When $t \to 0$, i.e., $t \ll t_s$, $H(t)$ behaves as $H(t) \sim 10\gamma/t$. In this limit, it follows from $w_{\mathrm{eff}} = -1 - 2\dot{H}/\left(3H^2\right)$ that the effective EoS parameter is given by $w_{\mathrm{eff}} = -1 + 1/(15\gamma)$. This behavior is identical with that in the Einstein gravity with matter whose EoS parameter is greater than -1.

On the other hand, when $t \to t_s$, we find $H(t) \sim 10/(t_s - t)$. In this case, the scale factor is given by $a(t) \sim \bar{a}(t_s - t)^{-10}$. When $t \to t_s$, therefore, $a \to \infty$, namely, the Big Rip singularity appears. In this limit, the effective EoS parameter is given by $w_{\mathrm{eff}} = -1 - 1/15 = -16/15$.

This behavior is identical with the case in which there is a phantom matter with its EoS parameter being smaller than -1. Thus, we have obtained an explicit model showing a crossing of the phantom divide.

It follows from $w_{\text{eff}} = -1 - 2\dot{H}/\left(3H^2\right)$ that the effective EoS parameter w_{eff} becomes -1 when $\dot{H} = 0$. Solving $w_{\text{eff}} = -1$ with respect to t by using $w_{\text{eff}} = -1 + U(t)$, namely, $U(t) = 0$, we find that the effective EoS parameter crosses the phantom divide at $t = t_{\text{c}}$ given by $t_{\text{c}} = t_s \left[-2\gamma + \sqrt{4\gamma^2 + \gamma/\left(\gamma+1\right)}\right]^{1/(2\gamma+1)}$. From Eq. (15), we see that when $t < t_{\text{c}}$, $U(t) > 0$ because $\gamma > 0$. Moreover, the time derivative of $U(t)$ is given by

$$\begin{aligned}\frac{dU(t)}{dt} = {} & -\frac{1}{15}\left[2\gamma\left(\gamma+1\right)\left(2\gamma+1\right)^2\right]\left[\gamma + \left(\gamma+1\right)\left(\frac{t}{t_s}\right)^{2\gamma+1}\right]^{-3} \times \\ & \left(\frac{1}{t_s}\right)\left(\frac{t}{t_s}\right)^{2\gamma}\left[1 - \left(\frac{t}{t_s}\right)^{2\gamma+1}\right].\end{aligned} \tag{16}$$

Eq. (17) tells that the relation $dU(t)/\left(dt\right) < 0$ is always satisfied because we only consider the period $0 < t < t_s$ as mentioned above. This means that $U(t)$ decreases monotonously. Thus, the value of $U(t)$ evolves from positive to negative. From $w_{\text{eff}} = -1 + U(t)$, we see that the value of w_{eff} crosses -1. Once the universe enters the phantom phase, it stays in this phase, namely, the value of w_{eff} remains less than -1, and finally the Big Rip singularity appears because $U(t)$ decreases monotonically.

As a consequence, $P(t)$ is given by $P(t) = \left\{\left(t/t_0\right)^{\gamma}/\left[1 - \left(t/t_s\right)^{2\gamma+1}\right]\right\}^5 \sum_{j=\pm} \tilde{p}_j t^{\beta_j}$. Using Eqs. (10), we obtain $Q(t) = -6H\left\{\left(t/t_0\right)^{\gamma}/\left[1 - \left(t/t_s\right)^{2\gamma+1}\right]\right\}^5 \sum_{j=\pm}\left(3H/2 + \beta_j/t\right)\tilde{p}_j t^{\beta_j}$.

When $t \to 0$, from $H(t) \sim 10\gamma/t$, we find $t \sim \sqrt{60\gamma\left(20\gamma-1\right)/R}$. In this limit, it follows from Eqs. (4) that the form of $F(R)$ is given by

$$\begin{aligned}F(R) \sim {} & \left\{\frac{\left[\frac{1}{t_0}\sqrt{60\gamma\left(20\gamma-1\right)}R^{-1/2}\right]^{\gamma}}{1 - \left[\frac{1}{t_s}\sqrt{60\gamma\left(20\gamma-1\right)}R^{-1/2}\right]^{2\gamma+1}}\right\}^5 \times \\ & R\sum_{j=\pm}\left\{\left(\frac{5\gamma-1-\beta_j}{20\gamma-1}\right)\tilde{p}_j\left[60\gamma\left(20\gamma-1\right)\right]^{\beta_j/2} R^{-\beta_j/2}\right\}.\end{aligned} \tag{17}$$

On the other hand, when $t \to t_s$, from $H(t) \sim 10/\left(t_s - t\right)$, we obtain $t \sim t_s - 3\sqrt{140/R}$. In this limit, it follows from Eqs. (4) that the form of $F(R)$ is given by

$$F(R) \sim \left[\frac{\left(J/t_0\right)^{\gamma}}{1 - \left(J/t_s\right)^{2\gamma+1}}\right]^5 R\sum_{j=\pm}\tilde{p}_j J^{\beta_j}\left\{1 - \sqrt{\frac{20}{7}}\left[\sqrt{\frac{15}{84}}t_s + \left(\beta_j - 15\right)R^{-1/2}\right]\frac{1}{J}\right\}, \tag{18}$$

where $J \equiv t_s - 3\sqrt{140/R}$. The above modified gravity may be considered as some approximated form of more realistic, viable theory. From Eq. (16), we see that in the above limit the scalar curvature diverges, and that the expression of $F(R)$ in (18) can be approximately written as

$$F(R) \approx \frac{2}{7}\left[\frac{1}{3\sqrt{140}\left(2\gamma+1\right)}\left(\frac{t_s}{t_0}\right)^{\gamma}\right]^5\left(\sum_{j=\pm}\tilde{p}_j t_s^{\beta_j}\right)t_s^5 R^{7/2}. \tag{19}$$

2.2 Corresponding scalar field theory

In this subsection, motivated by the discussion in Ref. [13], we consider the corresponding scalar field theory to modified gravity realizing a crossing of the phantom divide, which is obtained by making the conformal transformation of the modified gravitational theory. (In Ref. [14], the relations between scalar field theories and $F(R)$ gravity have been studied.) By introducing two scalar fields ζ and ξ, we can rewrite the action (1) to the following form [1]:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[\xi\left(R-\zeta\right)+F(\zeta)\right]+\mathcal{L}_{\mathrm{matter}}\right\}. \tag{20}$$

The form in Eq. (20) is reduced to the original expression in Eq. (1) by using the equation $\zeta = R$, which is derived by taking variation of the action (20) with respect to one auxiliary field ξ. Taking the variation of the form in Eq. (20) with respect to the other auxiliary field ζ, we obtain $\xi = F'(\zeta)$, where the prime denotes differentiation with respect to ζ. Substituting this equation into Eq. (20) and eliminating ξ from Eq. (20), we find

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(F'(\zeta)R+F(\zeta)-F'(\zeta)\zeta\right)+\mathcal{L}_{\mathrm{matter}}\right]. \tag{21}$$

This is the action in the Jordan-frame, in which there exists a non-minimal coupling between $F'(\zeta)$ and the scalar curvature R. We make the following conformal transformation of the action (21): $g_{\mu\nu} \to \hat{g}_{\mu\nu} = e^{\sigma}g_{\mu\nu}$, where $e^{\sigma} = F'(\zeta)$. Here, σ is a scalar field and a hat denotes quantities in the Einstein frame, in which the non-minimal coupling between $F'(\zeta)$ and the scalar curvature R in the first term on the right-hand side of Eq. (21) disappears. Consequently, the action in the Einstein frame is given by [15, 16]

$$S_{\mathrm{E}} = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2\kappa^2}\left(\hat{R}-\frac{3}{2}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma-V(\sigma)\right)+e^{-2\sigma}\mathcal{L}_{\mathrm{matter}}\right], \tag{22}$$

where $V(\sigma) = e^{-\sigma}\zeta(\sigma) - e^{-2\sigma}F\left(\zeta(\sigma)\right) = \zeta/F'(\zeta) - F(\zeta)/\left(F'(\zeta)\right)^2$ and $\hat{g}$ is the determinant of $\hat{g}^{\mu\nu}$. In deriving Eqs. (22), we have used $e^{\sigma} = F'(\zeta)$. In addition, $\zeta(\sigma)$ is obtained by solving $e^{\sigma} = F'(\zeta)$ with respect to ζ as $\zeta = \zeta(\varphi)$. Defining φ as $\varphi \equiv \sqrt{3/2}\sigma/\kappa$, the action (22) is reduced to the following form of the canonical scalar field theory:

$$S_{\mathrm{ST}} = \int d^4x\sqrt{-\hat{g}}\left[\frac{\hat{R}}{2\kappa^2}-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-V(\varphi)+e^{-2\sqrt{2/3}\kappa\varphi}\mathcal{L}_{\mathrm{matter}}\right]. \tag{23}$$

Taking the variation of the action (1) with respect to the metric $g_{\mu\nu}$, we find that the field equation of modified gravity is given by $F'(R)R_{\mu\nu} - (1/2)\,g_{\mu\nu}F(R) + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R) = \kappa^2 T^{(\mathrm{matter})}_{\mu\nu}$. When there is no matter, using the $(\mu,\nu) = (0,0)$ component and the trace part of the $(\mu,\nu) = (i,j)$ component of the above gravitational field equation, in the flat FRW background, we obtain

$$2\dot{H}F'(R)+6\left(-4H^2\dot{H}+4\dot{H}^2+3H\ddot{H}+\dddot{H}\right)F''(R)+36\left(4H\dot{H}+\ddot{H}\right)^2F'''(R)=0\,. \tag{24}$$

We now investigate the case in which $F(R)$ is given by $F(R) = c_1M^2\left(R/M^2\right)^{-n}$, where c_1 is a dimensionless constant and M denotes a mass scale. The form of $F(R)$ in Eq. (19) corresponds to the above expression with $n = -7/2$. In this case, the scale factor $a(t)$ and the scale curvature R are given by [13] $a(t) = \bar{a}\left(t_s - t\right)^{(n+1)(2n+1)/(n+2)}$ and $R = 6n(n+1)(2n+$

$1)(4n+5)(n+2)^{-2}(t_s-t)^{-2}$, respectively. It follow from $d\hat{t} = \pm e^{\sigma/2}dt$ that the relation between the cosmic time in the Einstein frame $\hat{t}$ and one in the Jordan frame is given by $\hat{t} = \mp\sqrt{-nc_1}\,(n+2)^n\,[6n(n+1)(2n+1)(4n+5)]^{-(n+1)/2}\,M^{n+1}\,(t_s-t)^{n+2}$. If $n<-2$, the limit of $t \to t_s$ corresponds to that of $\hat{t} \to \mp\infty$. For the case of Eq. (19), $n=-7/2$. Thus, we see that the Big Rip singularity does not appear in finite time for the scalar field theory, although it emerges in the corresponding modified gravitational theory. The metric in the Einstein frame is expressed as $d\hat{s}^2 = e^{\sigma}ds^2 = -d\hat{t}^2 + \hat{a}\left(\hat{t}\right)d\vec{x}^2$, where $\hat{a}(t)$ is the scale factor in the scalar field theory given by $\hat{a}(t) = \hat{\bar{a}}\hat{t}^{3[(n+1)/(n+2)]^2}$, where $\hat{\bar{a}}$ is a constant. For $n=-7/2$, because when $t \to t_s$, $\hat{t} \to \mp\infty$, the scale factor in the scalar field theory $\hat{a}(t)$ diverges at infinite time. Consequently, the (finite-time) Big Rip singularity in $F(R)$ gravity is transformed into the infinite-time singularity in the scalar field theory. This shows the physical difference of late-time cosmological evolutions between two mathematically equivalent theories.

3 Finite-time future singularities in $F(R)$ gravity

In this section, we examine several models of $F(R)$-gravity with accelerating cosmological solutions ending at the finite-time future singularities by using the reconstruction technique explained in the preceding section.

First, we consider the case of the Big Rip singularity [17], in which H behaves as $H = h_0/(t_s-t)$. Here, h_0 and t_s are positive constants and H diverges at $t=t_s$. In this case, if we neglect the contribution from the matter, the general solution of (9) is given by

$$P(\phi) = P_+\,(t_s-\phi)^{\alpha_+} + P_-\,(t_s-\phi)^{\alpha_-}\quad,\quad \alpha_\pm \equiv \frac{-h_0+1\pm\sqrt{h_0^2-10h_0+1}}{2}\,, \tag{25}$$

when $h_0 > 5+2\sqrt{6}$ or $h_0 < 5-2\sqrt{6}$ and

$$P(\phi) = (t_s-\phi)^{-(h_0+1)/2}\times \tag{26}$$
$$\left(A_1\cos\left((t_s-\phi)\ln\frac{-h_0^2+10h_0-1}{2}\right) + B_1\sin\left((t_s-\phi)\ln\frac{-h_0^2+10h_0-1}{2}\right)\right)\,,$$

when $5+2\sqrt{6} > h_0 > 5-2\sqrt{6}$. Here, P_+, P_-, A_1 and B_1 are constants. Using Eqs. (3), (4) and (10), we find that when R is large, the form of $F(R)$ is given by $F(R) \propto R^{1-\alpha_-/2}$ for $h_0 > 5+2\sqrt{6}$ or $h_0 < 5-2\sqrt{6}$ case and $F(R) \propto R^{(h_0+1)/4}\times$ (oscillating parts) for $5+2\sqrt{6} > h_0 > 5-2\sqrt{6}$ case.

Next, we investigate more general singularity $H \sim h_0\,(t_s-t)^{-\beta}$ [18], where h_0 and β are constants, and h_0 is assumed to be positive and $t < t_s$ because it should be for the expanding universe. Even for non-integer $\beta < 0$, some derivative of H and therefore the curvature becomes singular. We should also note that in this case the scale factor a behaves as $a \sim \exp\left\{[h_0/(\beta-1)]\,(t_s-t)^{-(\beta-1)} + \cdots\right\}$, where $\cdots$ expresses the regular terms. From this expression, we find that if β could not be any integer, the value of a, and therefore the value of the metric tensor, would become complex number and include the imaginary part when $t > t_s$, which is unphysical. This could tell that the universe could end at $t=t_s$ even if β could be negative or less than -1. We assume $\beta \neq 1$ because the case $\beta=1$ corresponds to the Big Rip singularity, which has been investigated. Furthermore, because the case $\beta=0$ corresponds to de Sitter space, which has no singularity, we take $\beta \neq 0$. When $\beta > 1$, the scalar curvature R behaves as $R \sim 12H^2 \sim 12h_0^2\,(t_s-t)^{-2\beta}$. On the other hand, when $\beta < 1$, the scalar curvature R behaves as $R \sim 6\dot{H} \sim 6h_0\beta\,(t_0-t)^{-\beta-1}$. We may obtain the asymptotic solution for P when $\phi \to t_s$:

(i) For $\beta > 1$, $P(\phi) \sim \exp\left\{[h_0/2(\beta-1)](t_s-\phi)^{-\beta+1}\right\}(t_s-\phi)^{\beta/2} \times$ $\times\left(A_2\cos\left(\omega(t_s-\phi)^{-\beta+1}\right)+B_2\sin\left(\omega(t_s-\phi)^{-\beta+1}\right)\right)$, where $\omega \equiv h_0/[2(\beta-1)]$, and A_2 and B_2 are constants. When $\phi \to t_s$, $P(\phi)$ tends to vanish.

(ii) For $1 > \beta > 0$, $P(\phi) \sim B_3\exp\left\{-[h_0/2(1-\beta)](t_s-\phi)^{1-\beta}\right\}(t_s-\phi)^{(\beta+1)/8}$, where B_3 is a constant.

(iii) For $\beta < 0$, $P(\phi) \sim A_3\exp\left\{-[h_0/2(1-\beta)](t_s-\phi)^{1-\beta}\right\}(t_s-\phi)^{-(\beta^2-6\beta+1)/8}$, where A_3 is a constant. Using Eqs. (3), (4) and (10), we find the behavior of $F(R)$ (at large R) as summarized in Table II.

In the above investigations, we found the behavior of the scalar curvature R from that of H. Conversely, we now consider the behavior of H from that of R. When R evolves as $R \sim 6\dot{H} \sim R_0(t_s-t)^{-\gamma}$, if $\gamma > 2$, which corresponds to $\beta = \gamma/2 > 1$, H behaves as $H \sim \sqrt{R_0/12}\,(t_0-t)^{-\gamma/2}$, if $2 > \gamma > 1$, which corresponds to $1 > \beta = \gamma-1 > 0$, H is given by $H \sim \{R_0/[6(\gamma-1)]\}(t_s-t)^{-\gamma+1}$, and if $\gamma < 1$, which corresponds to $\beta = \gamma-1 < 0$, we obtain $H \sim H_0 + \{R_0/[6(\gamma-1)]\}(t_s-t)^{-\gamma+1}$, where H_0 is an arbitrary constant and it does not affect the behavior of R. H_0 is chosen to vanish in $H \sim h_0(t_s-t)^{-\beta}$. If $\gamma > 2$, we find $a(t) \propto \exp\left[(2/\gamma-1)\sqrt{R_0/12}\,(t_s-t)^{-\gamma/2+1}\right]$, when $2 > \gamma > 1$, $a(t)$ behaves as $a(t) \propto \exp\left(\{R_0/[6\gamma(\gamma-1)]\}(t_s-t)^{-\gamma}\right)$, and if $\gamma < 1$, $a(t) \propto \exp\left(H_0t + \{R_0/[6\gamma(\gamma-1)]\}(t_s-t)^{-\gamma}\right)$. In any case, there appears a sudden future singularity [19] at $t = t_s$.

Since the second term in $H \sim H_0 + \{R_0/[6(\gamma-1)]\}(t_s-t)^{-\gamma+1}$ is smaller than the first one, we may solve Eq. (9) asymptotically as $P \sim P_0\left\{1+[2h_0/(1-\beta)](t_s-\phi)^{1-\beta}\right\}$ with a constant P_0, which gives $F(R) \sim F_0R + F_1R^{2\beta/(\beta+1)}$, where F_0 and F_1 are constants. When $0 > \beta > -1$, we find $2\beta/(\beta+1) < 0$. On the other hand, when $\beta < -1$, we obtain $2\beta/(\beta+1) > 2$. As we saw in $F(R) \propto R^{1-\alpha_-/2}$ above, for $\beta < -1$, H diverges when $t \to t_s$. Since we reconstruct $F(R)$ so that the behavior of H could be recovered, the $F(R)$ generates the Big Rip singularity when R is large. Thus, even if R is small, the $F(R)$ generates a singularity where higher derivatives of H diverge.

We assume that H behaves as $H \sim h_0(t_s-t)^{-\beta}$. For $\beta > 1$, when $t \to t_s$, $a \sim \exp\left[h_0(t_s-t)^{1-\beta}/(\beta-1)\right] \to \infty$ and ρ_{eff}, $|p_{\mathrm{eff}}| \to \infty$. If $\beta = 1$, we find $a \sim (t_s-t)^{-h_0} \to \infty$ and ρ_{eff}, $|p_{\mathrm{eff}}| \to \infty$. If $0 < \beta < 1$, a goes to a constant but ρ, $|p| \to \infty$. If $-1 < \beta < 0$, a and ρ vanish but $|p_{\mathrm{eff}}| \to \infty$. When $\beta < 0$, instead of $H \sim h_0(t_s-t)^{-\beta}$, one may assume $H \sim H_0 + h_0(t_s-t)^{-\beta}$. Hence, if $-1 < \beta < 0$, ρ_{eff} has a finite value $3H_0^2/\kappa^2$ in the limit $t \to t_s$. If $\beta < -1$ but β is not any integer, a is finite and ρ_{eff} and p_{eff} vanish if $H_0 = 0$ or ρ_{eff} and p_{eff} are finite if $H_0 \neq 0$ but higher derivatives of H diverge. We should note that the leading behavior of the scalar curvature R does not depend on H_0 in $H \sim H_0 + h_0(t_s-t)^{-\beta}$, and that the second term in this expression is relevant to the leading behavior of R. We should note, however, that H_0 is relevant to the leading behavior of the effective energy density ρ_{eff} and the scale factor a.

In Ref. [6], the finite-time future singularities has been classified as shown in Table I. The Type I corresponds to $\beta > 1$ or $\beta = 1$ case, Type II to $-1 < \beta < 0$ case, Type III to $0 < \beta < 1$ case, and Type IV to $\beta < -1$ but β is not any integer number. Thus, we have constructed several examples of $F(R)$ gravity showing the above finite-time future singularities of any type. It also follows from the reconstruction method that there appears Type I singularity for $F(R) = R + \tilde{\alpha}R^n$ with $n > 2$ and Type III singularity for $F(R) = R - \tilde{\beta}R^{-n}$ with

Table 1: Finite-time future singularities. Type I includes the case of ρ and p being finite at t_s. In case of Type IV, higher derivatives of H diverges. Type IV also includes the case in which p (ρ) or both of them tend to some finite values while higher derivatives of H diverge. Here, t_s is the time when a singularity appears and a_s is the value of $a(t)$ at $t = t_s$.

Type	Limit	a	ρ	$\vert p\vert$
Type I ("Big Rip")	$t \to t_s$	$a \to \infty$	$\rho \to \infty$	$\vert p\vert \to \infty$
Type II ("sudden")	$t \to t_s$	$a \to a_s$	$\rho \to \rho_s$	$\vert p\vert \to \infty$
Type III	$t \to t_s$	$a \to a_s$	$\rho \to \infty$	$\vert p\vert \to \infty$
Type IV	$t \to t_s$	$a \to a_s$	$\rho \to 0$	$\vert p\vert \to 0$

Table 2: Summary of the behavior of $F(R)$ gravity in case of $H \sim h_0 (t_s - t)^{-\beta}$. Here, $c_1 = [h_0/2(\beta - 1)] (12h_0)^{-(\beta-1)/(2\beta)}$ and $c_2 = [h_0/2(1-\beta)] (-6h_0\beta)^{(\beta-1)/(\beta+1)}$. We note that $-6h_0\beta R > 0$ when $h_0, R > 0$.

	Type I ("Big Rip")	Type II ("sudden")
β	$\beta > 1$	$-1 < \beta < 0$
$F(R)$	$F(R) \propto e^{c_1 R^{\frac{\beta-1}{2\beta}}} R^{-\frac{1}{4}}$	$F(R) \propto e^{-c_2 R^{\frac{\beta-1}{\beta+1}}} R^{\frac{\beta^2+2\beta+9}{8(\beta+1)}}$

Type III	Type IV
$0 < \beta < 1$	$\beta < -1$, β : not integer
$F(R) \propto e^{-c_2 R^{\frac{\beta-1}{\beta+1}}} R^{\frac{7}{8}}$	$F(R) \propto e^{-c_2 R^{\frac{\beta-1}{\beta+1}}} R^{\frac{\beta^2+2\beta+9}{8(\beta+1)}}$

$n > 0$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are constants. In fact, however, even if some specific model contains the finite-time future singularity, one can always reconstruct the model in the remote past in such a way that the finite-time future singularity could disappear. Positive powers of the curvature (polynomial structure) usually help to make the effective quintessence/phantom phase become transient and to avoid the finite-time future singularities. The corresponding examples have been examined in Refs. [18, 3].

4 Conclusion

In the present article, we have reviewed finite-time future singularities in modified gravity. We have reconstructed an explicit model of modified gravity realizing a crossing of the phantom divide. It has been shown that the Big Rip singularity appears in this modified gravitational theory, whereas that the (finite-time) Big Rip singularity in the modified gravity is transformed into the infinite-time singularity in the corresponding scalar field theory. In addition, we have examined several models of modified gravity which predict accelerating cosmologies ending at the finite-time future singularities of all four known types.

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The world of boundaries without Casimir effect

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Abstract

The Casimir effect is strongly dependent on the shape and structure of space boundaries. This dependence is encoded in the variation of vacuum energy with the different types of boundary conditions. The effect can be either atractive or repulsive. In the interfase between the both regimes we found a very interesting family of boundary conditions without Casimir effect. We characterize the types of boundary conditions for that do not induce any type of Casimir forces. The analytical characterization of all boundary conditions which do not generate any Casimir phenomenon of attraction or repulsion shows that the phenomenon is connected with the topological structure of the space of boundary conditons [4].

1 Introduction

Since the discovery of the Casimir effect [10] it is known that the phenomenon is highly dependent not only on the shape of the boundaries but also in the physical structure of the boundaries. Although the physical origin of the Casimir effect as a vacuum energy effect is well established [2, 3, 4, 5], there is not a clear intuition about the origin of the attractive or repulsive character of Casimir forces.

In this paper we analyze boundary conditions which are at the interface between attractive and repulsive regimes of the Casimir force, i.e. boundaries which do not undergo the Casimir effect. The analysis may be an useful tool for understanding the nature and origin of the Casimir force. The set boundary conditions without Casimir effect defines a subspace in the space of all consistent boundary conditions which was first introduced in QFT in Ref. [6].

This space of boundary conditions has a very interesting non-trivial topology with implications in the behavior of the Casimir energy [6]. In particular, for scalar field theories the

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variation of the Casimir energy under changes of boundary conditions reveals the existence of singularities generically associated to boundary conditions which either involve topology changes of the underlying physical space or edge states with unbounded below classical energy. The effect can be understood in terms of a new type of Maslov index associated to the non-trivial topology of the space of boundary conditions. In this space there are two subspaces (Cayley submanifolds) which are crossed by all non-contractible loops of boundary conditions. This fact permits the introduction of a Maslov index by bookkeeping the indexed number of such crossings by closed loops.

The new subspaces of boundary conditions with vanishing Casimir energy provide another splitting of the space of boundary conditions in two connected and simply connected sectors. In this sense the physical splitting of the space of boundary conditions in the sectors with attractive or repulsive Casimir forces is directly related to its topology.

2 Space of boundary conditions

In quantum theories the unitarity principle, imposes severe constraints on the boundary behaviour of quantum states in systems restricted to bounded domains [6]. The consistency of the quantum field theory requires, thus, a very stringent condition on the type of acceptable boundary conditions even in the case of massive theories in order to prevent this type of pathological behaviour of vacuum energy. In relativistic field theories, causality imposes further requirements [7]. The space of boundary conditions compatible with both constraints has interesting global geometric properties.

For a massless complex scalar free field ϕ the vacuum energy density is given by

$$\mathcal{E}_0 = \frac{1}{2}\,\mathrm{tr}\sqrt{\Delta} \tag{1}$$

Δ is the Laplace-Beltami operator $\Delta = -\ \partial^\mu\partial_\nu$. When the field theory is confined in a bounded domain Ω with smooth boundary $\partial\Omega$ the boundary conditions have to guarantee that Δ is a selfadjoint operator. The set $\mathcal{M}$ of all self-adjoint realizations of Δ is in one-to-one correspondence with the group of unitary operators of the boundary Hilbert space $L^2(\partial\Omega,\mathbb{C}^N)$. This correspondence is defined for any unitary operator $U\in\mathcal{U}(L^2(\partial\Omega,\mathbb{C}^N))$, by imposing on the fields the boundary condition

$$\varphi - i\partial_n\varphi = U\left(\varphi + i\partial_n\varphi\right)\ . \tag{2}$$

where φ. denotes the boundary value of ϕ and $\partial_n\varphi$ its normal derivative at the boundary $\partial\Omega$. This boundary condition defines a domain where Δ is a selfadjoint operator. Reciprocally, any selfadjoint realization of Δ on Ω is defined on this way [6].

In relativistic field theories, unitarity imposes further requirements [7, 8]. In particular Δ has not only to be a selfadjoint operator but also a positive operator. Otherwise, the quantum Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int d^3x\left(|\pi(x)|^2 + \varphi(x)\Delta\varphi(x)\right) \tag{3}$$

will not be a selfadjoint operator and the theory will be not unitary [8]. Two widely used boundary conditions, Dirichlet and Neumann, which correspond to $U=\mp\mathbb{I}$, respectively, satisfy these requirements. In the case of 1+1 dimensional theories defined on the space interval $\Omega=[0,L]\subset\mathbb{R}$, one finds in the space of self-adjoint boundary conditions $\mathcal{M}=U(2)$ periodic boundary conditions which correspond to $U=\sigma_x$ and also define a positive selfadjoint Laplacian Δ.

The identification of the space of boundary conditions $\mathcal{M}$ which define selfadjoints Laplacians with the unitary group $\mathcal{U}(L^2(\partial\Omega,\mathbb{C}^N))$ implies that it has a non-trivial fundamental homotopy group

$$\pi_1\ (\mathcal{M}) = \pi_1\ \left[\mathcal{U}(\ L^2(\partial\Omega,\mathbb{C}^N))\right] = \mathbb{Z}. \tag{4}$$

The non-simply connected structure of $\mathcal{M}$ disappears if we exclude any of the two Cayley submanifolds $\mathcal{C}_\pm$ defined by the unitary operators with al least one ± 1 eigenvalue. In fact, the generalized Maslov index of any loop of boundary conditions can be defined as the oriented sum of its crossings of the Cayley submanifold $\mathcal{C}_-$. However the subspace $\mathcal{M}\backslash\mathcal{C}_-$ is still a connected subspace of $\mathcal{M}$. Moreover, the two Cayley submanifolds are not disjoint because of the boundary conditions with eigenvalues $+1$ and -1, which includes boundary conditions that identify points of the boundary. The transition from normal boundary conditions to any of these conditions involves a topology change. Now, one interesting feature of the boundary conditions in the Cayley subspace $\mathcal{C}_-$ is that in its vicinity there exist a family of boundary conditions of Laplacians with unbounded below negative eigenvalues which correspond boundary states. This type of boundary conditions are not valid for quantum field theory because they will destroy unitarity of time evolution.

The subset of boundary conditions which are compatible with the general principles of quantum field theory is the subspace $\mathcal{M}_+$ of $\mathcal{M}$ which corresponds to positive hermitian Laplacian operators $\Delta \geq 0$. This is given by unitary matrices U whose eigenvalues $\lambda = \mathrm{e}^{i\alpha}$ are on the upper unit semi-circumference $0 \leq \alpha \leq \pi$. For a single real scalar field defined on the two-dimensional space-time $\mathbb{R} \times [0, L]$ the set of compatible boundary conditions is a four-dimensional manifold which can be covered by two charts parametrised by

$$L \begin{pmatrix} \dot\varphi(0) \\ \dot\varphi(L) \end{pmatrix} = A \begin{pmatrix} \varphi(0) \\ \varphi(L) \end{pmatrix} \tag{5}$$

where $A = -i(\mathbb{I} - U)/(\mathbb{I} + U)$ is any hermitian matrix with $A \geq 0$, and

$$\begin{pmatrix} \varphi(L) \\ L\dot\varphi(L) \end{pmatrix} = B \begin{pmatrix} \varphi(0) \\ L\dot\varphi(0) \end{pmatrix} \tag{6}$$

where $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any real matrix with $ad + bc = -1$, $ac \leq 0$ and $bd \leq 0$.

Notice that due to the bounded character of Ω the pathologies associated to massless theories in $1 + 1$ dimensions [10] do not appear [9, 11] for most of boundary conditions.

3 Casimir Energy

A scalar free field theory defined by a boundary condition of $\mathcal{M}_+$ has a unique vacuum state which in the functional Schrödinger representation corresponds to the Gaussian state

$$\Psi(\phi) = \mathcal{N}\ \mathrm{e}^{-\frac{1}{2}(\phi, \sqrt{\Delta}\,\phi)} \tag{7}$$

where $\mathcal{N}$ is a normalization constant. The corresponding energy density given by (1) is ultraviolet divergent but there are finite volume corrections to the vacuum energy density are which give rise to the Casimir effect.

The infrared properties of quantum field theory are very sensitive to boundary conditions [12]. In particular, the physical properties of the quantum vacuum state and the vacuum energy exhibit a very strong dependence on the type of boundary conditions.

In $1+1$ dimensions the space of physically consistent boundary conditions of a massless free field theory defined on the interval $[0,L]$ are given by the subspace $\mathcal{M}_+$ unitary matrices of of $U(2)$ with eigenvalues $\lambda = \mathrm{e}^{i\alpha}$ lying in the upper unit semi-circumference with $0 \leq \alpha \leq \pi$ [13]. In that case the energy levels of the Laplacian operator Δ are given by the zeros k_n of the spectral function

$$
\begin{aligned}
h_U(k) &= 4k \det U \cos kL - 2i(1+k^2) \det U \sin kL + 4k(U_{21}+U_{12}) \\
&-2i(1+k^2)\sin kL - 4k\cos kL + 2i(1-k^2)\mathrm{tr}U \sin kL
\end{aligned}
\tag{8}
$$

where $\lambda = k^2$. Notice that the spectral function is not only dependent on the invariants of the boundary unitary matrix $\det U$ and $\mathrm{tr}U$ but also in the entries $(U_{21}+U_{12})$, which implies that the spectrum of the quantum theory will be different for matrices with the same spectrum if they are non-equivalent as matrices. Therefore the vacuum energy can be given by

$$
E_0 = \frac{1}{4\pi i} \oint dz\, z\, \partial_z \log h_U(z) \tag{9}
$$

or equivalently

$$
E_0 = -\frac{1}{2\pi} \int_0^\infty dk\, k\, \frac{d}{dx} \log h_U(ik). \tag{10}
$$

The Casimir energy which is the finite volume correction of order $\mathcal{O}(1/L)$ of vacuum energy can be calculated by removing the leading order corrections of order $\mathcal{O}(L)$ and $\mathcal{O}(1)$ by subtracting the vacuum energy at a given reference size $L = L_0$ [14]

$$
\begin{aligned}
E_U^L - E_U^{L_0} = \; & -\frac{1}{2\pi}\int_0^\infty dk \left(k\, \frac{d}{dk} \log \frac{h_U^L(ik)}{(h_U^{L_0}(ik))^{\frac{L}{L_0}}} \right. \\
& \left. - \frac{L_0 - L}{L} \log(\det U + 1 + \mathrm{tr}U) \right)
\end{aligned}
\tag{11}
$$

when $\det U + 1 + \mathrm{tr}U \neq 0$. In the cases where this factort vanishes there is an similar formula which permits to obtain the Casimir. The calculation is straightforward in most of the cases. For instance, in the case of pseudo-periodic boundary conditions

$$
U_p = \cos\theta\sigma_x + \sin\theta\sigma_y = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} ; \varphi(L) = e^{i\theta}\varphi(0) \tag{12}
$$

we have that

$$
h_p = -8k(\cos kL - \cos\theta) \tag{13}
$$

and the Casimir energy

$$
E_0 = -\frac{\pi}{6L} \tag{14}
$$

is given by (e.g. see Ref. [15] and references therein)

$$
E_p = \frac{\pi}{L}\left(\frac{1}{12} - \min_{n\in\mathbb{Z}} \left(\frac{\theta}{2\pi} + n - \frac{1}{2} \right)^2 \right) \tag{15}
$$

which vanishes for $\theta = \pi\left(1 \pm \frac{1}{\sqrt{3}}\right) \pmod{2\pi}$. This is the simplest case of a boundary condition without Casimir effect. In the case of Dirichlet boundary conditions

$$
U_D = -\mathbb{I} \tag{16}
$$

$$h_D = -8i \sin kL \tag{17}$$

the Casimir energy is non null

$$E_D = \frac{\pi}{24L} \tag{18}$$

as in the case of Neumann

$$U = \mathbb{I} \tag{19}$$

$$h_N = -8ik^2 \sin kL \tag{20}$$

where the Casimir energy

$$E_N = D_D = \frac{\pi}{24L} \tag{21}$$

is the same.

A different family of boundary conditions is provided by quasi-periodic boundary conditions

$$\begin{aligned} &U_q = \cos \alpha \sigma_z + \sin \alpha \sigma_x \\ &\varphi(L) = \tan \frac{\alpha}{2}\, \varphi(0); \quad \partial_n \varphi(0) = \left(\tan \frac{\alpha}{2}\right)^{-1} \varphi(0), \end{aligned} \tag{22}$$

where

$$h_q = -8k(\cos kL - \sin \alpha). \tag{23}$$

The Casimir energy

$$E_q = \frac{\pi}{L}\left(\frac{1}{12} - \min_{n\in\mathbb{Z}}\left(\frac{\alpha}{2\pi} + n + \frac{1}{4}\right)^2\right) \tag{24}$$

shows the existence of a boundary condition with $\alpha = \pi\left(\frac{1}{2} \pm \frac{1}{\sqrt{3}}\right) \pmod{2\pi}$ which do not undergoes the Casimir effect.

Another family of quasi-periodic boundary conditions

$$\begin{aligned} &U_R = \begin{pmatrix} 0 & e^{i\alpha} \\ -e^{-i\alpha} & 0 \end{pmatrix} \\ &\partial_n \varphi(0) = -ie^{i\alpha}\, \varphi(L),\, \partial_n \varphi(L) = ie^{-i\alpha}\, \varphi(0) \end{aligned} \tag{25}$$

have an spectral function

$$h_R = -4i(1 + k^2) \sin kL + 8ik \sin \alpha \tag{26}$$

which induce a non-trival dependence of the Casimir energy on the parameter α which never vanishes.

Robin boundary conditions are defined by an unitary matrix which is proportional to the identity.

$$U_r = e^{i\alpha}\mathbb{I}: \quad \partial_n \varphi(0) = \tan \frac{\alpha}{2}\, \varphi(0),\ \partial_n \varphi(L) = \tan \frac{\alpha}{2}\, \varphi(L) \tag{27}$$

which smoothly interpolate between Dirichlet ($\alpha = \pi$) and Neumann ($\alpha = 2\pi$) boundary conditions. They are only physically consistent when α is restricted to the interval $\alpha \in [\pi, 2\pi]$.

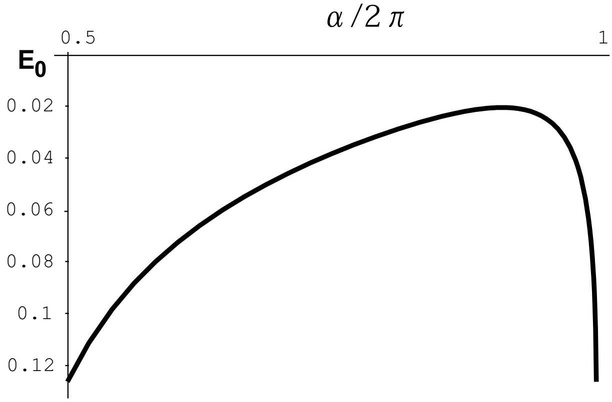

Figure 1. Casimir Energy for Robin boundary conditions

The spectral function in this case

$$h_r = 4e^{i\alpha}i\left(2\sin\alpha\cos kL - (1+k^2)\cos\alpha\sin kL + (1-k^2)\sin kL\right) \tag{28}$$

gives rise to the the following dependence of the Casimir energy in terms of the α parameter [16, 17, 18, 8], which again does not present any case with vanishing Casimir energy.

4 Boundary conditions with vanishing Casimir energy

The Casimir energy has a continuous dependence on the physical parameters which describe the space of compatible boundary conditions $\mathcal{M}_+$. The dependence is not completely smooth because of the presence of cusps for instance on the case of pseudo-periodic boundary conditions.

In the case of a single real massless scalar $\mathcal{M}_+$ is given by the boundary conditions (5)(6). The subspace of boundary conditions without Casimir effect is given by the unitary matrices in $\mathcal{M}_+$ which satisfy that

$$\lim_{L\to\infty}\int_0^\infty \frac{dk}{2\pi}\left(Lk\frac{d}{dk}\log\frac{h_U^{L_0}(ik)^{\frac{L}{L_0}}}{h_U^L(ik)} + \log(\det U + 1 + \mathrm{tr}U)^{(L_0-L)}\right) = 0 \tag{29}$$

Since the condition (29) is one single condition, the corresponding subspace $\mathcal{M}_+^0$ is a subspace of co-dimension one. We have explicitly found some boundary conditions belonging to this space. In particular, pseudo-periodic boundary conditions with $\theta = \pi\left(1 \pm \frac{1}{\sqrt{3}}\right)$ and quasi-periodic boundary conditions with $\alpha = \pi\left(\frac{1}{2} \pm \frac{1}{\sqrt{3}}\right)$ satisfy that property.

$\mathcal{M}_+^0$ is a 3-dimensional subspace of $\mathcal{M}_+$ which has a topological structure which differs from that of the Cayley subspaces $\mathcal{C}_\pm$. Its main property is that it splits the space of boundary conditions in two sectors according to the sign of the corresponding Casimir energies whereas $\mathcal{C}_\pm$ are connected manifodls. Moreover, its fundamental homotopy group $\pi_1\ of \mathcal{M}_+\backslash\mathcal{C}_\pm$ is trivial whereas $\pi_0\left(\mathcal{M}_+\backslash\mathcal{M}_+^0\right) = \mathbb{Z}_2$. Notice that most of closed loops of boundary conditions which cross $\mathcal{M}_=$ are not closed in $\mathcal{M}_+^0$ because contains boundary conditions which are not compatible with consistent quantum field theories.

Since our theories are massless they are classically conformal invariant but conformal invariance might be broken by boundary conditions [8, 13]. In particular, Robin boundary conditions break scale invariance. In 1+1 dimensions the only conditions which preserve conformal invariance are Dirichlet, Neumann, pseudo-periodic boundary conditions and quasi-periodic boundary conditions. All other boundary conditions flow towards any of these fixed points under the renormalization group flow. The behaviour of the renormalization group orbits around the fixed points is governed by the Casimir energy and presents different regimes. Dirichlet, Neumann and periodic boundary fixed points are stable whereas quasi-periodic and pseudo-periodic fixed points are in general unstable and marginally unstable, respectively.

For real scalar fields, Dirichlet, Neumann and periodic boundary conditions are the only stable points and the result holds for any dimension. Periodic boundary conditions, appear as attractors of systems with quasi-periodic and pseudo-periodic conditions. Any other boundary condition flows towards one of those fixed points. We have shown that among the conformally invariant boundary condition there are conditions without Casimir effect. This fact, illustrates a general feature of this type of special boundary conditions, they are conformally invariant and correspond to non-anomalous conformal theories

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Dark energy with time-dependent equation of state

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Abstract

The accelerating Friedmann flat universe, filled with an ideal fluid with a linear (oscillating) inhomogeneous equation of state (EoS) depending on time, is reviewed. The equations of motions are solved. It is shown that in some cases there appears a quasi-periodic universe, which repeats the cycles of phantom-type space acceleration [3].

1 Introduction

As is known, the present universe is subject to an acceleration, which can be explained in terms of an ideal fluid(dark energy) weakly interacting with usual matter and which has an uncommon equation of state. The pressure of such an ideal dark energy fluid is negative. In the present work we study a model where there is an ideal fluid with an inhomogeneous equation of state, $p = w(t)\rho + \Lambda(t)$ in which the parameters $w(t)$ and $\Lambda(t)$ depend linearly on time. In another version, these parameters are oscillating in time. Ideal fluids with an inhomogeneous equation of state were introduced in [1] (see also examples discussed in [2]). We show that, depending on the choices for the input parameters, it is possible for the universe to pass from the phantom era to the non-phantom era, implying the appearance of a cosmological singularity. Also possible are cases of quasi-periodic changes of the energy density and of the Hubble constant, with the appearance of quasi-periodic singularities, and the appearance of cosmological singularities.

The particular kind of equation of state in the present paper is one alternative amongst a variety of possibilities, proposed recently to cope with the general dark energy problem. Different examples include imperfect equation of state [4], general equation of state [5], inhomogeneous equation of state [1], [5] including time-dependent viscosity as a special case [6], and multiple-Lambda cosmology [7]. In the next section, a linear inhomogeneous EoS ideal fluid in a FRW universe is studied. In section 3, an oscillating inhomogeneous EoS ideal fluid is investigated. Some discussion is presented in the last section.

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2 Inhomogeneous equation of state for the universe and its solution

Let us assume that the universe is filled with an ideal fluid (dark energy) obeying an inhomogeneous equation of state (see also [8]),

$$p = w(t)\rho + \Lambda(t), \tag{1}$$

where $w(t)$ and $\Lambda(t)$ depend on the time t. This equation of state, when $\Lambda(t) = 0$ but with $\omega(t)$ a function of time, examined in [8] and [9].

Let us write down the law of energy conservation:

$$dot\rho + 3H(p + \rho) = 0, \tag{2}$$

and Friedman's equation:

$$\frac{3}{\chi^2}H^2 = \rho, \tag{3}$$

where ρ is the energy density, p- the pressure, $H = \frac{\dot{a}}{a}$ the Hubble parameter, $a(t)$- the scale factor of the three-dimensional flat Friedman universe, and χ - the gravitation constant.

Taking into account equations (1), (2) and (3), we obtain the gravitational equation of motion :

$$\dot{\rho} + \sqrt{3}\kappa(1 + \omega(t))\rho^{3/2} + \sqrt{3}\kappa\rho^{1/2}\Lambda(t) = 0. \tag{4}$$

Let us suppose in the following that both functions $w(t)$ and $\Lambda(t)$ linearly on time:

$$\omega(t) = a_1 t + b, \tag{5}$$

$$\Lambda(t) = ct + d. \tag{6}$$

This kind of behaviour may be the consequence of a modification of gravity (see [10] for a review).

Let $\Lambda(t) = 0$, $\omega(t) = a_1 t + b$. In this case the energy density takes the form:

$$\rho(t) = \frac{4(2a_1 + 1)^2}{3\chi^2} \cdot \frac{(a_1 t + b + 1)^{2/a_1}}{[(a_1 + b + 1)^{\frac{1}{a_1}+2} + S]^2}, \tag{7}$$

Hubble's parameter becomes:

$$H(t) = \frac{2}{3}(2a_1 + 1) \cdot \frac{(a_1 t + b + 1)^{1/a_1}}{(a_1 + b + 1)^{\frac{1}{a_1}+2} + S}, \tag{8}$$

where S is an integration constant.

The scale factor takes the following form:

$a(t) = \exp^{\frac{2}{3}(2a_1+1)I}$,

where

$$I = \frac{(-1)^{a_1}}{(2a_1+1)S^{a_1}} \ln|(a_1 t + b + 1)^{\frac{1}{a_1}+2} + S| - \frac{1}{(2a_1+1)S^{a_1}} \cdot \tag{9}$$
$$\sum_{k=0}^{a_1-1} \cos\frac{(a_1+1)(2k+1)}{2a_1+1} \cdot \pi \cdot \ln([(a_1 t + b + 1)^{\frac{2}{a_1}} + S]^2 -$$
$$2S(a_1 t + b + 1)^{\frac{1}{a_1}} \cdot \cos\frac{2k+1}{2a_1+1}\pi + S^2) + \frac{2}{(2a_1+1)S^{a}} \cdot$$
$$\sum_{k=0}^{a_1-1} sin\frac{(a_1+1)(2k+1)}{2a_1+1} \cdot \pi \cdot \arctan\frac{(a_1 t + b + 1)^{\frac{1}{a_1}} - S\cdot\cos\frac{2k+1}{2a_1+1}\cdot\pi}{S\cdot\sin\frac{2k+1}{2a_1+1}\cdot\pi},$$

$1 \le a_1 \le 2a_1 - 1$.

At $t_1 = -\frac{b+1}{a_1}$ or $t_2 = \frac{1}{a_1}(\frac{Sa_1}{a_1+1})^{\frac{1}{2+\frac{1}{a_1}}} - \frac{b+1}{a_1}$, one has $\dot{H} = 0$. With $a_1 > 0$, $b > -1$ and $t < t_2$, one gets $\dot{H} > 0$, that is, the accelerating universe is in the phantom phase (see, for example, [11]), and with $t > t_2$, one gets $\dot{H} < 0$, the universe is in the non-phantom phase. At the moment when the universe passes from the phantom to the non-phantom era, Hubble's parameter equals

$$H_m = \frac{2}{3S} \sqrt[2a_1+1]{a_1(a_1+1)^{2a_1}}. \tag{10}$$

In the phantom phase $\dot{\rho} > 0$ the energy density grows; in the non-phantom phase $\dot{\rho} < 0$ the energy density decreases. However, the derivative of the scale factor $\dot{a} > 0$, therefore the universe expands. Note, as has been shown in [8], that in the phantom phase the entropy may become negative. If $t \to +\infty$, then $H(t) \to 0$ and $\rho(t) \to 0$, so that the phantom energy decreases.

3 Inhomogeneous oscillating equation of state

Instead of assuming the form Eq. (5) and Eq. (6)for the time-dependent parameters, one might assume that there is an oscillating dependence on time. Let us investigate the following form: $w(t) = -1 + \omega_0 cos\omega t$. From equations (3) and (4) we get

$$H = \frac{2\omega}{3(\omega_1 + \omega_0 sin\omega t)}, \tag{11}$$

where ω_1 is an integration constant. If $|\omega_1| < \omega_0$, the denominator can in this case be zero, what implies a future cosmological singularity. But if $|\omega_1| > \omega_0$, singularity is absent.

In view of the fact that (cf. [8])

$$\dot{H} = -\frac{2\omega^2 \omega_0 cos\omega t}{3(\omega_1 + \omega_0 sin\omega t)}, \tag{12}$$

we see that with $\omega_0 cos\omega t < 0$ ($\omega_0 cos\omega t > 0$)the universe is located in the phantom (non-phantom) phase, corresponding respectively to $\dot{H} > 0$ ($\dot{H} < 0$). If the oscillation period of the universe is large, it is possible to have a unification of inflation and phantom dark energy [11, 12]. The density of dark energy takes the form

$$\rho(t) = \frac{4\omega^2}{3\chi^2(\omega_1 + \omega_0 sin\omega t)^2}, \tag{13}$$

this being a periodic function so that the universe oscillates between the phantom and non-phantom eras.

Let us assume now that $\Lambda(t) \neq 0$. For simplicity we take $\Lambda(t) = \Lambda_0 sin\omega t$, i.e. a periodic function. If $\Lambda_0(t) < 0$, equation (4) has the following solution:

$$\frac{\sqrt{\rho(t)} + \sqrt{\frac{|\Lambda_0|}{\omega_0}}}{\sqrt{\rho(t)} - \sqrt{\frac{|\Lambda_0|}{\omega_0}}} = exp[\sqrt{3\chi^2|\Lambda_0|\omega_0}(\frac{sin\omega t}{\omega} + C_1)], \tag{14}$$

where C_1 is an integration constant.

Finally, we obtain for the energy density:

$$\rho(t) = \{\frac{\sqrt{\frac{|\Lambda_0|}{\omega_0}} exp[\sqrt{3\chi^2|\Lambda_0|\omega_0}(\frac{sin\omega t}{\omega} + C_1)] + \sqrt{\frac{|\Lambda_0|}{\omega_0}}}{exp[\sqrt{3\chi^2|\Lambda_0|\omega_0}(\frac{sin\omega t}{\omega} + C_1)] - 1}\}. \tag{15}$$

The Hubble parameter becomes, according to (3),

$$H(t) = \sqrt{\frac{\chi^2\rho(t)}{3}}. \tag{16}$$

At the moments when the denominator of (15) is zero, the energy density diverges. This corresponds to a future cosmological singularity. Depending on the choice of parameters in the equation of state for the dark energy, $H(t)$ can thus correspond to either a phantom, or a non-phantom, universe. In both cases the universe expands with (quintessential or super) acceleration.

4 Summary

In this work we have studied a model of the universe in which there is a linear inhomogeneous equation of state, with a linear or oscillating dependence on time. The consequences of various choices of parameters in the linear functions are examined: there may occur a passage from the non-phantom era of the universe to the phantom era, resulting in an expansion and a possible appearance of singularities. In the absence of inhomogeneous terms it is possible to have a repetition of the passage process. When the universe goes from the phantom era to the non-phantom era with expansion one may avoid singularities, or there may appear singularities, but the passage occurs without repetition. The presence of a linear inhomogeneous term in the equation of state leads either to a compression of the universe in the evolution process or to a quasi-periodic change in the energy density and in the Hubble parameter, and also to a quasi-periodic appearance of singularities. By this the universe either passes into the non-phantom era, or stays within the same era as it was originally. Thus, the universe filled with an inhomogeneous time-dependent equation-of-state ideal fluid may currently be in the acceleration epoch of quintessence or phantom type. Moreover, it is easy to see that the effective value of the equation-of-state parameter may easily be adjusted so as to be approximately equal to -1 at present, what corresponds to current observational bounds.

5 Acknowledgment

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This article represents brief review of our paper [13].

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The inhomogeneous equation of state and the road towards the solution of the cosmological constant problem

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Abstract

We present a cosmological model containing a cosmological constant Λ and a component with an inhomogeneous equation of state. We study the form of the inhomogeneous equation of state for which the model exhibits the relaxation of the cosmological constant, i.e. it asymptotically tends to the de Sitter regime characterized by a small positive effective cosmological constant. The effect of the relaxation of the cosmological constant is observed both for negative and positive values of Λ and for a range of model parameters. A special emphasis is put on the study of the details of the CC relaxation mechanism and the robustness of the mechanism to the variation of model parameters. It is found that within the studied model the effective cosmological constant at large scale factor values is small because the absolute value of the real cosmological constant is large [2].

1 Introduction

The understanding of cosmology has undergone a phase of intensive development during the last decade. It has been largely propelled by the arrival of more precise and abundant observational data [1, 2, 3]. The picture of universe implied by the new data, however, revealed that the unknown part of the composition of the universe is far greater that previously believed. It is interesting to notice that the progress achieved in a great deal consists in establishing the fact that we know much less about our universe than we thought before the advent of the new observational data. The term unknown

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composition of the universe should not be understood too literally. It might really be the case that the universe contains a new physical component (or several of them), but it is also possible that the perceived unknown composition is just a manifestation of additional dimesions or modifications of interactions such as gravity.

One of the most striking consequences of the unknown part of the composition of the universe is its present accelerated expansion which started at redshift values of the order 1. The accelerated expansion of the present universe is strongly confirmed by the observational data. Presently available data, however, provide much less information on the actual cause of the acceleration. As already stated, the acceleration might happen owing to the existence of an unknown component with the negative pressure, referred to as *dark energy (DE)*, or might be a consequence of the fact that our universe has a number of macroscopic dimensions different than 4 or that the laws of gravitational interaction are modified at cosmological scales. These possibilities represent some of the most studied options leading to the accelerated expansion, but they certainly do not exhaust the list of proposals for the explanation of the accelerated expansion. The concept of dark energy seems especially useful in the modelling of the acceleration mechanisms. Namely, even if the acceleration is not due to some physical component with a negative pressure, the framework of dark energy can be used as a very good effective description of the alternative acceleration mechanisms which especially facilitates the comparisons of different approaches to the explanation of the accelerated expansion of the universe.

It is also important to stress that the unknown composition of the universe is not entirely connected to the accelerated expansion of the universe. The observational data imply that our universe contains a significant component of nonrelativistic matter, also called *dark matter (DM)* which is important for the explanation of the growth and formation of the structures such as galaxies and clusters of galaxies that we observe in the universe today. The nature of dark matter also has not yet been firmly established.

The dark energy component is primarily characterized by its negative pressure. A very large number of DE models have been proposed lately, of various degree of complexity, predictive potential and connection to fundamental physical theories [4]. The observational data still provide a lot of space for dynamical DE models, but the central place of the allowed parametric space is occupied by a very simple DE model, a so called *ΛCDM model*. This model assumes that the DE component is actually a small positive cosmological constant (CC). The cosmological constant is a well known concept present in the theory of general relativity (GR) from the very first years of the development of the theory [5, 6, 7]. An important observation is that the GR allows the existence of the CC, but it does not determine its size. So, in any cosmological model based on GR we are not concerned with the question whether the CC should be there or not, but with the problem of its size and sign. Since, therefore, the CC should already be an ingredient of cosmological model based on GR, it is very convenient to use such an object as a source of the acceleration of the universe. Indeed, an assumption of an existence of a small positive CC, together with the existence of dark matter, fits the observational data very well. From the observational side things work well: we have a model with few parameters that uses familiar concepts and fits the data well.

Fundamental quantum physical theories, however, predict various contributions to the observed value of the cosmological constant. In quantum field theory (QFT) there exist very large zero-point energy contributions for each of the quantum fields. There are also contributions from condensates such as Higgs condensate or QCD condensates. These contributions should be added to, in principle arbitrary value of CC allowed in GR. Any attempt of calculation of contributions to the cosmological constant reveals that their size is many orders of magnitude larger than the observed value of the CC. The number of orders of magnitude differs with the choice of effective QFT cutoff, but in any case we have differences which raise a lot of suspicion, to put it mildly. But, in the end, it is not that individual contributions matter, but their sum. In principle, for the classical contribution to the CC we can choose the needed value and reproduce the observed value of Λ. The problem is that all contributions have to cancel to very many decimal places for this mechanism to be effective. This problem, referred to as *fine-tuning* plagues this explanation of the observed value of the cosmological constant. It is sometimes also called "the old CC problem".

If theoretical considerations reveal such difficulties for the cosmological constant as a DE candidate, maybe we should opt for some of dynamical DE models. Even should the future observational data prove that dark energy (as a true component or as an effective representation of some other acceleration mechanism) is dynamical, it only relegates the CC problem to another level. Then we have to understand why the size of the CC is much smaller that the observed DE energy density or, possibly, why it is zero. The possibility that the CC is exactly zero would open the way to the solution of the CC problem by invoking some new symmetry. However, presently there is no proof for the existence of such a symmetry.

Therefore, the CC problem is difficult and it goes even beyond the issue of dark energy. Indeed, apart from the drastic problem of the size of the observed CC, there is another problem related to the cosmological constant. Namely, presently available observational data imply that the energy densities of nonrelativistic matter and the cosmological constant are of the same order of magnitude at present epoch of the expansion of the universe. The energy density of nonrelativistic matter scales very differently with the expansion than the CC. Therefore it is quite peculiar that these energy densities that have been very different in size in the past and will presumably be very different in the future (in the context of the ΛCDM model), are presently of comparable size. This problem is also called "coincidence problem". In this paper we shall mainly deal with "the old CC problem" whereas the "coincidence problem" will be only remotely commented.

This paper further elaborates the mechanism of the relaxation of the cosmological constant proposed in [8]. The relaxation of the cosmological constant is defined as a dynamical solution of the "old CC problem" without the fine-tuning of the parameters of the model. Essentially, in [8] we model the dark energy sector and study the asymptotic behavior at large scale factor values. The relaxation of the cosmological constant corresponds to the asymptotic de Sitter regime with a small positive effective cosmological constant. The mechanism is implemented in the framework of a cosmic component with an inhomogeneous equation of state [9]. In this paper we expand the model of [8], study some of its limitations and examine the robustness of the CC relaxation mechanism.

2 The cosmological constant relaxation model

We consider a two component cosmological model containing a cosmological constant with the energy density ρ_Λ and an additional cosmological component with the energy density ρ. It is assumed that the universe is spatially flat, $k = 0$. The expansion of the universe is defined by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}(\rho_\Lambda + \rho)\,. \tag{1}$$

The evolution of the second component with the expansion of the universe is defined by the standard equation of continuity

$$d\rho = -3(\rho + p)\frac{da}{a}\,, \tag{2}$$

where its equation of state (EOS) has a nonstandard form

$$p = w\rho - 3\zeta_0(H^2 + \beta)^\alpha\,. \tag{3}$$

Here we take $\zeta_0 > 0$ and α is an arbitrary real parameter. The form of EOS given in (3) fits into the framework of the inhomogeneous equation of state [9]. The concept of the inhomogeneous equation of state was also studied in [10, 11, 12, 13]. The role in the inhomogeneous DE equation of state in the process of structure formation was examined in [14]. In the remainder of this paper we study how the inhomogeneous equation of state contributes to the relaxation of the cosmological constant. The reference [9] (see especially the Appendix) shows that a possible way to understand the inhomogeneous equation of state is as an effective description of the modified gravity theories or time-dependent nonlinear viscosity.

The modified theories of gravity study the extension of GR as a possible source of the acceleration mechanism active at present cosmological era [15, 16]. An example of $f(R)$ gravity, free of instabilities, [15, 16, 17] was presented in [8] showing that the mechanism of the CC relaxation could be realized directly in $f(R)$ modified gravity theories. A systematic study of modified gravity theories consistent with the solar system precision gravity tests [18] is needed to establish the robustness of the CC relaxation mechanisms within the modified gravity theories.

The concept of bulk viscosity, as a dissipation mechanism of imperfect cosmic fluid consistent with the symmetries of the FRW universe, was used for the study of various phenomena in cosmology [19, 20, 21]. A very interesting possibility is that the bulk viscosity could potentially account for the present acceleration of the universe without the presence of dark energy [22, 23]. Here we consider generalization of the phenomenon of bulk viscosity. Namely, H is not a variable of state of the imperfect fluid and a general dependence of the fluid pressure on H represents a step out of standard framework of bulk viscosity. A more appropriate name would be time-dependent or nonlinear viscosity. This generalized concept of viscosity, however, proves to be very useful in the study of the present accelerated cosmic expansion [24], peculiar properties of dark energy, including the phenomenon of the CC boundary crossing [25] as well as other interesting phenomena [26, 27]. Furthermore, it is of particular interest to investigate how the concept of viscosity combines with the concepts of braneworlds and modified gravity [28, 29, 30, 31, 32, 33].

The expressions (1), (2) and (3) can be easily combined to obtain a dynamical equation for the evolution of the Hubble function

$$dH^2 + 3(1+w)\frac{da}{a}\left(H^2 - \frac{8\pi G\rho_\Lambda}{3} - \frac{8\pi G\zeta_0}{1+w}(H^2+\beta)^\alpha\right) = 0\,. \tag{4}$$

Further we scale all quantities of interest and introduce the following notation

$$h = (H/H_X)^2,\ \ s = a/a_X,\ \ \lambda = 8\pi G\rho_\Lambda/3H_X^2,\ \ \xi = 8\pi G\zeta_0 H_X^{2(\alpha-1)}/(1+w)\,,\ \ b = \beta/H_X^2\,. \tag{5}$$

Here $H(a_X) = H_X$. it is important to state that the value of a_X can in principle take any value. It is not intrinsically constrained within the present model. Using this notation we can rewrite (4) as

$$s\frac{dh}{ds} + 3(1+w)(h - \lambda - \xi(h+b)^\alpha) = 0\,, \tag{6}$$

with $h(1) = 1$ defining the initial condition.

A thorough analysis of the model for the case $b = 0$ was performed in [8] [3]. There it was shown that the CC relaxation mechanism was active for $\alpha < 0$. Here we proceed with the analysis of the same interval for α and examine the effects of the nonzero values of the parameter b.

[3]Note that in [8] it was convenient to choose the parameter α somewhat differently that in the present paper.

Next we consider the value $\alpha = -1$ as a representative and an analytically tractable case. The equation (6) now reads

$$\frac{(h+b)\, dh}{(h-h_{*1})(h-h_{*2})} = -3(1+w)\frac{ds}{s} \, . \tag{7}$$

Here h_{*1} and h_{*2} stand for the zeros of the denominator of the expression at the left hand side of (7). Their respective values are given by the following expressions:

$$h_{*1} = \frac{1}{2}\left(\lambda - b + \sqrt{(\lambda+b)^2 + 4\xi}\right) \tag{8}$$

and

$$h_{*2} = \frac{1}{2}\left(\lambda - b - \sqrt{(\lambda+b)^2 + 4\xi}\right) \, . \tag{9}$$

The differential equation (7) can be easily integrated and we arrive at the closed form solution for the dynamics of the scaled Huuble parameter with the scale factor:

$$\left(\frac{h-h_{*1}}{1-h_{*1}}\right)^{A_1} \left(\frac{h-h_{*2}}{1-h_{*2}}\right)^{A_2} = s^{-3(1+w)} \, , \tag{10}$$

Here $A_1 = (b+h_{*1})/(h_{*1}-h_{*2})$ and $A_2 = -(b+h_{*2})/(h_{*1}-h_{*2})$.

Before the analysis of the results presented above, we make a short summary of the results of paper [8] which correspond to a specific value $b = 0$. This is a starting point of our analysis since in this paper we are interested in how the nonvanishing value of parameter b modifies the CC relaxation mechanism observed in [8]. The dynamics of the Hubble function h depends on all model parameters α, λ, ξ and w.

The Hubble function h as a function of the scale factor for a case of negative λ with a large absolute value and other representative parameter values ($\xi > 0$, $\alpha < 0$ and $w > -1$) is given in Fig. 1. For these intervals of parameters the dynamics of h is characterized by a very abrupt transition between a phase of expansion at small scale factor values where $h \sim a^{-3(1+w)}$ and a de Sitter phase at large scale factor values characterized by a small effective positive CC with $h_{asym} \sim \Lambda_{eff} \sim \xi/|\lambda|$. With other parameters fixed, the scale of h before the transition grows with $|\lambda|$ and the value of Λ_{eff} decreases with $|\lambda|$. The dynamics of h before the transition is not affected by the size of ξ whereas its asymptotic value at large a grows with ξ. The choice of exponent α does not affect the behavior at small values of the scale factor, whereas the asymptotic value of h at large a decreases as α becomes more negative. Finally, as already stated, the value of w affects the behavior before the transition and the large a behavior of h does not depend on w. For other choices of parameters the model may exhibit other interesting types of dynamics which however do not correspond to the CC relaxation mechanism.

The dynamics of the Hubble function h for a large positive λ and representative parameter values ($\xi < 0$, $\alpha < 0$ and $w < -1$) is given in Fig. 2. In this regime, both for small and large scale factor values we find de Sitter regimes, $h \sim \lambda$ at small a and $h \sim -\xi/\lambda$ at large a. These asymptotic regimes are again interconnected by an abrupt transition. The scale of the de Sitter regime preceding the transition grows and the scale of de Sitter regime following the transition decreases with the size of λ. The value of α does not affect the behavior at small a values and the scale of the de Sitter regime at large a grows as α becomes more negative. The asymptotic value of h grows with the size of $|\xi|$ at large a whereas the dynamics at small a is not sensitive to the value of ξ. The value of parameter w does not affect the dynamics of h at large a whereas the approach to de Sitter regime at small a is sensitive to w. As in the case of negative λ, for other parameter intervals the behavior of the model is different and the CC relaxation mechanism is not effective. It is also important to stress that in the case of positive λ, the energy density ρ should be negative. This is a strong argument to consider the second component as an effective description of some other fundamental mechanism.

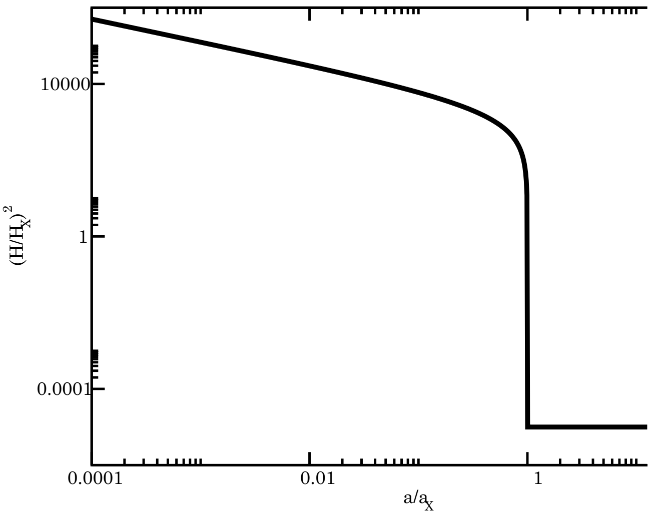

Figure 1: The dependence of h on the scale factor a for the negative cosmological constant in the regime $\alpha < 0$. The parameter values used are $b = 0$, $\alpha = -1$, $\lambda = -2000$, $\xi = 0.02$ and $w = -0.8$.

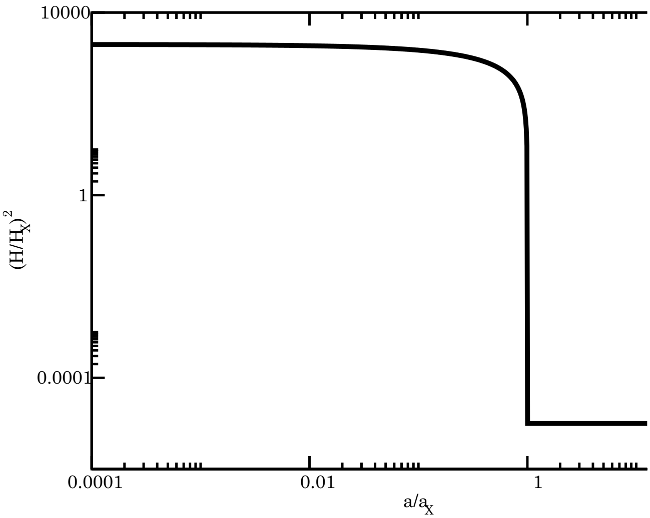

Figure 2: The dynamics of h as a function of the scale factor a for a positive λ and $\alpha < 0$. The values of the used parameters are $b = 0$, $\alpha = -1$, $\lambda = 2000$, $\xi = -0.02$ and $w = -1.2$.

Next we turn to the case of nonvanishing b. Since the values (8) and (9) determine the asymptotic behavior of the model, let us further study their dependence on the parameter b. We generally assume that the parameter $|\lambda|$ is by far the largest parameter of the model. More precisely, we suppose that $|\lambda|^2 \gg |\xi|$ and $|\lambda| \gg |b|$. These assumptions allow us to make an expansion of the square root terms in (8) and (9) with the following results:

$$h_{*1} = \frac{1}{2}(\lambda + |\lambda|) + \frac{b}{2}\left(\frac{|\lambda|}{\lambda} - 1\right) + \frac{b^2 + 4\xi}{4|\lambda|}\,, \tag{11}$$

$$h_{*2} = \frac{1}{2}(\lambda - |\lambda|) + \frac{b}{2}\left(-\frac{|\lambda|}{\lambda} - 1\right) - \frac{b^2 + 4\xi}{4|\lambda|}\,. \tag{12}$$

The expressions differ for the cases of positive and negative λ. For $\lambda > 0$ we obtain

$$h_{*1} = \lambda + \frac{b^2 + 4\xi}{4|\lambda|}\,, \tag{13}$$

$$h_{*2} = -b - \frac{b^2 + 4\xi}{4|\lambda|}\,. \tag{14}$$

On the other hand, for $\lambda < 0$ we have

$$h_{*1} = -b + \frac{b^2 + 4\xi}{4|\lambda|}\,, \tag{15}$$

$$h_{*2} = \lambda - \frac{b^2 + 4\xi}{4|\lambda|}\,. \tag{16}$$

$>$From the expressions (13) to (16) we can determine the effect of the parameter b on the asymptotic values.

For a positive λ, at very small values of $|b|$ when $|b| \ll |\xi/\lambda|$ we have $h_{*1} \simeq \lambda$ and $h_{*2} \simeq -\xi/\lambda$. On the other hand, for a sufficiently large $|b|$, where $|b| \gg |\xi/\lambda|$, we have $h_{*1} \simeq \lambda$ and $h_{*2} \simeq -b$. The asymptotic behavior at large scale factor values is determined by the value h_{*2} and we see that at sufficiently large values of $|b|$, in the sense defined above, the parameter b determines the asymptotic behavior of the Hubble function h. It is important to notice that in this case de Sitter regime at large values of the scale factor is realized only for negative values of b.

For negative values of λ at small values for the parameter b, with $|b| \ll |\xi/\lambda|$ we obtain and $h_{*1} \simeq \xi/|\lambda|$ and $h_{*2} \simeq \lambda$. For $|b| \gg |\xi/\lambda|$ we further have $h_{*1} \simeq -b$ and $h_{*2} \simeq \lambda$. These results again show that for a sufficiently large value of $|b|$, this parameter determines h_{*1} which in turn controls the asymptotic dynamics of the Hubble function at large scale factor values. Again, to have a de Sitter regime at large scale factor values b has to be negative.

A more careful analysis of the model immediately shows that for $h \rightarrow -b$ the pressure of the component with the inhomogeneous EOS diverges. The dynamics of the model reveals that this singular point is never reached. The expressions (8) and (9) show that the point where p would diverge is never reached during the evolution of the model. For a large $|\lambda|$, the dynamics of h stabilizes at a value slightly above $-b$.

3 Discussion

The principal role of the considerations given above and the plots in Figures 3 and 4 is to gauge the role of the parameter b in the model defined by (1)-(3). Our primary goal is to find out if and in which extent does the finite value of b change the behavior of the model compared to the previously studied case corresponding to $b = 0$. The qualitative behavior of the model retains the pattern observed for the vanishing value of b: both

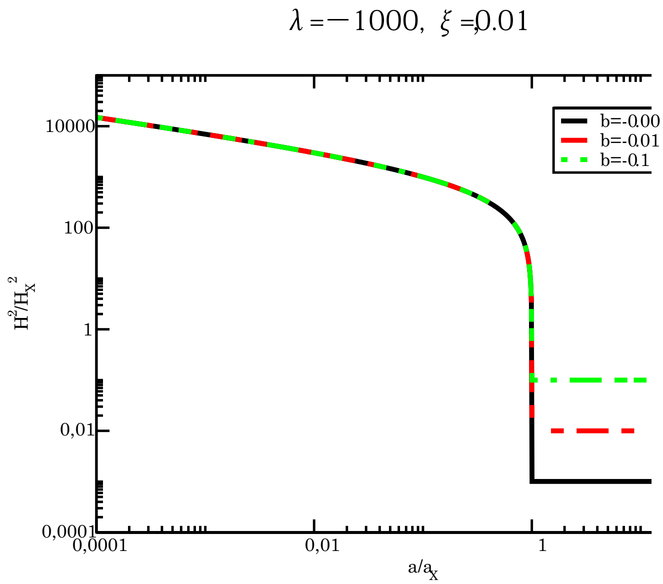

Figure 3: The behavior of h as a function of the scale factor a for different values of the parameter b. The used parameter values are $\alpha = -1$, $\lambda = -1000$, $\xi = 0.01$ and $w = -0.9$. In this parameter regime, the value of b controls the asymptotic behavior of h at large a.

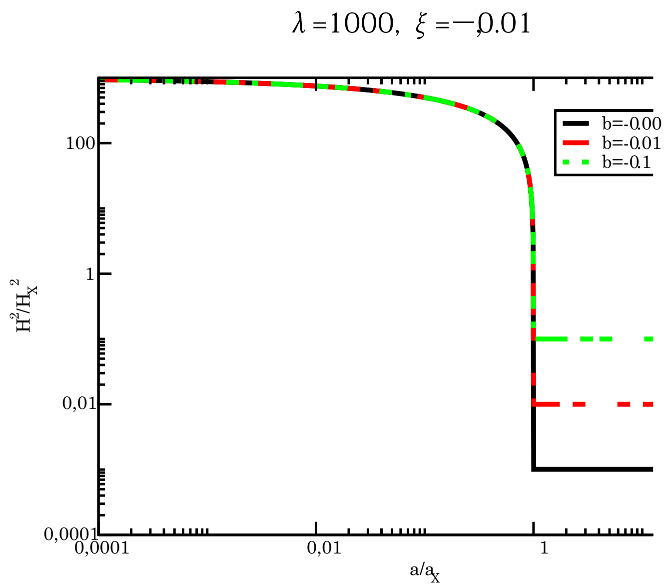

Figure 4: The dependence of the dynamics of h as a function of the scale factor on parameter b. The values of the parameters used are $\alpha = -1$, $\lambda = 1000$, $\xi = -0.01$ and $w = -1.1$. The asymptotic behavior of h at large a is determined by the size of b.

for positive and negative values of the scaled CC parameter λ there is a distinct and abrupt transition from the expansion at a high energy density to the de Sitter regime. Therefore, this specific signature of the CC relaxation mechanism is not lost with the addition of the additional parameter b. The asymptotic value of H^2, however, depends on the interplay of all model parameters. For a sufficiently small value of b, the asymptotic value is determined by the ratio of parameters ξ and λ ($h = |\xi/\lambda|$). As b grows, the asymptotic value becomes fully dominated by the value of parameter b.

Our aim is to look into a mechanism of the CC relaxation without fine-tuning. The case of vanishing b possesses certain appeal since there the effective positive CC is small because λ is large in absolute value and also the parameter ξ is not expected to be large. For parameter values which we would expect based on the fundamental theories the expected value of the effective CC at large scale factor values is a small positive number. There is no need for the parameters to be fine-tuned.

For a large absolute value of λ, sufficiently large b determines the asymptotic behavior of H^2. If we eventually aim at explaining the observed value of the cosmological constant, the value of b should be small. Since there is no clear reason why the value of b should be so small, some form of fine-tuning reenters into the model. We have to introduce a small parameter b just to match the value of observed Λ_{eff}. Still, it is very important to stress that this value is very different from the real value Λ. Although a very large Λ is present in the model, it does not determine decisively the asymptotic behavior of the system. Furthermore, the parameter b plays the main role only because λ is very large in absolute value. In a way the spirit of the CC relaxation mechanism is preserved: A universe with a large Λ finally tends to a de Sitter state characterized by a small Λ_{eff}. The principal difference to the $b = 0$ case is that for a sufficiently large $|b|$ there is no strong argument why Λ_{eff} should be small.

The model of this paper represents and extension of the model studied in [8], but it is still just a starting point towards a realistic cosmological model with the resolved CC problem. Essentially, both the model of this paper and [8] model the dark energy sector of the universe. Clearly, other components such as radiation and matter have to be added to create a realistic cosmological model and reproduce the standard eras of the evolution of the universe such as radiation dominated and matter dominated eras. The abruptness of the observed transition might pose a significant challenge to the construction of such a complete cosmological model. However, even in a model which contains the matter and radiation components, the presented CC relaxation mechanism should be efficient asymptotically since the energy densities of these components decay quickly with the expansion. Other important issues for further work are the timing of the transition, possible links to inflation and the growth of inhomogeneities in a universe in which the CC relaxation mechanism is active.

As discussed in the Introduction, there is another important problem related to the size of the dark energy (or CC) density. The coincidence problem is not directly addressed in the present model since the matter and radiation components are not present in the model. Only in the model with all relevant components this issue could be addressed properly.

Another very important issue is the physical motivation for the EOS given in (3). As already stated, the motivation could come from several directions of which we singled out modified gravity and generalized nonlinear viscosity. The elaboration of these topics might prove essential for a microscopic foundation of the mechanism exhibited by our model.

4 Conclusions

The extension of the original model of the CC relaxation [8] presented in this paper allows us the study of the limits of the CC relaxation mechanism. This is achieved through the introduction of the new parameter b. For a sufficiently large $|b|$, we no longer have an explanation of the smallness of Λ_{eff} without fine-tuning. Still, even for larger values of $|b|$ the asymptotic value is not determined by a large λ, but some other small parameter, in particular b. In this sense, the spirit of the CC relaxation mechanism persists even for larger values of $|b|$. These conclusions show that the concept of the CC relaxation mechanism is robust, although for small or vanishing value of b the solution of the CC problem is more natural. These findings further support the study of other types of inhomogeneous EOS as a road to a complete cosmological model in which the CC problem is naturally solved.

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The impact fractally matrix resonators "ISTOK" on the Casimir effect

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In this article there represented results of laboratory research into the influence of fractal-matrix resonators (FMR) series "ISTOK " on closely spaced rigid bodies in nonzero temperature. There offered a variant of solution of the main problem of Casimir effect in nanotechnologies, which is developed on a micromechanical devices level as an adhesion of its separate elements[2].

Till recently Casimir effect considered to be a fundamental effect of quantum theory of field. Thats why the rise of attractive power on nanometer scale considered to be difficult to remove. It is Federico Capasso's opinion that it is Casimir effect which should be considered to be the main problem for the miniature micromechanical devices.

Nowadays there are two main methods of understanding Casimir effect basis:

1. Display of Van der Waals force.
2. Quantum fluctuations of vacuum

If we consider Casimir effect (for a review, see [1]) as a display of well-known Van der Waals forces we can analyze in more detail this effect as a sort of intermolecular interaction. Van der Waals forces is a kind of attractive forces, which work between all atoms and molecules. The importance of these forces is evident from its two unique properties. First, these forces are universal. Such attraction mechanism works between all atoms and molecules. Secondly, these forces keep considerable size at comparatively large spaces between molecules. Moreover, it results in rise of attraction between two solid objects, divided by a small gap [2].

Let‘s examine quantum fluctuations of vacuum as the main reason for Casimir effect. Two metal plates situated close one to another (at a distance of a micron or less) form a resonator, on the length of which lays a whole number of $1/2$ waves. As a result, resonant electromagnetic waves reinforce, and the others are suppressed. The number of suppressed waves is much more. As a result, the pressure of virtual photons on inside is much less than outside pressure. Thus, the closer the surfaces one to anther are, the less wave-length between them in resonance there are. There are much more suppressed waves. As a result, attractive force between the surfaces rises.

Authors of this article got interesting results of the influence of modulated laser emission on source material for FMR. Distinctive feature of derived resonators is their

[1] E-mail: istokbcn@msn.com

[2] This article is **dedicated to 70th aniversary of Professor Iver Brevik**

ability for directional influence on closely spaced bodies. This influence takes place in nonzero temperatures, has high stability and can vary in wide limits depending on initial material and activation regime. Laboratory experiments and researches give grounds to presume the possibility of the influence on the main problem of Casimir effect in nanotechnologies, which is developed on a micromechanical devices level and makes their separate elements stick together.

The problem is that on these scales there are quite classical effects, connected with surface tension and Van der Waals forces. Experimenters try to diminish the influence of the effect by means of selection of certain materials, from which elements and their geometric forms are made. The new method of solving the problem of Casimir effect offered by the authors of this article is that microelements in mechanisms should be specially influenced on "ISTOK" technology [3].

The technology is based on the influence (activation) with a special modulated low-intensive laser emission with wave-length 635 - 670 nm on solid and liquid substances. Activation process takes place in laboratory conditions in nonzero temperature.

Distinctive feature of derived resonators is their ability to have directed effect on closely spaced bodies. This effect takes place in none zero temperatures, it has a high level of stability and can vary within wide limits depending on initial material and activation regime.

Preliminary results showed that metal plates and minerals, activated according to this technology get special properties:

- they influence on bodies located in immediate proximity, changing their physico-chemical properties
- they spread their influence at a distance up to several centimeters
- they influence freely through different substances, such as glass, paper, wood and some other kinds of polymers
- they do not change newly acquired properties for an indefinably long period of time
- they partly transmit newly acquired properties to other bodies (secondary reradiation effect).

Changes in physical and chemical properties of different substances and processes have been registered in laboratory conditions in immediate proximity to resonators' surfaces. This fact suggests that the change in display of Van der Waals forces in the examples is one of the factors of the influence on these processes. There may be another factor - the presence of long-lived charges on the surface of received resonators, the charges develop In the form of fluctuation of the electromagnetic field.

Short list of observed effects of FMR influence:

- changes in crystallization process in solutions
- changes in time of solution evaporation
- lowering of ammonium hydrate concentration in water
- changes in molecular diffusion process, evaporation and crystallization of saturated solution of copper sulfate
- changes of spectral absorption level in spirit-based liquids

The influence of FMR on diffusion process at a distance of several centimeters may be explained by changes of intermolecular interactions which happened as a result of fluctuations of electromagnetic field on the resonator's surface. It is necessary to point out the direction vector of this fluctuation is situated transversely to the resonator's surface and depends on the shape of this surface, material, activation regime and do not change in time. Taking into account the fact that the resonator has a definite direction, it is possible to create a combination of two or more resonators for getting effects with opposite sign. In other words, if we apply this technology we can control the force and direction of Casimir effect. And this isn't contrary to the works already conducted by other scientists. According to the suggestions of scientists from Los Angeles national laboratory in New Mexico, Casimir force may be changed into repulsive with the help of so called metamaterials - matters that are produced in laboratory environment and have impossible characteristics for nature conditions.

So several factors influence on Casimir effect: Quantum fluctuations of vacuum, Van der Waals forces. Some extra factors have been analyzed, such as geometric pattern, medium and structure of material.

The interesting is to take into account the research by I. Brevik on effect of Casimir effect at nonzero temperatures [5].

It‘s possible to suppose that activation of materials and mechanisms on "ISTOK" technology turns out to be just an extra factor of this influence. If this hypothesis is proved by laboratory measurements of Casimir effect it will have a large practical importance.

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Curriculum Vitae

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January 1963	Sivilingeniør of Physics, Norwegian Institute of Technology (NTH), Trondheim,
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ACADEMIC POSITIONS:

1963 - 1966	NTH, scientific assistant/fellow in Theoretical physics,
1966 - 1968	NORDITA (Nordisk Institut for Teoretisk Fysik), fellow,
1968 - 1972	NTH, fellow/lecturer/scientific assistant,
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RESEARCH PERIODS ABROAD:

NORDITA, several periods, as fellow and assistant professor, 2.5 years in all.,
Brief research periods at the University of Göttingen,Massachusetts Institute of Technology, and the University of Barcelona.

AREAS OF RESEARCH ACTIVITY:

Hydrodynamics: air-bubble plumes,

wave-current interactions, MHD turbulence.
Electromagnetic theory of continuous media:
energy-momentum tensor, Casimir effect,
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Cosmology: the early universe.

AWARDS:

1996 American Biographical Institute's "Gold Record of Achievement",
2008 Outstanding Referee, American Physical Society,
Listed in Marquis' "Who's Who in the World", and others.

ORGANIZATIONAL MEMBERSHIPS:

Royal Norwegian Society of Sciences and Letters
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Publication List
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Books

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