Б. Гейер, П. М. Лавров

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Книга посвящена рассмотренню современных методов ковариантного квантования калибровочных теорий. Дано детальное представление методов квантования Фаддеева-Попова и Баталина-Вилковыского; более кратко описаны методы Sp(2)-ковариантного, триплектического, супеполевого и osp(1,2)-симметричного квантований. Представляет интерес для научных работников, аспирантов, студентов старших курсов физико-математических факультетов, специализирующихся в области теоретической физики высоких энергий.

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The book is devoted to the present-day methods of covariant quantization for gauge theories. A detailed treatment is given to the Faddeev-Popov and Batalin-Vilkovisky quantization approaches; the methods of Sp(2)-covariant, triplectic, superfield and osp(1,2)-symmetric quantization are considered more briefly. The book is intended for researchers and PhD students in area of high energy theoretical physics.

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Preface

All the fundamental interactions that exist in Nature (electromagnetic, gravitational, strong and weak) can be described in terms of gauge theories. The quantization of gauge theories is one of the most essential means providing insight into the quantum properties of the fundamental forces. The formalisms of Hamiltonian and Lagrangian quantization of gauge theories present two different approaches to the quantum description of dynamical systems [77, 84, 75, 47, 120, 170, 188, 174, 80, 104, 103, 158, 62, 205].

There are many reasons for the interest in the covariant quantization of general gauge theories in the framework of Lagrangian formalism. First of all, in contrast to canonical quantization, it is possible to retain the covariance of description at all stages of calculations. This formalism provides a systematic method of obtaining the conservation laws on the basis of the Noether theorem. The most essential ingredient of covariant quantization is the path integral technique, being the most popular method currently available, and providing the most economic way of obtaining the Feynman rules directly from the classical Lagrangian. In solving the problem of quantization, we achieve a better understanding of the structure and quantum properties of general gauge theories.

The covariant quantization of gauge theories has made a long way starting from the famous works of Feynman [83], Faddeev, Popov [79], and DeWitt [74].

Many authors have contributed to developing the methods of covariant quantization, as well as to providing them with various applications. More references can be found in [125] (Henneaux and Teitelboim), [205] (Weinberg) and [108] (Gomis, Paris and Samuel).

The main purpose of this book is to introduce the reader to modern approaches to covariant quantization of gauge theories. We shall proceed according to the following plan of exposition.

The first subject to be considered is *"Canonical quantization of constraint systems"*. Regarding this question as one of the necessary educational elements for anyone engaged in the study of quantum theory, we give a brief review of the principal results obtained in this area, following the books [77] (Dirac) and [103] (Gitman and Tyutin).

Considered next is *"Faddeev–Popov quantization"*, the first success in the quantization of non-trivial gauge theories, like Yang–Mills ones, in the Lagrangian formalism proposed by Faddeev and Popov [79].

The third subject area to be covered is *"Batalin–Vilkovisky method"*. This method, developed by Batalin and Vilkovisky [40, 41], provides a unique closed approach to covariant quantization, based on a special kind of global supersymmetry, the so-called BRST symmetry, discovered by Becchi, Rouet and Stora [45, 46], and, independently, also by Tyutin [195].

We next proceed with *"Sp(2)-covariant Lagrangian quantization"*. This method, proposed by Batalin, Lavrov and Tyutin [25, 26, 27], handles the quantization of general gauge theories using a realization of so-called extended BRST symmetry, including BRST symmetry and antiBRST symmetry, discovered for Yang–Mills theories by Curci and Ferrari [66], and, independently, also by Ojima [166].

Following is *"Triplectic quantization"*. This quantization method, proposed by Batalin, Marnelius and Semikhatov [28, 35, 29], gives a completely anticanonical form to the $Sp(2)$-covariant procedure. A modified version of triplectic quantization has recently been suggested by Geyer, Gitman and Lavrov [96].

Then we deal with *"Superfield BRST quantization"*. This approach, proposed by Lavrov, Moshin and Reshetnyak [145], realizes the BRST transformations for general gauge theories as supertranslations in superspace with respect to an additional (Grassmann) coordinate, thereby giving an elegant geometric interpretation to the Ward indentities in the quantum theory of gauge fields.

The above subject area is followed by *"Superfield extended BRST quantization"*. This method, discovered by Lavrov [144], succeeds in presenting BRST and antiBRST transformations as supertranslations in superspace along additional Grassmann coordinates, thus giving a geometric interpretation to the Ward indentities in the method of $Sp(2)$-covariant quantization.

As regards superfield quantization, it should be noted that the geometric content of the BRST and antiBRST transformations in Yang–Mills theories, as supertranslations in superspace along additional (Grassmann) coordinates, was realized many years ago by the studies [53] (Bonora and Tonin), [54] (Bonora, Pasti and Tonin), [6] (Alvarez-Gaume and Baulieu), and [44] (Baulieu); however, no satisfactory superfield description of the quantization procedure was proposed. Moreover, the crucial point of these superfield methods was the manifest structure of Yang–Mills theories, and therefore the treatment of arbitrary gauge theories remained an unsolved peoblem.

Finally considered is "*osp(1,2)-covariant quantization*". This approach, recently proposed by Geyer, Lavrov and Mülsch [98, 99] on the basis of invariance under the global supergroup *osp(1,2)*, generalizes the method of $Sp(2)$-quantization and ensures the symplectic invariance of the quantum action in general gauge theories.

This reveiw was originated by a lecture course given by one of the authors (P.M.L.) to students and aspirants at the Graduate College "Quantum Field Theory" of Leipzig University (Germany) and at the Institute for Physics of Juiz de Fora University (Brazil). We are greatly indebted to D. Mülsch for useful discussions and participation at the initial stage of this work, and also to S. Falkenberg, one of the most attentive and responding listeners of this lecture course, a young promising scientist. Untimely and tragic death both of them was a heavy loss for us. We would like to thank I.A. Batalin, I.L. Buchbinder, A.A. Deriglazov, A.V. Galajinsky, D.M. Gitman, S.M. Kuzenko, P.Yu. Moshin, V.I. Mudruk, A.P. Nersessian, J.A. Neto, V.F. Popov, S.D. Odintsov, W. Oliveira, A.A. Reshetnyak, I.L. Shapiro, S. Theisen, I.V. Tyutin, B.L. Voronov for useful discussions on the related topics.

Chapter 1

Canonical Quantization of Constraint Systems

At present, the problem of quantization of an arbitrary Lagrangian system in the Hamiltonian formalism should be considered as solved. Here, we state only the main results following from the (appropriately modified) canonical quantization and reformulate it in terms of the Feynman path integral formalism (for a more detailed consideration, see [77, 120, 188, 174, 103]). Despite preferring the time coordinate, resulting in the loss of manifest covariance in the quantization of relativistic theories, canonical quantization is a natural starting point for all further considerations. For the sake of simplicity, we first present it for the case of finite degrees of freedom and, at the end of each subsection, we generalize to field theories; also various illustrations are given in this respect.

1.1 Lagrange equations

Let the system under consideration be described by the Lagrangian L

$$L = L(q, \dot{q}), \quad \dot{q} \equiv \frac{dq}{dt}, \tag{1.1.1}$$

where $(q^i, \dot{q}^i),\ i = 1, 2, ..., n$, are generalized coordinates and velocities, respectively. The fundamental equations of the classical theory, the equations of motion, follow from the principle of stationary action, $\delta S = 0$, applied to the action functional,

$$S_L = S[q] = \int dt L(q, \dot{q}). \tag{1.1.2}$$

They are given by the Lagrange equations

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{1.1.3}$$

1.2 Hessian matrix

Going over to the Hamiltonian formulation of the classical theory, we introduce the generalized momenta p_i, corresponding to the generalized coordinates q^i, according to

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = p_i(q, \dot{q}). \tag{1.2.4}$$

Since these equations need to be resolved for $\dot{q}^i = \dot{q}^i(p, q)$, the further analysis relies on the properties of the Hessian matrix,

$$H_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \ . \tag{1.2.5}$$

Two different cases are possible

$$\text{A)} \qquad \det |H_{ij}| \neq 0, \tag{1.2.6}$$

$$\text{B)} \qquad \det |H_{ij}| = 0, \tag{1.2.7}$$

which will be dealt with independently.

1.3 Hamiltonian equations of unconstrained systems

Consider, firstly, the simple case corresponding to condition (1.2.6). Then the relations $p_i = p_i(q, \dot{q})$ can be solved uniquely in terms of the velocities

$$p_i = p_i(q, \dot{q}) \quad \Longleftrightarrow \quad \dot{q}^i = \dot{q}^i(p, q), \tag{1.3.8}$$

and thus we have a dynamical system without constraints. Let us introduce the quantity

$$H = p_i \dot{q}^i - L(q, \dot{q}), \tag{1.3.9}$$

which, in view of (1.3.8), can be represented as a function of the variables (p, q), and is referred to as the *Hamiltonian* $H = H(p, q)$ of the system. The transition from $L(q, \dot{q})$ to $H(p, q)$, given by Eq. (1.3.9), is a Legendre transformation. One easily establishes the following properties of this transformation:

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \quad \Leftrightarrow \quad \dot{q}^i = \frac{\partial H(p, q)}{\partial p_i}, \tag{1.3.10}$$

$$\left. \frac{\partial L(q, \dot{q})}{\partial q^i} \right|_{\dot{q}} = - \left. \frac{\partial H(p, q)}{\partial q^i} \right|_{p} . \tag{1.3.11}$$

Note that in the phase space (p, q) more curves may be specified than in the coordinate space. Indeed, singling out a continuous trajectory $q(t)$ in the coordinate space produces the (piecewise) continuous curve $\dot{q}(t)$; and therefore the corresponding trajectory $(q(t), p(t) = p(q(t), \dot{q}(t)))$ in the phase space is uniquely defined. On the other hand, singling out a curve $(p(t), q(t))$ in the phase space specifies the curve $\dot{q}(t) = dq(t)/dt$. However, also such curves $q(t)$ may occur that the relation $p(t) = \partial L/\partial \dot{q} = p(q(t), \dot{q}(t))$ is violated, and therefore there is no corresponding trajectory in the coordinate space.

Despite this fact, the extremum of the action holds at the same trajectories in both the coordinate and the phase space. Namely, let us introduce the Hamiltonian action by

$$S_H = S[p,q] = \int dt(p_i \dot{q}^i - H(p,q)). \tag{1.3.12}$$

Taking the variation of the action S_H considered as a functional of the variables p and q, we obtain the following equations which determine the classical extremals in the phase space:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}; \tag{1.3.13}$$

these relations are the Hamilton equations. In addition, one easily proves the equivalence of the equations of motion in both the Lagrangian and the Hamiltonian formalism (Eqs. (1.1.3) and (1.3.13)); in other words,

$$\delta S_H = 0 \quad \Leftrightarrow \quad \delta S_L = 0.$$

To establish this fact, it is necessary to use the above-mentioned properties of the Legendre transformations.

1.4 Poisson bracket

It is advantageous to introduce the Poisson bracket, defined for any two quantities F, G in the phase space, by the rule

$$\{F,G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \ . \tag{1.4.14}$$

The Poisson bracket (1.4.14) obeys the following properties:
(1) Antisymmetry

$$\{F,G\} = -\{G,F\}, \tag{1.4.15}$$

(2) Jacobi identity

$$\{F,\{G,H\} + cyclic\ \ perms.(F,G,H) \equiv 0, \tag{1.4.16}$$

(3) Linearity

$$\{F+G,H\} = \{F,H\} + \{G,H\} \tag{1.4.17}$$

(4) Leibniz rule

$$\{FH,G\} = F\{H,G\} + \{F,G\}H. \tag{1.4.18}$$

One easily verifies the following equalities:

$$\{q^i,q^j\} = \{p_i,p_j\} = 0, \quad \{q^i,p_j\} = \delta^i_j. \tag{1.4.19}$$

By using the Poisson bracket, the Hamilton equations (1.3.13) can be presented in the form

$$\dot{q}^i = \{q^i,H\}, \quad \dot{p}_i = \{p_i,H\}; \tag{1.4.20}$$

and the time evolution of any physical quantity $A(p, q)$ is given by

$$\dot{A} = \{A, H\} \ . \tag{1.4.21}$$

Obviously, eqs. (1.4.15) – (1.4.21) introduce an algebraic structure of the classical theory. It will be the basis of canonical quantization (see, for example [80, 103, 62]) where these algebraic properties, given in terms of Poisson brackets, are translated into quantum brackets defined as commutation relations between the corresponding operators according to

$$\{F, G\} \Longrightarrow (i\hbar)^{-1}[\hat{F}, \hat{G}]. \tag{1.4.22}$$

1.5 Quantization

That quantization procedure is governed by the following *postulates*:

(1) A state of the system is described by a (normalized) vector $|\psi\rangle$ in the Hilbert space $\mathcal{H}$ with the inner product $\langle\psi_1|\psi_2\rangle$. Generically, the Hilbert space is realized as a Fock space constructed with the help of a (unique) vacuum state $|0\rangle$.
Physical *observables* A are represented by Hermitian operators $\hat{A}$ acting on the Hilbert space.

(2) The expectaction value of an observable A with respect to a state $|\psi\rangle$ is given by $\langle\psi|\hat{A}|\psi\rangle$.

(3) The initial coordinates and momenta $q^i(t),\ p_i(t)$ in the Heisenberg picture are described by Hermitian operators $\hat{q}^i(t),\ \hat{p}_i(t)$, which satisfy the (equal time) canonical commutation relations

$$[\hat{q}^i(t), \hat{q}^j(t)] = 0, \quad [\hat{p}_i(t), \hat{p}_j(t)] = 0, \quad [\hat{q}^i(t), \hat{p}_j(t)] = i\hbar\delta^i_j, \tag{1.5.23}$$

where $\hbar$ is the Planck constant.

(4) The time evolution of operators $\hat{A}(t)$ is determined, similarly to (1.4.21), by the equation

$$i\hbar\frac{d\hat{A}(t)}{dt} = [\hat{A}(t), \hat{H}], \tag{1.5.24}$$

where the Hermitian operator $\hat{H}$, the quantum Hamiltonian, is obtained from the classical Hamiltonian H by substituting the operators $\hat{q}^i(t),\ \hat{p}_i(t)$ in place of the coordinates $q^i(t)$ and momenta $p_i(t)$.

In performing quantization according to the above rules, one faces the problem of the arrangement of the operators $\hat{q}, \hat{p}$ in the values representing physical quantities, like the Hamiltonian $\hat{H} = H(\hat{q}, \hat{q})$. Notice that different forms of the correspondence principle give rise to restrictions on the zeroth and first terms in the expansion of physical quantities in powers of $\hbar$. However, there remains considerable arbitrariness in the arrangement of operators. In what follows, it is assumed that a certain arrangement of the non–commuting operators $\hat{q}, \hat{p}$ has been applied. (Note that in the path integral quantization, to be introduced further, the so-called Weyl ordering is preferred.)

1.6 Green's functions

The fundamental objects of the quantum theory are the Green's functions of the position operators $\hat{q}^i(t)$ defined by the following vacuum expectation values

$$G^{(n)}_{i_1...i_n}(t_1, ..., t_n) \quad = \quad \langle 0|T\left(\hat{q}^{i_1}(t_1)...\hat{q}^{i_n}(t_n)\right)|0\rangle. \tag{1.6.25}$$

Here, the symbol T denotes chronological ordering, which implies that the operators must be arranged from left to right so that their (mutually different) time arguments decrease. Moreover, under the sign of the T–product all operators commute with each another. The set of Green's functions $\{G^{(n)}_{i_1...i_n}(t_1, ..., t_n), n = 1, 2, ..., \}$ contains the complete information one expects to derive from quantum theory. In other words, possessing the knowledge of the whole set of Green's functions, one can recover the Hilbert space and the algebra of observables. This is the meaning of the GNS construction, well known in general quantum field theory (see, for example [119]).

1.7 Generating functional of Green's functions

Instead of considering the set of Green's functions $G^{(n)}$ separately, one can introduce an object combining all of them. Let $J_i(t)$ be a set of scalar functions belonging e.g. to some space of test functions and referred to as external sources, then the generating functional $Z(J)$ of (complete) Green's functions is defined by

$$Z(J) = \sum_n \frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \int dt_1 ... \int dt_n G^{(n)}_{i_1...i_n}(t_1, ...t_n) J_{i_1}(t_1)...J_{i_n}(t_n), \tag{1.7.26}$$

such that

$$G^{(n)}_{i_1...i_n}(t_1, ..., t_n) \quad = \quad \left(\frac{i}{\hbar}\right)^n \frac{\delta^n}{\delta J_{i_1}(t_1)...\delta J_{i_n}(t_n)} Z(J)\Big|_{J=0}. \tag{1.7.27}$$

It is well known [84, 174, 170, 104] that this functional $Z(J)$ can be written as a functional or path integral over trajectories in the phase (or configuration) space with the integrand depending on the phase of the action integral,

$$Z(J) \quad = \quad \int \mathcal{D}q\mathcal{D}p \ \exp\left\{\frac{i}{\hbar}\int d\tau[p_i\dot{q}^i - H(p, q) + J_i q^i]\right\} \tag{1.7.28}$$

where the expression

$$\int \mathcal{D}q\mathcal{D}p \ \equiv \ \int \prod_\tau \left(\frac{dq^i(\tau)dp_i(\tau)}{2\pi\hbar}\right) \tag{1.7.29}$$

is referred to as functional integration over the entire phase space without boundary conditions, i.e. the integration is performed over all the trajectories without restrictions. Taking into account eqs. (1.6.25) and (1.7.27), we obtain

$$\langle 0|T\left(\hat{q}^{i_1}(t_1)...\hat{q}^{i_n}(t_n)\right)|0\rangle \quad = \quad \int \mathcal{D}q\mathcal{D}p\left(q^{i_1}(t_1)...q^{i_n}(t_n)\right) \exp\left\{\frac{i}{\hbar}\int d\tau[p_i\dot{q}^i - H(p, q)]\right\}.$$

Let us point to the fact that the time ordering in this approach appears automatically. Sometimes the generating functional (1.7.28) is called the vacuum-to-vacuum transition amplitude in the presence of the external current J, i.e. $Z(J) = \langle 0|0\rangle^J$. Since the functional integral (1.7.28) contains the entire information of the quantum theory obtained from a classical system with the Hamiltonian H, this procedure is called path integral quantization in phase space (The reader, not familiar with the path integral formulation of quantum theory, may profit from the short exposition of the essential steps leading to eq. (1.7.28) given in **Appendix B**).

It is important that the generating functional $Z(J)$ should also be expressed directly in terms of the Lagrangian of the theory,

$$Z(J) = \int \mathcal{D}q \exp\left\{\frac{i}{\hbar}\int \left(L(q,\dot{q}) + J_i q^i\right) dt\right\}. \tag{1.7.30}$$

This can be seen from the following consideration. Let us make a shift $p \to \overline{p}(q,\dot{q}) + p$ of the integration variable, where $\overline{p}(q,\dot{q})$ is the solution of the Hamilton equation of motion (1.3.13), i.e.

$$\dot{q}^i \;=\; \{q^i, H\} \;=\; \left.\frac{\partial H}{\partial p_i}\right|_{p=\overline{p}}.$$

Of course, this results from the translation invariance of the measure (1.7.29) and takes the classical solution $\overline{p}(q,\dot{q})$ as reference curve for the quantum fluctuations p around it. If the Hamiltonian H is build up from the Lagrangian L, which is assumed to be the case, then such a solution always exists and is given by (see eq. (1.3.10))

$$\overline{p_i}(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}^i}\ .$$

Here, the identity holds

$$(p_i\dot{q}^i - H)|_{p=\overline{p}} \;\;\equiv\;\; L(q,\dot{q}). \tag{1.7.31}$$

Therefore,

$$S_H|_{p\to p+\overline{p}} = S_L + \int dt \triangle H,$$

where $S_L = S[q]$ is the classical Lagrangian action, and

$$\triangle H = -\sum_{n=2}^{\infty}\frac{1}{n!}p^n \left.\frac{\partial^n H}{\partial p^n}\right|_{p=\overline{p}}.$$

Making use of (1.7.31), one can write:

$$Z(J) = \int \mathcal{D}q \triangle(q) \exp\left\{\frac{i}{\hbar}(S[q] + J_i q^i)\right\} \tag{1.7.32}$$

where

$$\triangle(q) = \int \mathcal{D}p \exp\left(\frac{i}{\hbar}\int dt \triangle H(p,q)\right).$$

Formula (1.7.32) expresses the generating functional $Z(J)$ of a nonsingular theory in the form of a functional integral in configuration space. It acquires an especially simple form if the interaction does not contain derivatives of the coordinates with respect to time. In this case, H is a quadratic form in the momenta with constant coefficients, so that $\triangle(q)$ is simply a constant and we arrive at the above-mentioned form (1.7.30) for $Z(J)$. Consequently, it is obvious that the path integral over the configuration space is more restricted than the one over the entire phase space.

The above results for theories with finite degrees of freedom can be generalized to field theories, i.e. to the case of infinite degrees of freedom. The trajectory $q^i(t)$ in configuration spaces is replaced by a field function $\phi(x)$, where $x = (t, \mathbf{x})$. The Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, ...)$ is considered as a function of the fields ϕ and their partial derivatives $\partial_\mu\phi \equiv \partial\phi/\partial x^\mu$. The degrees of freedom are now labeled by the continuous index $\mathbf{x}$ as well as by additional labels indicating the field components with respect to certain symmetries of the theory; obviously, the number of degrees of freedom is infinite. The corresponding momenta are given by $\pi(x) = \partial\mathcal{L}/\partial_0\phi(x)$. To define the appropriate path integral, one can start from a multiple integral on a discrete, and to begin with, finite lattice of space-time points. This amounts to defining the quantum field theory as a limit of a theory possessing only a finite number of degrees of freedom. Note that a consistent definition of functional integral in quantum field theory can be given, at least in perturbation theory, without any reference to the limiting process [182, 80].

By analogy with the above results, we may postulate the following path integral representation for the generating functional of Green's function of a quantum field theory without constraints:

$$Z(J) = \int \mathcal{D}\pi\mathcal{D}\phi \exp\left\{\frac{i}{\hbar}(S_H[\pi, \phi] + J\phi)\right\} \tag{1.7.33}$$

where $S_H[\pi, \phi]$ is the classical Hamiltonian action (1.3.12); here we have used the notation $J\phi \equiv \int dx J(x)\phi(x)$. The corresponding functional integral over configuration space reads:

$$Z(J) = \int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}(S_L[\phi] + J\phi)\right\} \tag{1.7.34}$$

Example: real scalar field

The standard field theoretic example of the above situation is the model of a real scalar field $\varphi(x)$ with the action

$$S[\varphi] = \int dx \mathcal{L}(\varphi, \partial_\mu\varphi), \quad \mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right) - V(\varphi) \tag{1.7.35}$$

where $V(\varphi) \equiv \mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\varphi^3 + \frac{\lambda'}{4!}\varphi^4 + ...$ is a potential. The Hamiltonian action is given by

$$S[\pi, \varphi] = \int dx(\pi\dot{\varphi} - \mathcal{H}(\pi, \varphi)), \quad \mathcal{H}(\pi, \varphi) = \frac{1}{2}\left(\pi^2 + \partial_i\varphi\,\partial_i\varphi + m^2\varphi^2\right) + V(\varphi), \tag{1.7.36}$$

and, by construction, we have $\triangle H = 0$. The equality of the two expressions for $Z(J)$, the phase space and the configuration space functional integrals, is based on the fact that the integration over π is Gaussian. The explicit form for $Z(J)$ can be given as

$$\begin{aligned} Z(J) &= \exp\left\{ - \frac{i}{\hbar}\int dx\, V\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)\right\}\int \mathcal{D}\varphi \exp\left\{-\frac{i}{\hbar}\int dx \left[\left(\frac{1}{2}\varphi(\Box + m^2\right)\varphi - J\varphi\right]\right\} \\ &= \exp\left\{ - \frac{i}{\hbar}\int dx\, V\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)\right\}\exp\left\{\frac{i}{\hbar}\int dxdy\,\frac{1}{2}J(x)D_c(x-y)J(y)\right\}, \end{aligned} \tag{1.7.37}$$

where $D_c(x-y) = \langle x|(\Box + m^2)^{-1}|y\rangle$ is the free (causal) propagator. Obviously, contrary to the Hamiltonian action eq. (1.7.36) the Lagrangian action eq. (1.7.35) is covariant. Therefore, the latter is preferable if symmetry properties of the theory need to be formulated.

Example: Dirac field

Another standard example is the Dirac field ψ, whose Lagrangian action is given by

$$S[\psi,\overline{\psi}] = \int dx \mathcal{L}_{\mathcal{D}}(\psi,\overline{\psi}), \quad \mathcal{L}_{\mathcal{D}}(\psi,\overline{\psi}) = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \mathcal{L}_{\text{int}}, \tag{1.7.38}$$

where the coupling (e.g. to the Maxwell field) is given by

$$\mathcal{L}_{\text{int}} = \overline{\psi}\gamma_\mu \psi A^\mu, \tag{1.7.39}$$

with ψ being the spinor field, γ – the Dirac matrices, and A^μ – the electromagnetic potential. However, as is well known, the canonical quantization of Dirac fields is to be formulated in terms of anticommutators instead of the commutators below. Consequently, in the path integral the classical fields $\psi, \overline{\psi}$ are anticommuting or Grassmann variables. (A short exposition of functional integrals with Grassmann variables is given in **Appendix C**).

1.8 Constraints

The second possibility mentioned above is more complicated. In this case $\det |H_{ij}| = 0$ and the equations (1.2.4), $p_i = \partial L/\partial \dot{q}^i$, are solvable with respect to $\dot{q}^i$ *only partially.* That is, eq.(1.2.4) may give rise to some (linearly independent) relations involving no $\dot{q}^i$, which called first stage or primary constraints; a system with constraints is called a singular system.
In addition, let us remark, that in this case the equations of motion

$$H_{ij}(q,\dot{q})\ddot{q}^j = K_i(q,\dot{q}) \equiv \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j$$

cannot be solved uniquely. The solution of the Cauchy problem for the second order differential equations of motion depends on arbitrary functions, thus demonstrating the *gauge freedom* of the theory.

The constraints

$$\phi_\alpha(p,q) = 0, \quad \alpha = 1, ..., r\ , \tag{1.8.40}$$

are functions of q^i and p_i. They define the physical surface in the phase space of the system. Let us consider a variation of the Hamiltonian $H = p_i\dot{q}^i - L$, i.e.

$$\delta(p_i\dot{q}^i - L) = \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i + \left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right)\delta \dot{q}^i. \tag{1.8.41}$$

If eq. (1.2.4) is applied, it follows from eq. (1.8.41) that the Hamiltonian can be expressed in terms of q^i and p_i. Of course, this statement holds under the validity of the constraints, Eq.(1.8.40). Hence we should consider a generalized Hamiltonian,

$$H^* = p_i\dot{q}^i - L + \lambda^\alpha \phi_\alpha \equiv H + \lambda^\alpha \phi_\alpha, \tag{1.8.42}$$

where the Lagrange multipliers λ^α remain undetermined. The (modified) canonical equations of motions may be formulated as

$$\dot{p} = \{p, H^*\}, \quad \dot{q} = \{q, H^*\} \quad \text{together with} \quad \phi^\alpha(p,q) = 0; \tag{1.8.43}$$

of course, before calculating the Poisson bracket the constraints must be considered non-vanishing. The time development of a quantity A is given by $\dot{A} = \{A, H^*\}$. Since the constraints are supposed to hold for any t, $\dot{\phi}_\alpha = 0$, consistency requires

$$\{\phi_\alpha, H^*\} \equiv \{\phi_\alpha, H\} + \lambda^\beta \{\phi_\alpha, \phi_\beta\} = 0. \tag{1.8.44}$$

Part of Eqs. (1.8.44) can be satisfied by an appropriate choice of λ^α, but the remainder may contain new conditions, which are to be regarded as second-stage constraints. We should therefore consider eq. (1.8.44) for these new constraints. Repeating this procedure up to the L-th stage, we find no further secondary constraints, and obtain a set of independent constraints Eq. (1.8.40) with $\alpha = 1, .., s$ $(r \leq s \leq n)$.

According to Dirac, a function f of the variables (p, q) is called a first-class function if its commutator (the Poisson bracket) with any constraint is proportional to constraints $\{f, \phi\} \simeq \phi$. Accordingly, one introduces the notion of first-class constraints. Consequently, any set of constraints ϕ for which the matrix $\| \{\phi, \phi\} \|_{|\phi=0}$ is nonsingular, will be referred to as a set of second-class constraints. The number of second-class constraints is necessarily even. This follows from the fact that a nonsingular antisymmetric matrix always has even rank.

1.9 Second class theories

Let us first consider theories for which the antisymmetric matrix $\| \{\phi_\alpha, \phi_\beta\} \|$ composed by the Poisson brackets of all the constraints ϕ_α is nonsingular,

$$\text{Det} \, \| \{\phi_\alpha, \phi_\beta\} \|_{|\phi=0} \quad \neq \quad 0. \tag{1.9.45}$$

In this case it is possible to solve eq. (1.8.44) for the Lagrangian multipliers. Then, introducing for any functions $F(p,q)$ and $G(p,q)$ a modification of the Poisson bracket,

$$\{F, G\}_D = \{F, G\} - \{F, \phi^\alpha\}\{\phi_\alpha, \phi_\beta\}^{-1}\{\phi^\beta, G\}, \tag{1.9.46}$$

the so-called *Dirac bracket*, one represents the equations of motion in the form

$$\dot{p} = \{p, H\}_D, \quad \dot{q} = \{q, H\}_D \quad \text{together with} \quad \phi^\alpha(p,q) = 0, \tag{1.9.47}$$

with the initial Hamiltonian H. The theory is quantized by using the same postulates (1) – (4), with the Poisson bracket replaced by the Dirac bracket and, in addition, with the constraints required to hold (on the physical states).

This procedure can be implemented within the path integral formalism. Omitting all calculations, let us only give the final result for the generating functional of Green's functions for theories with second-class constraints

$$Z(J) = \int \mathcal{D}p\mathcal{D}q \, \text{Det}^{1/2}\{\phi_\alpha, \phi_\beta\}\delta(\phi_\alpha) \exp\left\{\frac{i}{\hbar}[S_H(p,q) + Jq]\right\}. \tag{1.9.48}$$

Here, the δ-functional ensures that all the constraints $(\alpha = 1, ..., s)$ are fulfilled, thereby reducing the integration to the independent variables, whereas the Jacobi determinant results from the corresponding change of variables.

Example: massive vector field

As an example of a theory with second-class constraints, we consider the theory of a massive vector field A^μ. This theory is described by the following action:

$$S(A) = \int dx\Big(-\frac{1}{4}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu\Big) = \int dx\mathcal{L}(A_\mu, \partial_\nu A_\mu) \tag{1.9.49}$$

where the field strenght $\mathsf{F}_{\mu\nu}$ is given by

$$\mathsf{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The canonical momenta are

$$P_\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}^\mu} = -\mathsf{F}_{0\mu}.$$

and, therefore, the theory contains a primary constraint:

$$\phi_1 \equiv P_0 = 0. \tag{1.9.50}$$

Then for the Hamiltonian H we obtain

$$H = \frac{1}{2}P_i^2 - P_i\partial_i A_0 + \frac{1}{4}\mathsf{F}_{ik}^2 - \frac{m^2}{2}(A_0^2 - A_i^2). \tag{1.9.51}$$

Hence,

$$H^* = H + \lambda^1 P_0.$$

Commuting the primary constraint (1.9.50) with the Hamiltonian H^*, i.e. using the canonical equations of motion (1.8.44), we find a secondary constraint

$$\phi_2 \equiv \partial_i P_i - m^2 A^0 = 0. \tag{1.9.52}$$

There are no further secondary constraints. Hence $\phi = (\phi_1, \phi_2)$ is the complete system of constraints. The matrix composed by the constraints,

$$\|\{\phi,\phi\}\| = \begin{pmatrix} 0 & m^2 \\ -m^2 & 0 \end{pmatrix}\delta(\vec{x} - \vec{y}), \tag{1.9.53}$$

is nonsingular in this case. Thus, we have a theory with second-class constraints.

Let us construct for this theory the generating functional of Green's functions. Note that the matrix (1.9.53) in this case does not depend on the fields, and therefore we need not write the corresponding determinant in the functional integral (1.9.48). Then, the generating functional has the form

$$Z = \int \mathcal{D}A^\mu\mathcal{D}P_\mu\delta(P_0)\delta(\partial_i P_i - m^2A^0)\exp\left[\frac{i}{\hbar}\int dx(P_\mu\dot{A}^\mu - H + J_iA^i)\right], \tag{1.9.54}$$

where the Hamiltonian H is defined by Eq.(1.9.51). We have introduced the sources J_i only to the fields A^i, since the corresponding Green functions are sufficient to describe all physical quantities of the theory.

Let us now perform some operations in the integral (1.9.54). In the Hamiltonian H, eq. (1.9.51), we substitute the term $-P_i\partial_i A_0$ by $m^2A_0^2$. This can be done, owing to the presence of the δ-function of the secondary constraint in the integral (1.9.54). Furthermore, let us introduce

$$\delta(\partial_i P_i - m^2 A^0) = \int \mathcal{D}B \exp\Big[-\frac{i}{\hbar}\int dx B(\partial_i P_i - m^2 A^0)\Big],$$

and then integrate over P_0 and A^0. The result can be written as (being B replaced by A^0):

$$\int \mathcal{D}A^\mu \mathcal{D}P_i \exp\Big\{ \frac{i}{\hbar}\int dx\Big[-\frac{1}{2}P_i^2 + P_i(\dot{A}^i + \partial_i A^0) + \frac{m^2}{2}(A_0^2 - A_i^2) - \frac{1}{4}F_{ik}^2 + J_i A^i\Big]\Big\}.$$

Now, the integral over the momenta can be easily calculated. As a result, we obtain

$$Z(J) = \int \mathcal{D}A \exp\Big[\frac{i}{\hbar}(S(A) + J_\mu A^\mu)\Big]. \tag{1.9.55}$$

Here, for the sake of formal symmetry, we have introduced a source also to the field A^0. From (1.9.55) one can observe the validity of the naive Feynman rules in perturbation calculations of the Green functions for this model.

1.10 First class theories

We shall now consider theories for which the matrix $\|\{\phi,\phi\}\|$ is singular on the physical surface,

$$\det\|\{\phi,\phi\}\|_{\phi=0} = 0, \quad \mu = [\phi] - \mathrm{rank}\|\{\phi,\phi\}\| > 0, \tag{1.10.56}$$

where $[\phi]$ denotes, at any fixed space-time point x, the number of constraints ϕ (we assume that $[\phi]$ does not depend on any x). In this case, the analysis of the classical theory is more complicated than the one considered above. In short, the main results may be outlined as follows [103]:

First, the theory possesses μ first-class constraints. Among them, there should be $\mu_1 \neq 0$ primary first-class constraints. Second, the solution of the Hamiltonian equations of motion essentially contains μ_1 arbitary functions of time. In turn, the solutions of the Lagrangian equations contain exactly μ_1 arbitrary functions of time, equal to the number of primary first-class constraints in the Hamiltonian formalism. Third, all the constraints $\{\Phi\}$ can be divided into two groups, i.e.

$$\Phi = (\phi;\, \varphi),$$

where ϕ are all first-class constraints and φ are all second-class constraints.

Of course, a quantization procedure analogous to the second class constraints does not work for the first class ones because the Dirac bracket would not be well-defined. However, introducing independent gauge functions χ, as many as the number of the first class constraints, makes a functional formulation of gauge theories possible. There is only one restriction on the gauge functions; namely, it is necessary that the determinant of all the first-class constraints ϕ with all the gauges χ shoul be non-vanishing:

$$\mathrm{Det}\{\phi,\chi\} \neq 0.$$

In this case the generating functional of Green's functions can be expressed in the form

$$Z(J) = \int \mathcal{D}p\mathcal{D}q\,\mathrm{Det}^{1/2}\{\varphi,\varphi\}\mathrm{Det}\{\chi,\phi\}\delta(\chi)\delta(\varphi)\delta(\phi)\exp\left(\frac{i}{\hbar}[S_H(p,q)+Jq]\right). \quad (1.10.57)$$

Let us point to the formal equivalence with the quantization of second-class theories, if the gauge functions are considered as a completion of the first-class constraints, ϕ_α, according to $\Phi' = (\phi,\chi;\ \varphi)$ then we have

$$\mathrm{Det}^{1/2}\{\Phi',\Phi'\}\delta(\Phi') = \mathrm{Det}^{1/2}\{\varphi,\varphi\}\mathrm{Det}\{\chi,\phi\}\delta(\chi)\delta(\varphi)\delta(\phi).$$

Example: free electromagnetic field

To give an illustrative example, we consider the theory of a free electromagnetic field A^μ, which is described by the action

$$S(A) = \int dx\Big\{-\frac{1}{4}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}\Big\}. \quad (1.10.58)$$

From Eq.(1.10.58) it follows that there is one primary constraint here,

$$\phi_1 \equiv P_0 = 0. \quad (1.10.59)$$

The Hamiltonians H and H^* are of the form

$$H = \frac{1}{2}P_i^2 - P_i\partial_i A_0 + \frac{1}{4}\mathsf{F}_{ik}^2, \quad H^* = H + \lambda P_0. \quad (1.10.60)$$

Commuting the constraint (1.10.59) with the Hamiltonian H^*, we obtain

$$\{H^*,\phi_1\} = \partial_i P_i.$$

Thus a second-stage constraint appears,

$$\phi_2 \equiv \partial_i P_i = 0. \quad (1.10.61)$$

The commutator of the constraint ϕ_2 with the Hamiltonian H^* is equal to zero, which means that no further constraints arise in this case. Clearly, we are dealing with a theory with first-class constraints, since the constraints ϕ_1 and ϕ_2 commute with each other. Obviously, this theory obtains from the former example in the limit $m^2 = 0$. Let us remind that $P_i = E_i$ are the components of the electric field, whereas $B_i = (1/2)\varepsilon_{ijk}\mathsf{F}_{jk}$ are the components of the magnetic field, and that $P_0 = 0$ is a trivial constraint, whereas $\partial_i E_i = 0$ is the Gauss law.

To construct the generating functional in this case, let us consider the following choice of gauge functions:

$$\chi_1 \equiv A_0, \quad \chi_2 \equiv \partial_i A_i, \quad (1.10.62)$$

leading to the non-covariant canonical, or radiation, gauge with $A_0 = 0$ (temporal gauge) and $\partial_i A_i = 0$ (Coulomb gauge). Notice that the matrix of the first-class constraints $\phi = (\phi_1,\phi_2)$ and gauge functions $\chi = (\chi_1,\chi_2)$ has the form

$$\|\{\chi,\phi\}\| = \begin{pmatrix} 1 & 0 \\ 0 & -\triangle \end{pmatrix}\delta(\vec{x}-\vec{y});$$

it does not depend either on the fields A^μ or on the momenta P_μ, and again the Jacobian is simply a constant. Then, for the generating functional $Z(J)$ we have

$$\begin{aligned} Z(J) \quad &= \quad \int \mathcal{D}A^\mu \mathcal{D}P_\mu \delta(A^0)\delta(P_0)\delta(\partial_i A_i)\delta(\partial_i P_i) \exp\left\{\frac{i}{\hbar}\int dx \Big[P_\mu \dot{A}^\mu - \right. \\ &\left. -\tfrac{1}{2}P_i^2 + P_i\partial_i A_0 - \tfrac{1}{4}\mathsf{F}_{ik}^2 + J_i A^i\Big]\right\}. \end{aligned}$$

Next, integrating over A^0 and P_0 and then representing $\partial_i P_i$ in the form of a functional integral,

$$\delta(\partial_i P_i) = \int \mathcal{D}A^0 \exp\Big(-\frac{i}{\hbar}\int dx A^0 \partial_i P_i\Big),$$

we obtain

$$Z(J) = \int \mathcal{D}A^\mu \mathcal{D}P_i \delta(\partial_i A_i) \exp\left\{\frac{i}{\hbar}\int dx \Big[P_i \dot{A}^i - \frac{1}{2}P_i^2 - P_i \partial^i A_0 - \frac{1}{4}\mathsf{F}_{ik}^2 + J_i A^i\Big]\right\}.$$

Again, the integral over the momenta is Gaussian and can easily be calculated. As a final result, we obtain the expression for Z in the form of a functional integral in configuration space:

$$Z(J) = \int \mathcal{D}A^\mu \delta(\partial_i A_i) \exp\left\{\frac{i}{\hbar}\Big(S(A) + J_\mu A^\mu\Big)\right\} \tag{1.10.63}$$

where $S(A)$ is the initial action (1.10.58). In the integral (1.10.63) we have introduced the source J_0 to the field A^0 for the sake of symmetry, although the corresponding Green functions are not necessarily present in the calculation of the physical quantities.

It is useful to rewrite the integral (1.10.63) in the following form:

$$Z(J) = \int \mathcal{D}A^\mu \mathcal{D}B \exp\left\{\frac{i}{\hbar}\Big(S(A) + B\chi + J_\mu A^\mu\Big)\right\} \tag{1.10.64}$$

by introducing an auxiliary scalar field B multiplying the gauge function $\chi = \partial_i A_i$. Here, owing to the δ-function, we have the generating functional represented by the singular gauge. It is often convenient to use a non-singular form of gauge in the functional integral for Green's functions:

$$Z(J) = \int \mathcal{D}A^\mu \mathcal{D}B \exp\left\{\frac{i}{\hbar}\Big(S(A) + B\chi + \frac{\alpha}{2}B^2 + J_\mu A^\mu\Big)\right\}, \tag{1.10.65}$$

where α is a gauge parameter. It is interesting to observe that the same form of the generating functional is obtained in the case of covariant gauges, as in the Feynman case. Then one has to substitute into (1.10.64) the Lorentz gauge

$$\chi = \partial_\mu A^\mu, \tag{1.10.66}$$

with the result

$$Z(J) = \int \mathcal{D}A^\mu \mathcal{D}B \exp\left\{\frac{i}{\hbar}\Big(S(A) + B\partial_\mu A^\mu + \frac{\alpha}{2}B^2 + J_\mu A^\mu\Big)\right\}. \tag{1.10.67}$$

Integrating in (1.10.67) over the field B, we obtain the generating functional in the generalized Feynman gauge:

$$Z(J) = \int \mathcal{D}A^\mu \exp\left\{\frac{i}{\hbar}\Big(S(A) - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + J_\mu A^\mu\Big)\right\} \tag{1.10.68}$$

1.11 Naive Feynman rules

The quantization of electrodynamics started in the late 1920's, beginning from the procedure of canonical quantization. Some gauges, e.g. the Lorentz gauge, could not be imposed as operator identitis on the whole state space, but only on the subspace of physical states. This resulted in the concept of Hilbert spaces with indefinite metric and a certain projection operator onto the physical states – which reappeared afterwards in the case of more complicated gauge theories as the BRST operator. In the mid 1940's, the Feynman path integral quantization of classical electrodynamics, circumvented this canonical approach, introducing the diagrammatic rules for the computation of S–matrix elements directly in terms of the effective action $S_{\rm eff}$. This effective action is obtained by adding an appropriate gauge part $S_g(A)$ to the action (1.10.58), for example, in the form

$$S_g(A) = -\frac{1}{2\alpha}\Big(\partial_\mu A^\mu\Big)^2 \tag{1.11.69}$$

where α is the gauge parameter. Then the generating functional $Z(J)$ for the quantized electrodynamics has the form

$$Z(J) = \int DA \exp\left\{\frac{i}{\hbar}\Big(S_{\rm eff}(A) + JA\Big)\right\}, \tag{1.11.70}$$

where

$$S_{\rm eff}(A) = S(A) + S_g(A). \tag{1.11.71}$$

The interaction with (charged) spinor fields, e.g. electron-positron field, did not meet with serious troubles: the generalization of the above canonical quantization method to the case of anticommuting variables is straightforward. However, incorporating them into the path integral formalism led to intrioducing (classical) anticommuting variables (so-called Grassmann variables), defining integrals over them, and replacing determinants in the functional integral for Green's functions by superdeterminats (for the reader's convenience some properties to be used in the rest of this review are given in **Appendix B**).

Until the early 1960's the idea existed that the naive Feynman rules could be constructed by using the functional integral over all fields of the initial theory with the action modified by the gauge, if necessary. As has been illustrated by the specific examples, after integrating over the momenta, the expression for the generating functional of Green's functions, obtained by (modified) canonical quantization, can be written in the form

$$Z(J) = \int D\phi \exp\left\{\frac{i}{\hbar}\Big(S_{\rm eff}(\phi) + J\phi\Big)\right\}, \tag{1.11.72}$$

where the set of integration fields ϕ includes both the initial fields of the theory and some additional fields, like Nakanishi–Lautrup field, Faddeev–Popov ghost and antighost fields (see the next chapter), meanwhile the effective action $S_{\rm eff}$ is a non-degenerate functional of all fields ϕ.

It appears to be an attractive idea to construct the effective action directly from the action S of the original classical singular (gauge) theory without having recourse to the procedure of canonical quantization. Such an approach, referred to as Lagrangian quantization, has an additional advantage – that the formalism can be manifestly covariant.

Chapter 2

Faddeev-Popov and BRST Quantization

2.1 Yang-Mills fields

In 1954 an important step in the theory of gauge fields was taken by C.N. Yang and R.L. Mills [207]. They introduced the concept of non-abelian gauge fields A_μ and constructed the action for these theories analogous to electrodynamics. [1]

A Yang–Mills field can be associated with any compact semi-simple Lie group G, i.e., a compact group without (nontrivial) Abelian invariant subgroup. (For a short exposition of terminology and definitions being relevant in the following, see **Appendix A**.) The number of independent parameters $\xi^a, a = 1, \dots, n$, which characterize an arbitrary element $g(\underline{\xi})$ of this group, i.e., the dimension of G, is denoted by n. Among the representations of this group and of the corresponding Lie algebra $Lie(G)$, there exists a distinguished representation by $n \times n$ matrices, the regular or adjoint representation. Any element ξ in the adjoint representation M of the Lie algebra can be represented by a linear combination of the n skew-hermitian generators $T_a \equiv ad(X_a)$, cf. Eq. (A.0.17),

$$\xi = \xi^a T_a, \quad \text{with} \quad T_a^+ = -T_a,$$

and any element $g \in G$ of the Lie group is given by $g(\underline{\xi}) = \exp\{\xi^a T_a\}$. The generators T_a can be normalized according to

$$\mathrm{tr}\,(T_a T_b) = \delta_{ab}, \tag{2.1.1}$$

defining the Cartan metric of G, cf. Eq. (A.0.18). In that case the group manifold is Euclidean and, by convention, the group indices will be written as upper ones only. Then, the structure constants f^{abc} of the Lie algebra may be chosen *completely antisymmetric* and the product in the Lie algebra, i.e. the Lie bracket, is given by:

$$[T^a, T^b] = f^{abc} T^c. \tag{2.1.2}$$

The Yang–Mills field is given by a Lorentz vector $A_\mu(x)$ on Minkowski spacetime, $x \in M_4$, taking its values in $Lie(G)$. It is convenient to consider $A_\mu(x)$ as a matrix in the adjoint

[1] The idea of a non-abelian gauge theory with gauge group $SU(2)$ has been formulated already by O. Klein at a Conference in 1938 in Warsaw [138]

representation of this algebra. In this case the field $A_\mu(x)$ is defined by its coefficients,

$$A_\mu(x) = A^a_\mu(x) T^a,$$

with respect to the basis $\{T^a\}$ of the generators. The gauge transformations of the field $A_\mu(x)$ are defined by the rule

$$A_\mu(x) \to A^g_\mu(x) = g^{-1}(x) A_\mu(x) g(x) + g^{-1}(x)\partial_\mu g(x), \qquad (2.1.3)$$

where $g(x) = \exp\big(\xi^a(x) T_a\big)$ – at any value of x – is a matrix taking its values in the adjoint representation of the group G. It is easy to see that these transformations compose a group, which is called the group of gauge transformation or, in short, the gauge group, $\mathcal{G} = \prod_x G_x$.[2]

It is often convenient to deal with the infinitesimal form of the gauge transformations. Let the matrices $g(x)$ differ infinitesimally from the unit matrix, i.e.,

$$g(x) = 1 + \xi(x) + \ldots = 1 + \xi^a(x) T^a + \ldots,$$

where $\xi(x)$ belongs to the Lie algebra of the group G_x, and the ellipses indicate the terms of second and higher order in the infinitesimal group parameters ξ^a. Then the change of A_μ under that transformation will be

$$\delta A_\mu(x) = \partial_\mu \xi(x) + [A_\mu(x), \xi(x)] =: D_\mu(A)\xi(x),$$

with the covariant derivative $D_\mu(A) \equiv \partial_\mu + [A_\mu(x), \,\boldsymbol{\cdot}\,]$ and, in components,

$$\delta A^a_\mu = \partial_\mu \xi^a + f^{abc} A^b_\mu \xi^c = D^{ab}_\mu \xi^b, \quad D^{ab}_\mu = \delta^{ab}\partial_\mu + f^{acb} A^c_\mu.$$

One can easily check that

$$[D_\mu, D_\nu]\xi = [\mathrm{F}_{\mu\nu}, \xi] \quad \text{with} \quad \mathrm{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

being the field strength tensor of the Yang-Mills field A_μ. This generalizes the electromagnetic field strengh $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with the electromagnetic potentials A_μ being the gauge field related to the abelian group $U(1)$. One can represent $\mathrm{F}_{\mu\nu}$ as

$$\mathrm{F}_{\mu\nu} = \mathrm{F}^a_{\mu\nu} T^a, \quad \text{where} \quad \mathrm{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu.$$

Notice that the tensor $\mathrm{F}_{\mu\nu}$ transforms homogeneously under the gauge transformations (2.1.3),

$$\mathrm{F}^g_{\mu\nu} = g^{-1} \mathrm{F}_{\mu\nu} g,$$

which implies that the classical action functional [3]

$$S(A) = -\frac{1}{4}\int dx\, \mathrm{tr}\left(\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}\right) = -\frac{1}{4}\int dx\, \mathrm{F}^a_{\mu\nu}\mathrm{F}^{a\mu\nu} \qquad (2.1.4)$$

is invariant under gauge transformations,

$$S(A^g) = S(A).$$

---

[2] A mathematical more adequate formalism uses the notion of a principle bundle P over the manifold M_4 whose fibres at each point $x \in M_4$ are (identical copies of) the group G. The gauge field $A_\mu(x)$ defines a connection of the cotangent bundle T^*P, and the field strenght $F_{\mu\nu}(x)$ is the associated curvature. For an introduction to that formulation of classical gauge theories, see, e.g. [159].

[3] Here, it should be mentioned that we already dropped the gauge coupling g of the Yang-Mills field which usually appears in the covariant derivative and the field strenght through $\mathsf{g}f^{abc}$ by including it into the gauge field A_μ. Then the action functional should contain an additional factor $1/\mathsf{g}^2$ which, for simplicity, in the following will be set equal to one.

2.2 Orbits

By analogy with electrodynamics, one may try to quantize the theory under consideration in the form of the Lagrangian path integral with the classical action $S(A)$, Eq. (2.1.4), modified by the gauge fixing action with some gauge parameter α,

$$S_{\rm gf}(A) = -\frac{1}{2\alpha}\int dx\, {\rm tr}\left(\partial^\mu A_\mu\right)^2 = -\frac{1}{2\alpha}\int dx\left(\partial^\mu A^a_\mu\right)\left(\partial^\nu A^a_\nu\right).$$

It is exactly like this that Feynman [83] analyzed in 1963 the problem of S-matrix unitarity in Yang-Mills theories and also in Einstein gravity, thus discovering the non-unitarity of the physical S-matrix in these theories. He also showed that the failure could be cured by introducing so-called 'ghost fields". Therefore a natural question arises: How can a properly defined functional $Z(J)$ be constructed in configuration space for Yang-Mills theories? The answer to this question was found in 1967 by L.D. Faddeev and V.N. Popov [79] and by B.S. DeWitt [74].

Following the naive Feynman rules of quantization, one expects that the vacuum functional may be expressed in the form of a functional integral over all field configurations,

$$\langle 0|0\rangle \sim \int \mathcal{D}A_\mu \exp\left\{\frac{i}{\hbar}S(A)\right\}, \tag{2.2.5}$$

where the integration measure $\mathcal{D}A_\mu$ is required to be invariant under gauge transformations,

$$\mathcal{D}A_\mu = \mathcal{D}A^g_\mu.$$

The simplest (formal) *Ansatz* for the integration measure, satisfying this property, is

$$\mathcal{D}A_\mu = \prod_{\mu,a,x} dA^a_\mu(x).$$

However, since the integration is taken over all possible configurations A_μ, this implies multiple counting of physically equivalent configurations, i.e., those being equal up to a gauge transformation.

Therefore, let us divide the configuration space of the gauge fields into equivalence classes $\{A^g_\mu(x) : g(x) \in \mathcal{G}\}$, called orbits of the gauge group. Namely, an orbit of the group includes all the field configuration which arise when all possible transformations $g(x)$ of the gauge group $\mathcal{G}$ are applied to a given initial field configuration $A_\mu(x)$. Obviously, the integrand of the functional integral (2.2.5) is ill-defined – the action remains constant along any orbit of the gauge group. Consequently, the integral is proportional to an infinite constant – the volume $\Pi_x {\rm vol}(G_x) = {\rm vol}(G) \times M_4$ of the gauge group $\mathcal{G}$.

2.3 Factoring out the volume of gauge group

Faddeev and Popov suggested a procedure of factoring out this infinite constant of the path integral. The idea is to split the integration over the full configuration space into an integration over the equivalence classes of configurations $\{A^g_\mu\}$ and a further integration over the configurations of any individual orbit. This corresponds to a change of 'coordinates" from $\{A_\mu(x)\}$ to $\{A^{g_0}_\mu(x), g(x)\}$, i.e., to exactly one representant $A^{g_0}_\mu$ of each orbit and the group manifold (along the orbit).

First, let $\mathcal{D}g$ denote an invariant measure, e.g., the (continuous product of) Haar measure, on the gauge group $\mathcal{G}$,

$$\mathcal{D}g = \mathcal{D}(gg'); \quad \mathcal{D}g = \prod_x dg(x).$$

Furthermore, let us introduce a functional $\triangle[A_\mu]$ as follows:

$$1 = \triangle[A_\mu] \int \mathcal{D}g\, \delta\big(\chi[A_\mu^g]\big). \tag{2.3.6}$$

Here, $\delta\big(f\big)$ represents the continuous product of the usual Dirac δ-functions, $\prod_x \delta(f(x))$, one corresponding to each space-time point. Concerning the functional $\chi[A_\mu]$, we assume that the equation (with respect to $g(x) \in \mathcal{G}$)

$$\chi[A_\mu^g] = 0$$

has exactly one solution, g_0, for any initial field $A_\mu \equiv A_\mu^e$, e: unit of $\mathcal{G}$. [4] Then, in the configuration space $\{A_\mu\}$ the equation $\chi[A_\mu^g] = 0$ defines a hypersurface that intersects any of the orbits exactly once. In other words, $\chi[A_\mu^g] = 0$ defines a 'gauge' by fixing a field $A_\mu^{g_0}(x)$ which represents the orbit. In general, the functional $\chi[A_\mu]$ should be of the form $\chi^a(x, [A_\mu])$. In fact, to fix a gauge we need at each space-time point one equation for each group parameter ξ^a.

Notice that $\triangle[A_\mu]$ is invariant under gauge transformations. This can be demonstrated by

$$\triangle^{-1}[A_\mu^g] = \int \mathcal{D}g'\delta\chi\Big([A_\mu^{gg'}]\Big) = \int \mathcal{D}(gg')\delta\Big(\chi[A_\mu^{gg'}]\Big) = \int \mathcal{D}g''\delta\Big(\chi[A_\mu^{g''}]\Big) = \triangle^{-1}[A_\mu],$$

where we have used the invariance of the group measure $\mathcal{D}g$. In fact, $\triangle[A_\mu]$ is a functional on the space of orbits.

Now, our aim is to replace the integration over all field configurations by an integration restricted to the hypersurface $\chi[A_\mu] = 0$. In that case each orbit would contribute with only one field configuration, and we are left with an integration over physically distinct fields only. This is achieved as follows. We start by inserting (2.3.6) into the path integral (2.2.5). Then we change the order of integration, which implies

$$\int \mathcal{D}g \int \mathcal{D}A_\mu \triangle [A_\mu]\delta\big(\chi[A_\mu^g]\big) \exp\left\{\frac{i}{\hbar}S(A)\right\}.$$

An important observation is that the total expression under the integral $\int \mathcal{D}g$ is, in fact, independent of g. To demonstrate this, we use the gauge invariance of $\int \mathcal{D}A_\mu$, $\triangle[A_\mu]$ and $S(A)$, replacing them by $\int \mathcal{D}A_\mu^g$, $\triangle[A_\mu^g]$ and $S(A^g)$, respectively; the result,

$$\int \mathcal{D}A_\mu^g \triangle [A_\mu^g]\delta\big(\chi[A_\mu^g]\big) \exp\left\{\frac{i}{\hbar}S(A^g)\right\}$$

can be made manifestly g-independent by a change of notation: $A_\mu^g \to A_\mu$. Consequently, the group integration $\int \mathcal{D}g$ factorizes out producing an infinite constant: the volume of the complete gauge group. We finally obtain

$$\Big(\int \mathcal{D}g\Big) \int \mathcal{D}A_\mu \triangle [A_\mu]\delta\big(\chi[A_\mu]\big) \exp\left\{\frac{i}{\hbar}S(A)\right\}.$$

[4] For non-abelian gauge theories this requirement cannot be fulfilled globally, i.e., there never exists a global section of the bundle T^*P which cuts every fibre only once thereby fixing one and only one representant of each orbit [111]. However, this is not what is required for quantizing small fluctuations of the theory. It is only necessary that the solution of Eq. (2.3.7) is unique in the neighbourhood of the classical extremals.

2.4 Faddeev-Popov determinant

Next, we need to calculate $\triangle[A_\mu]$. By definition it holds $1 = \int \mathcal{D}\chi\delta\big(\chi[A_\mu^g]\big)$. After formally changing variables $g \leftrightarrow \chi$ (which is possible at least if χ depends linearly on A_μ):

$$\triangle^{-1}[A_\mu] = \int \mathcal{D}g\delta\big(\chi[A_\mu^g]\big) = \int \mathcal{D}\chi \left(\mathrm{Det}\frac{\delta\chi[A_\mu^g]}{\delta g}\right)^{-1} \delta\big(\chi[A_\mu^g]\big),$$

we obtain

$$\triangle[A_\mu] = \mathrm{Det}\left(\frac{\delta\chi[A_\mu^g]}{\delta g}\right)\Bigg|_{\chi[A_\mu^g]=0}. \tag{2.4.7}$$

$\triangle[A_\mu]$ is called the Faddeev-Popov determinant. It is the Jacobian of a 'coordinate transformation" from χ to g.

It is convenient to use the gauge invariance of $\triangle[A_\mu]$ to choose A_μ such that it already satisfies the gauge condition $\chi[A_\mu] = 0$. Then in (2.4.7) we can take the constraint $\chi[A_\mu^g] = 0$ at $g = e$, which simplifies practical calculations:

$$\triangle[A_\mu] = \left(\mathrm{Det}\frac{\delta\chi[A_\mu^g]}{\delta g}\right)\Bigg|_{g=e}. \tag{2.4.8}$$

In the vicinity of $g = e$ we should only deal with the infinitesimal transformations: $g(\xi) = 1 + \xi^a T^a$ (where $\xi^a(x) \ll 1$). We can now rewrite (2.4.8) in a more explicit form, with all relevant indices

$$\triangle[A_\mu] = \left(\mathrm{Det}\frac{\delta\chi^a(x,[A_\mu^\xi])}{\delta\xi^b(y)}\right)\Bigg|_{\xi=0} \equiv \mathrm{Det}M^{ab}(x,y).$$

We have to calculate the determinant of a matrix in both space-time and the group indices $M^{ab}(x,y)$. For the Lorentz covariant gauge

$$\chi^a[A_\mu] \equiv \partial_\mu A^{\mu a}(x) = 0$$

we obtain

$$M^{ab}(x,y) = \partial_\mu D^{\mu ab}\delta(x-y).$$

The generating functional of Green's functions takes on the form,

$$Z(J) = \int \mathcal{D}A_\mu \mathrm{Det}\Big(\partial^\mu D_\mu^{ab}\Big)\delta(\partial^\mu A_\mu^a)\exp\left\{\frac{i}{\hbar}[S(A) + J_\mu^a A^{\mu a}]\right\}. \tag{2.4.9}$$

The same result (2.4.9) can be obtained (see, for example, [103]) by using the method of canonical quantization (cf. Eq. (1.10.57) for the general expression in phase space).

2.5 Ghost and antighost fields

The standard method of dealing with the Faddeev-Popov determinant is to replace it by an additional functional integration over auxiliary, *mutually independent* complex scalar fields

$C^a(x)$ (ghost fields) and $\bar{C}^a(x)$ (antighost fields), which are Grassmann variables: [5]

$$\mathrm{Det}\Big(\partial_\mu D^{\mu ab}\Big) = \int \mathcal{D}C\mathcal{D}\bar{C}\exp\Big\{\frac{i}{\hbar}\bar{C}^a M^{ab}C^b\Big\} \equiv \int \mathcal{D}C\mathcal{D}\bar{C}\exp\Big\{\frac{i}{\hbar}\int dxdy\bar{C}^a(x)M^{ab}(x,y)C^b(y)\Big\}.$$

Introducing additional auxiliary fields $B^a(x)$ (Nakanishi–Lautrup fields), we can also represent $\delta(\partial_\mu A^{\mu a})$ in the form of a functional integral,

$$\delta(\partial_\mu A^{\mu a}) = \int \mathcal{D}B\exp\left\{\frac{i}{\hbar}B^a\partial_\mu A^{\mu a}\right\}$$

so that finally we obtain

$$Z(J) = \int \mathcal{D}\phi\exp\left\{\frac{i}{\hbar}(S_{\text{eff}}(\phi) + J_A\phi^A)\right\}. \tag{2.5.10}$$

In Eq.(2.5.10) we have introduced the effective action,

$$S_{\text{eff}}(\phi) = S(A) + B^a\chi^a + \bar{C}^a\frac{\delta\chi^a}{\delta A^{\mu c}}D^{\mu cb}C^b, \quad \text{with} \quad \chi^a(x) = \partial_\mu A^{\mu a}(x), \tag{2.5.11}$$

and the entire set of dynamical fields in the Lagrangian formalism which constitute the so-called extended configuration space of the Yang-Mills theory under consideration:

$$\phi^A = (A^{\mu a}, B^a, C^a, \bar{C}^a), \tag{2.5.12}$$

2.6 Faddeev-Popov action

For the sake of formal symmetry we have introduced also sources for the auxiliary fields $B^a, C^a, \bar{C}^a$ in the integral (2.5.10). Furthermore, let us denote by $\varepsilon(\phi)$ the Grassmann parity and by $gh(\phi)$ the ghost number of a field ϕ, which are given for the various quantities in Yang–Mills theories as follows:

$$\varepsilon(A^a_\mu) = \varepsilon(B^a) = 0, \qquad \varepsilon(C^a) = \varepsilon(\bar{C}^a) = 1, \qquad \varepsilon(\xi^a) = 0, \qquad \varepsilon(S) = 0, \tag{2.6.13}$$

$$gh(A^a_\mu) = gh(B^a) = 0, \quad gh(C^a) = 1, \quad gh(\bar{C}^a) = -1, \quad gh(\xi^a) = 0, \quad gh(S) = 0. \tag{2.6.14}$$

This finishes the Faddeev–Popov method to obtain an effective action, the Faddeev–Popov action S_{FP}, for any Yang-Mills theory:

$$S_{\text{FP}}(\phi) = S(A) + B^a\partial_\mu A^{\mu a} + \bar{C}^a\partial_\mu D^{\mu ab}C^b. \tag{2.6.15}$$

Besides of the Lagrange multiplier field B which easily may be integrated out – leading to the Landau gauge – the unphysical ghost and antighost fields C and $\bar{C}$ occure. In the case of electromagnetism these fields decouple since, because of the abelianess of the gauge group, i.e., $f^{abc} = 0$, the Faddeev–Popov determinant $\mathrm{Det}\partial_\mu\partial^\mu$ is field independent and the ghost field equation of motion, $\Box C = 0$, decouples from the gauge field A_μ. In that case the ghost and antighost fields also may be integrated out leading to

$$\|\{\chi,\phi\}\| = \begin{pmatrix} \partial_0^2 & 0 \\ 0 & -\triangle \end{pmatrix}\delta(\vec{x}-\vec{y});$$

[5] Here, and in the following we use the convention of DeWitt [73], see also [75], where indices are assumed to contain also the spacetime variables, if necessary, and summation over the indices then also include integration over these continuous variables

in the discussion of Chapter **1** the constant Faddeev-Popov determinant has been dropped!

For nonsingular gauges, like the Feynman gauge which obtains after addition of the term $(\alpha/2)B^2$ to Eq. (2.6.15), the Nakanishi-Lautrup field could be also integrated out, thus leading to the following action

$$S_{\mathrm{FP}}(\phi) = S(A) - \frac{1}{2\alpha}\left(\partial_\mu A^{\mu a}\right)^2 + \bar{C}^a \partial_\mu D^{\mu ab} C^b. \tag{2.6.16}$$

The action (2.6.15) or, likewise, the action (2.6.16) is the starting point to determine the Feynman rules of Yang-Mills theories, in order to compute scattering amplitudes, decay rates, and so on, for the physical field A_μ. Ghost and antighost fields occure in internal lines only, but their contribution adds up such that the S-matrix comes out to be unitary.

Here, it should be remarked, that Yang–Mills theories may be quantized also by the canonical formalism, leading to the same physical results – but manifest relativistic covariance would be lost. There appear also no problems to include interactions with matter fields, like Dirac or scalar fields, as is necessary in formulating Quantum Chromodynamics or the Electroweak Standard Model (see below). – In addition, it should be mentioned that any gauge fixing functional $\chi[A_\mu]$ works as long as it is *local* and *linear* in the fields. For the non-covariant axial gauges, like $\chi \equiv n^\mu A_\mu$, n^μ: const., it is even possible to decouple the (anti)ghost fields again – however, other complications are introduced instead.

As is well known from QED that, because of gauge symmetry, the Green's functions, in general, are not independent from each other. There occure relations between them which are governed by some Ward identities, relating, e.g., the (3–point) vertex function and the derivative of the (2-point) propagator. The same situation, but much more involved, also occurs for nonabelian Yang–Mills theories. Furthermore, it has to be proven that after renormalization the quantized theory shows the same symmetries as the classical (effective) action. Of course, the gauge invariance with respect to $\mathcal{G}$ is broken by the gauge fixing and ghost terms, $S_{\mathrm{gf}} + S_{\mathrm{gh}}$, but there must be some relic of the original symmetry group G!

2.7 BRST symmetry and BRST cohomology

The next important step in the development of gauge theories was taken by C. Becchi, A. Rouet and R. Stora [46] and also, independently, by I.V. Tyutin [195]. They discovered a remarkable invariance of the action (2.6.15) under some *nonlinear global supertransformations* in the extended configuration space, the so-called BRST-transformations:

$$\delta_{\mathrm{B}} S_{\mathrm{FP}}(\phi) = 0, \tag{2.7.17}$$

with

$$\delta_{\mathrm{B}} A_\mu^a = D_\mu^{ab}(A) C^b \lambda, \qquad \delta_{\mathrm{B}} C^a = -\tfrac{1}{2} f^{abc} C^b C^c \lambda, \tag{2.7.18}$$

$$\delta_{\mathrm{B}} \bar{C}^a = B^a \lambda, \qquad \delta_{\mathrm{B}} B^a = 0, \tag{2.7.19}$$

where λ is a constant Grassmann parameter ($\varepsilon(\lambda) = 1$). The first set of fields, the so-called *minimal pair*, transforms nonlinear in the fields, whereas the second set of fields, the so-called *trivial pair*, transforms linear. For the initial fields of the theory, A_μ (as well as possible matter fields), the BRST-transformations are gauge transformations with gauge parameters $\xi^a(x) = C^a(x)\lambda$. Owing to this fact, the initial action $S(A)$ is invariant under these transformations.

Furthermore, the transformation rule of the ghost fields encodes the (global) symmetry group G through their structure constants.

Let us now define an operator s acting on the fields ϕ^A:

$$\delta_{\mathrm{B}}\phi^A = (\mathsf{s}\phi^A)\lambda. \tag{2.7.20}$$

This operator is the *generator of BRST-transformations* in the Lagrangian formalism. One can verify that it is nilpotent, i.e.,

$$\mathsf{s}^2 = 0. \tag{2.7.21}$$

Indeed, [6]

$$\begin{aligned}
\mathsf{s}^2 B^a &= \mathsf{s}(\mathsf{s}B^a) = 0,\\
\mathsf{s}^2 \bar{C}^a &= \mathsf{s}(\mathsf{s}\bar{C}^a) = (\mathsf{s}B^a) = 0,\\
\mathsf{s}^2 C^a &= \mathsf{s}\big(\tfrac{1}{2} f^{abc} C^c C^b\big) = \tfrac{1}{2} f^{abc} C^c f^{bde} C^d C^e =\\
&= -\frac{1}{6}(f^{acb}f^{bde} + f^{adb}f^{bec} + f^{aeb}f^{bcd})C^c C^d C^e = 0,\\
\mathsf{s}^2 A^a_\mu &= \mathsf{s}(D^{ab}_\mu C^b) = \tfrac{1}{2} D^{ab}_\mu (f^{bcd} C^d C^c) + \frac{\delta D^{ab}_\mu}{\delta A^c_\nu}(D^{cd}_\nu C^d) C^b =\\
&= \tfrac{1}{2} f^{abd}\partial_\mu (C^d C^b) + f^{adb}(\partial_\mu C^d) C^b + \tfrac{1}{2} f^{aeb} f^{bcd} A^e_\mu C^d C^c + f^{acb} f^{ced} A^e_\mu C^d C^b\\
&= \tfrac{1}{2}(f^{acb}f^{bde} + f^{adb}f^{bec} + f^{aeb}f^{bcd}) A^e_\mu C^d C^c = 0.
\end{aligned}$$

Here, we have used the equalities following from the Jacobi identity (cf. Eq. (A.0.8))

$$f^{abc}f^{cde} + f^{adc}f^{ceb} + f^{aec}f^{cbd} \equiv 0;$$

computing $(\delta D^{ab}_\mu/\delta A^c_\nu)(D^{cd}_\nu C^d)$ it must be observed that the sum over ν implicitly contains an integration over, say, y leading to $\int dy\, \big(f^{acb}\delta(x-y)\big)\, D^{cd}_\nu(y) C^d(y)$! In addition, we remark that

$$S_{\mathrm{gf}} + S_{\mathrm{gh}} = \mathsf{s}\Psi(\phi) \qquad \text{with} \qquad \Psi(\phi) = \bar{C}^a \partial^\mu A^a_\mu \tag{2.7.22}$$

holds. Because of the nilpotency of s this makes the proof of BRST invariance of S_{FP} quite trivial. The same would hold for any other choice of the so-called gauge fermion Ψ of ghost number $gh(\Psi) = -1$, also if its dependence on A would not be linear! (In Feynman gauge the gauge fermion reads $\Psi(\phi) = \bar{C}^a(\partial^\mu A^a_\mu + \frac{\alpha}{2}B^a)$). Let us point also to another fact. If the Nakanishi-Lautrup fields are integrated out from the action the BRST transformation of the trivial pair reduces to the following single rule for the antighost $\bar{C}$ only:

$$\mathsf{s}\bar{C}^a = -\frac{1}{\alpha}\chi^a[A_\mu]. \tag{2.7.23}$$

However, then the BRST operator fails to be nilpotent when applied to $\bar{C}$! On the other hand, nilpotency of the BRST operator is a very essential ingredient for a consistent physical interpretation of quantum gauge theories as will be shown now.

[6] Here, it should be kept in mind that the BRST operator s according to its definition, Eq. (2.7.20), *acts from the right*, cf. [103, 108]. This has to be taken into account when its action on Grassmann odd variables is to be considered! Warning: Some textbooks, e.g., [205, 168, 80] introduce s as (usual) left operation by writing λ in Eqs. (2.7.18) and (2.7.19) also to the left!

The physical content of the theory is given by the entire set of BRST-invariant functionals $\Phi(\phi)$ of ghost number zero, $\mathsf{s}\Phi(\phi)=0$, modulo the set of BRST-variations of any functional $\Psi(\phi)$ of ghost number -1, $\mathsf{s}\Psi(\phi)$, which are invariant because of the nilpotence of s. Generally speaking, *physical observables as well as states are nontrivial cohomology classes of the BRST operator.* [7] Let us qualify this statement more explicitly.

From the BRST invariance of the FP action (2.6.15) it follows – as for any global symmetry – the existence of a conserved Noether current

$$J^\mu_{\rm B}=\frac{\delta_r S}{\delta\phi^A_\mu}\,\mathsf{s}\phi^A \quad \text{with} \quad \partial_\mu J^\mu_{\rm B}=0,$$

where $\delta_r X(\phi)/\delta\phi^A_\mu$ denotes the *right derivative* of a functional $X(\phi)$. Its explicit form reads

$$\begin{aligned} J^\mu_{\rm B} &= -F^{a\mu\nu}(D^{ab}_\nu C^b)+B^a(D^{\mu ab}C^b)-\tfrac{1}{2}f^{abc}(\partial^\mu\bar{C}^a)C^bC^c, \\ &= B^aD^{\mu ab}C^b-(\partial^\mu B^a)C^a+\tfrac{1}{2}f^{abc}(\partial^\mu\bar{C}^a)C^bC^c-\partial_\nu(F^{a\mu\nu}C^a), \end{aligned} \tag{2.7.24}$$

where, for the second line, the field equations of A_μ have been used. In general, the total divergence does not contribute to the corresponding conserved charge, the BRST charge:

$$Q_{\rm B}=\int d^3\mathbf{x}J^0_{\rm B} \quad \text{with} \quad \frac{dQ_{\rm B}}{dt}=0.$$

The theory contains another conserved current J^μ_C, the ghost current, and a conserved charge Q_C, the ghost or FP charge, which is associated with the invariance of the FP-action under the *scale* transformation

$$C^a\to e^\theta C^a, \quad \bar{C}^a\to e^{-\theta}\bar{C}^a,$$

where θ is a constant Grassmann even parameter:

$$\begin{aligned} J^\mu_{\rm C} &= i\big(\bar{C}^a(D^{\mu ab}C^b)-(\partial^\mu\bar{C}^a)C^a\big) \quad \text{with} \quad \partial_\mu J^\mu_{\rm C}=0, \\ Q_{\rm C} &= \int d^3\mathbf{x}J^0_{\rm C} \quad \text{with} \quad \frac{dQ_{\rm C}}{dt}=0. \end{aligned}$$

In exactly the same manner as the fields are supplemented with the usual charges corresponding to some phase transformation, all the fields here can be supplemented with the (conserved) ghost number $gh(\phi)$, Eq. (2.6.14), the eigenvalue of the ghost operator on the corresponding field operator.

Furthermore, both currents are related:

$$J^\mu_{\rm B}=-\mathsf{s}J^\mu_{\rm C}-\partial_\nu(F^{a\mu\nu}C^a). \tag{2.7.25}$$

As is well known from Quantum Electrodynamics the full state space of the theory is a Hilbert space $\mathcal{V}$ with *indefinite metric*: scalar photons have zero norm and longitudinal ones may have negative norm. However, there exists a projection operator onto the physical Hilbert space having positive definite metric [118, 52]. The same situation occurs for the quantum states of non-abelian gauge theories which, in addition, suffer from the (virtual)

[7] In mathematical terms one says that Φ, which is in the *kernel* of s, is a BRST-closed functional and that Ψ is a BRST-exact functional, since $\mathsf{s}\Psi$ is in the *image* of s. This is completely analogous to the de Rham cohomology of the (nilpotent) exteriour differential $d, d^2=0$, in differential geometry where a form ω of degree p is called closed if $d\omega=0$, and a form ω is called exact if it can be written as $\omega=d\nu$ with some form ν of degree $p-1$. Because of this similarity between d and s the latter is also called an *anti-derivation*.

appearance of ghost and antighost particles. In order to be able to determine the subspace of physical states of the theory, $\mathcal{V}_{\text{phys}} \subset \mathcal{V}$, it is necessary to study the irreducible representations of the BRST and the FP charge in the full state space which, for the first time, has been done by Kugo and Ojima [140]. This plays a crucial role in the formulation a unitary *physical* S-matrix.

Both the BRST charge and the ghost charge, together with the fields ϕ^A, are to be represented by corresponding operators. In fact, the ghost numbers are the eigenvalues of $i\hat{Q}_C$, e.g.,

$$[i\hat{Q}_C, \hat{C}^a] = \hat{C}^a, \qquad [i\hat{Q}_C, \hat{\bar{C}}^a] = -\hat{\bar{C}}^a, \tag{2.7.26}$$

and the (nilpotent) BRST charge operator generates the BRST transformations of the field operators,

$$\mathsf{s}\hat{\phi}^A = [i\hat{Q}_B, \hat{\phi}^A]_{\mp}, \tag{2.7.27}$$

with the commutator and the anticommutator in the case of bosonic and fermionic fields, respectively. Both charge operators are hermitian. They satisfy the following BRST algebra:

$$\{\hat{Q}_B, \hat{Q}_B\} = 2(\hat{Q}_B)^2 = 0, \tag{2.7.28}$$
$$[\,i\hat{Q}_C, \hat{Q}_B\,] = \hat{Q}_B, \tag{2.7.29}$$
$$[\,\hat{Q}_C, \hat{Q}_C\,] = 0. \tag{2.7.30}$$

2.8 Physical state space

A physical state $\mathcal{V}_{\text{phys}}$ is defined by the BRST charge

$$\hat{Q}_B|\varphi\rangle = 0, \qquad \forall|\varphi\rangle \in \mathcal{V}_{\text{phys}} \equiv \ker \hat{Q}_B. \tag{2.8.31}$$

Because of its nilpotence there exist only two types of representations of $\hat{Q}_B$, namely, the *singlet states* $|s\rangle$ and the *doublet states* ($|p\rangle$, $|d\rangle$) being called parent (p) and daughter (d) states:

$$\begin{aligned} \hat{Q}_B|s\rangle &= 0, \\ \hat{Q}_B|p\rangle = |d\rangle &\neq 0, \qquad \hat{Q}_B|d\rangle = 0. \end{aligned}$$

In addition it holds

$$\begin{aligned} \langle\varphi|d\rangle &= \langle\varphi|\hat{Q}_B|p\rangle = 0, \\ \langle d|d\rangle &= \langle p|(\hat{Q}_B)^2|p\rangle = 0. \end{aligned}$$

Therefore, any physical state should be a singlet state modulo some daughter state,

$$|\varphi\rangle = |s\rangle + |d\rangle,$$

i.e., *physical states are nontrivial cohomology classes of* $\hat{Q}_B$. In addition, genuine physical states should have vanishing ghost number, i.e., it should hold

$$\hat{Q}_C|\text{phys}\rangle = 0.$$

States with nonvanishing ghost number occure pairwise with opposite ghost number N with $\langle N|-N\rangle = 1$ which also proves that states $|\psi_\lambda\rangle = |N\rangle + \lambda|-N\rangle$, λ complex, may have negative norm (for $\mathrm{Re}\,\lambda$ negative). [8] In general the following state configurations are possible:

1. BRST singlet with $N = 0$ (genuine physical state),

2a. BRST singlet with $N \neq 0$ together with an FP conjugate d–state (unpaired singlet),

2b. FP conjugate pairs of BRST singlets with $N \neq 0$ (singlet pair),

3. Two FP conjugate BRST doublet states with $N \neq 0$ (quartet state).

This exhausts all the possible representations of the BRST algebra in indefinite inner product spaces (for a comprehensive review, see [140, 158]).

2.9 AntiBRST symmetry

As it became obvious from the previous considerations ghost and antighost fields enter the FP quantization not symmetrically – besides the fact that they could be renamed. However, for the FP action (2.6.15) it was discovered by Curci and Ferrari [66] and, independently, by Ojima [166] that in addition to BRST-symmetry there exists another global supersymmetry, the so-called antiBRST symmetry, which also leaves the quantum Yang–Mills action invariant, provided the gauge-fixing functional χ is *linear* in the fields. These antiBRST-transformations read

$$\bar{\delta}_{\mathrm{B}} A_\mu^a = D_\mu^{ab}(A)\bar{C}^b\bar{\lambda}, \qquad \bar{\delta}_{\mathrm{B}} \bar{C}^a = \tfrac{1}{2} f^{abc}\bar{C}^b\bar{C}^c\bar{\lambda}, \tag{2.9.32}$$

$$\bar{\delta}_{\mathrm{B}} C^a = (-B^a + f^{abc}C^b\bar{C}^c)\bar{\lambda}, \qquad \bar{\delta}_{\mathrm{B}} B^a = -f^{abc}\bar{C}^b B^c\bar{\lambda}, \tag{2.9.33}$$

where for the first pair of transformations C and $\bar{C}$ are exchanged relative to Eqs. (2.7.18) but for the second pair the transformations look more complicated than Eqs. (2.7.19). The antiBRST operator $\bar{\mathsf{s}}$ acting on the fields ϕ^A as

$$\bar{\delta}_{\mathrm{B}}\phi^A = (\bar{\mathsf{s}}\phi^A)\bar{\lambda}.$$

satisfies, together with s, the following algebra:

$$\mathsf{s}^2 = \mathsf{s}\bar{\mathsf{s}} + \bar{\mathsf{s}}\mathsf{s} = \bar{\mathsf{s}}^2 = 0.$$

Later on, in Chapters **4, 5, 7** and **8**, we make use of that possible extension of the formalism.

2.10 Zinn-Justin equation

The BRST symmetry of the effective action should be maintained also for the renormalized Green's functions of the theory. The formulation of that requirement in renormalized perturbation theory is not trivial since (most of) the BRST transformations are nonlinear thus leading to serious problems in properly renormalizing products of operators being defined on the *same* spacetime point. The way out has been given by Kluberg-Stern and Zuber [139] who

[8] The existence of pairs of states with conjugate imaginary eigenvalues iN and $-iN$ of $\hat{Q}_C$ is in accordance with the indefinite metric of the state space.

introduced (external) classical sources, later on called antifields ϕ^*_A by Batalin and Vilkovisky [40], coupling to the nontrivial BRST transforms of the fields ϕ. Therefore, instead of the FP action (2.6.15) we introduce an extended action $S_{\rm ext}$

$$S_{\rm ext}(\phi,\ \phi^*) = S_{FP}(\phi) + A^*_{\mu a} D^{\mu ab} C^b + \bar{C}^*_a B^a + \tfrac{1}{2} C^*_a f^{abc} C^b C^c, \qquad (2.10.34)$$

with the set of antifields ϕ^*_A

$$\phi^*_A = (A^*_{\mu a},\ B^*_a,\ C^*_a,\ \bar{C}^*_a) \quad \text{with} \quad \epsilon(\phi^*_A) = \epsilon(\phi^A) + 1, \quad gh(\phi^*_A) = -1 - gh(\phi^A).$$

Obviously, in the present situation the introduction of the antifields B^* and $\bar{C}^*$ is *not* mandatory; however, for general gauge theories to be considered in the next chapter they occure necessarily. Here, by definition, the antifields ϕ^*_A are *BRST invariant sources* of the BRST-transforms of the fields ϕ^A:

$$\mathsf{s}\phi^A = \frac{\delta S_{\rm ext}}{\delta \phi^*_A}, \qquad \mathsf{s}\phi^*_A = 0.$$

Later on, in the same manner we introduce also antifields coupled to the antiBRST transforms of the fields ϕ^A (cf. Chapter **4**).

By construction, the extended action is invariant under the BRST-transformations

$$\mathsf{s} S_{\rm ext}(\phi,\ \phi^*) = 0$$

which, equivalently, may be expressed by the Zinn-Justin equation:[9]

$$\frac{\delta S_{\rm ext}}{\delta \phi^A} \frac{\delta S_{\rm ext}}{\delta \phi^*_A} = 0. \qquad (2.10.35)$$

Let us remind the reader that this equation is a shorthand notation of the following integrated expression:

$$\int dx \left\{ \frac{\delta S_{\rm ext}}{\delta A^{a\mu}(x)} \frac{\delta S_{\rm ext}}{\delta A^*_{a\mu}(x)} + \frac{\delta S_{\rm ext}}{\delta C^a(x)} \frac{\delta S_{\rm ext}}{\delta C^*_a(x)} + \frac{\delta S_{\rm ext}}{\delta \bar{C}^a(x)} \frac{\delta S_{\rm ext}}{\delta \bar{C}^*_a(x)} \right\} = 0.$$

For the first time, the property of the BRST invariance of the extended action $S_{\rm ext}$, as well as for the 1PI-vertex functional Γ, for gauge theories of Yang-Mills type in the form of Eq. (2.10.35) was realized by J. Zinn-Justin in his lectures in 1975 [208]. Let us emphasize that Eq. (2.10.35) is very general. It does not contain any information about the gauge group. The special aspects of the theory are given only by the classical action $S_0(A)$ and the gauge fixing procedure *à* la Faddeev-Popov.

2.11 Slavnov–Taylor identity

The Zinn-Justin equation allows for a short derivation of the Ward identity of the BRST symmetry in terms of the extended generating functional of Green's functions $\mathcal{Z}(J, \phi^*)$,

$$\mathcal{Z}(J, \phi^*) = \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} \Big(S_{\rm ext}(\phi, \phi^*) + J_A \phi^A \Big) \right\},$$

[9] From now on we use the convention that *any derivation with respect to the fields* – if not stated otherwise – is understood as acting from the *right*, whereas any derivative with respect to the antifields acts – as usual – from the left.

with the evident property

$$\mathcal{Z}(J,\phi^*)|_{\phi^*=0} = Z(J),$$

where $Z(J)$ is given by Eq. (2.5.10) with an effective action written with an arbitrary (linear) gauge function $\chi(A)$. By virtue of Eq. (2.10.35) it follows immediatly

$$\int \mathcal{D}\phi \frac{\delta S_{\rm ext}}{\delta \phi^A}\frac{\delta S_{\rm ext}}{\delta \phi^*_A} \exp\left\{ \frac{i}{\hbar}\Big(S_{\rm ext}(\phi,\phi^*) + J_A\phi^A\Big)\right\} = 0$$

which may be rewritten as

$$\int \mathcal{D}\phi \frac{\delta S_{\rm ext}}{\delta \phi^A}\frac{\delta}{\delta \phi^*_A} \exp\left\{ \frac{i}{\hbar}\Big(S_{\rm ext}(\phi,\phi^*) + J_A\phi^A\Big)\right\} = 0.$$

Taking into account the explicit form of $S_{\rm ext}$, Eq. (2.10.34), we obtain

$$\frac{\delta^2 S_{\rm ext}}{\delta\phi^*_A \delta\phi^A} = 0,$$

and, therefore, we have

$$\frac{\delta}{\delta\phi^*_A}\int \mathcal{D}\phi \frac{\delta S_{\rm ext}}{\delta \phi^A} \exp\left\{ \frac{i}{\hbar}\Big(S_{\rm ext}(\phi,\phi^*) + J_A\phi^A\Big)\right\} = 0.$$

Supposing (as usual) that any integral over a total derivative vanishes as long as the expression under the derivative vanishes at the boundary, $\int \mathcal{D}\phi (\delta/\delta\phi^A) \exp\left\{ \frac{i}{\hbar}\big(S_{\rm ext}(\phi,\phi^*) + J_A\phi^A\big)\right\} = 0$, then, integrating in the last expression by parts, gives the Ward identity of BRST symmetry in terms of the generating functional $\mathcal{Z}$:

$$J_A \frac{\delta \mathcal{Z}(J,\phi^*)}{\delta\phi^*_A} = 0. \tag{2.11.36}$$

For the generating functional of *connected Green's function*, $\mathcal{W}(J,\phi^*) = (\hbar/i)\ln \mathcal{Z}(J,\phi^*)$, the Ward identity resulting from (2.11.36) simply reads

$$J_A \frac{\delta \mathcal{W}(J,\phi^*)}{\delta\phi^*_A} = 0. \tag{2.11.37}$$

For the generating functional of the *1PI-vertex functions* being defined through the Legendre transformation of $\mathcal{W}$, [10] where the antifields are independent 'spectators",

$$\Gamma(\phi,\phi^*) = \mathcal{W}(J,\phi^*) - J_A\phi^A,$$

$$\phi^A = \frac{\delta \mathcal{W}(J,\phi^*)}{\delta J_A}, \qquad \frac{\delta\Gamma(\phi,\phi^*)}{\delta\phi^A} = -J_A, \qquad \frac{\delta \mathcal{W}(J,\phi^*)}{\delta\phi^*} = \frac{\delta\Gamma(\phi,\phi^*)}{\delta\phi^*},$$

the corresponding Ward identity has the form of the Zinn-Justin equation,

$$\frac{\delta\Gamma}{\delta\phi^A}\frac{\delta\Gamma}{\delta\phi^*_A} = 0. \tag{2.11.38}$$

This *nonlinear* identity is often refered to as the Slavnov–Taylor identity [191, 186]. It plays a crucial role in the proof of the renormalizability of Yang-Mills type theories based on the BRST-symmetry [208, 209]; for a comprehensive review based on the BRST-algebraic approach, see, e.g., [168].

[10] Now, the functions ϕ which enter Γ are classical C^∞–functions like the (external) antifields ϕ^*.

2.12 Renormalization

The vertex functional Γ is the basic object to study renormalizability of a quantum field theory. In general, it is a very complicated functional of the fields and antifields whose loop expansion reads

$$\Gamma(\phi, \phi^*) = \Sigma(\phi, \phi^*) + \sum_{N=1}^{\infty} \Gamma_N(\phi, \phi^*), \tag{2.12.39}$$

with $\Sigma = S_{\rm ext}(\phi, \phi^*)$ being the tree approximation. Proving renormalizability means to show, order by order in perturbation theory, that the Slavnov-Taylor (ST) identity, Eq. (2.11.38), together with the gauge fixing condition,

$$\frac{\delta\Gamma}{\delta B^a(x)} = \chi^a(A), \tag{2.12.40}$$

and the ghost equation of motion,

$$\frac{\delta\Gamma}{\delta\bar{C}^a(x)} + \partial_\mu \frac{\delta\Gamma}{\delta A^*_{a\mu}(x)} = 0, \tag{2.12.41}$$

are satisfied; obviously, they hold in the tree approximation. The last equation shows that Γ depends only on the combination $\tilde{A}^*_{a\mu}(x) = A^*_{a\mu}(x) + \partial_\mu \bar{C}^a(x)$; taking this for granted the ghost equation of motion simply reads $\delta\Gamma/\delta\bar{C}^a(x) = 0$.

As long as the gauge function χ depends on A_μ only linearly then, by applying the quantum action principle [156, 157, 64, 141, 142], the gauge condition and the ghost equation of motion may be proven to hold, i.e., any possible obstruction of the vertex functional can be compensated, order by order, by adding corresponding conterterms into the vertex functional Γ without disturbing its structure in terms of fields (and antifields). If the gauge condition depends nonlinear on the fields one has to introduce additional sources to deal with this situation, too.

However, also the nonlinear Slavnov-Taylor identity should not be broken by anomalous terms. In order to formulate that condition let us introduce the following equivalent notations for the action of the nonlinear Slavnov-Taylor operator $\mathcal{S}(\cdot)$ on the vertex functional Γ (cf. also Chapter **3**)

$$\mathcal{S}(\Gamma) = \frac{\delta\Gamma}{\delta\phi^A}\frac{\delta\Gamma}{\delta\phi^*_A} \equiv (\Gamma, \Gamma). \tag{2.12.42}$$

Obviously, the ST operator has ghost number $gh(\mathcal{S}) = 1$.

Absence of any obstruction means that, for any order $\hbar^n$, it should hold

$$\sum_{n'=0}^{n} (\Gamma_n, \Gamma_{n-n'}) = 0.$$

If, at order n, the ST identity would be broken by an integrated local polynomial in the fields and antifields $\Delta(\Phi, \Phi^*)$ with $gh(\Delta) = 1$,

$$\mathcal{S}(\Gamma) = \hbar^n \Delta + O(\hbar^{n+1}),$$

then it has to be shown that this could be remedied by an appropriate counterterm of Γ. This leads to the consideration of another cohomology problem. Namely, any possible breaking

Δ has to satisfy some consistency condition. To show this let us introduce the linearized Slavnov–Taylor operator,

$$\mathcal{B}_\Gamma = \frac{\delta\Gamma}{\delta\phi^A}\frac{\delta}{\delta\phi^*_A} + \frac{\delta\Gamma}{\delta\phi^*_A}\frac{\delta}{\delta\phi^A} \tag{2.12.43}$$

which identically obeys

$$\mathcal{B}_\Gamma \mathcal{S}(\Gamma) \equiv 0. \tag{2.12.44}$$

Furthermore, if the ST identity is fulfilled then $\mathcal{B}_\Gamma$ is nilpotent, i.e.,

$$\mathcal{S}(\Gamma) = 0 \Rightarrow (\mathcal{B}_\Gamma)^2 = 0. \tag{2.12.45}$$

Here, it should be mentioned that these statements are true not only for Γ but also for an arbitrary functional F. Especially, the latter statement holds for the classical action Σ. Therefore, because of (2.12.44), the consistency condition for Δ reads

$$\mathcal{B}_\Sigma \Delta = 0. \tag{2.12.46}$$

Of course, this defines a cohomology problem for $\mathcal{B}_\Sigma$ in the sector of integrated local field polynomials of ghost number one:

- If $\Delta = \mathcal{B}_\Sigma \widetilde{\Delta}$, then this cohomology is trivial and the vertex functional can be redefined by subtracting the counterterm $\hbar^n \widetilde{\Delta}$;
- if, however,

  $$\Delta = r\mathcal{A} + \mathcal{B}_\Sigma \widetilde{\Delta} \quad \text{with} \quad r\mathcal{A} \neq \mathcal{B}_\Sigma \widetilde{\mathcal{A}} \tag{2.12.47}$$

  with nonvanishing r, then the Slavnov–Taylor identity is broken by the anomaly $\mathcal{A}$,

  $$\mathcal{S}(\Gamma) = r\hbar^n \mathcal{A} + O(\hbar^{n+1}), \tag{2.12.48}$$

  and the classical symmetry cannot be implemented at the quantum level.

Therefore, renormalizability means absence – or at least mutual compensation – of anomalies.

2.13 BRST quantization

The procedure described up to now extends the Faddeev-Popov–DeWitt quantization in so far as it also works in cases where the (anti)ghost fields occure not only bilinear. Let us, therefore, summarize the various components of that so-called BRST quantization:

- The first step consists in finding the *most general classical action* S_{eff} being invariant under the BRST transformations (possibly, also under further local symmetries like general coordinate transformations as for general relativity) and allowing for a renormalizable quantum action, i.e., the various terms of the Lagrangian should have dimensions not exceeding the dimension of spacetime. In principle, this is equivalent of asking for the most general classical solution $\Sigma(\phi, \phi^*)$ of the Zinn-Justin equation.
- The second step consists in proving absence of anomalies of the Slavnov–Taylor identity – as well as of further Ward identities corresponding to additional symmetries – for the renormalized vertex functional $\Gamma(\phi, \phi^*)$ thus ensuring preservation of the symmetry at the quantum level.

- Additionaly, the existence of a conserved, nilpotent BRST–charge operator $\widehat{Q}_{\rm B}$ and of an (anti)ghost-free physical subspace should be proven consisting of states $|{\rm phys}\rangle$ which are annihilated by $\widehat{Q}_{\rm B}$ and have positive definite norm; furthermore, the S–matrix should be proven to be unitary in that physical state space.

2.14 Further generalizations and special properties of FP quantization

Now we give further generalization of the FP method in order to be able to investigate more complicated gauge theories than pure Yang–Mills theories. Thereby, also we prepare the ground for introducing the Batalin-Vilkovisky method of the next Chapter. In addition, we give some aspects of the extended action related to the gauge fixing procedure.

(i) General conditions allowing for the application of FP quantization

At first we generalize to the case where Yang-Mills theories are coupled to some matter fields, e.g., scalar or spinor fields. Thereby, we formulate the general conditions which should be fulfilled in order to be able to apply the FP method. Let us start from some initial action $S_0(A)$ of the fields $A^i = \{A^{a\mu}(x), \varphi^r(x), \psi^r(x), \ldots\}$, with Grassmann parities $\varepsilon(A^i) \equiv \varepsilon_i$, being invariant under the gauge transformations ($X_{,} \equiv \delta X/\delta A^i$)

$$\delta A^i = R^i_\alpha(A)\xi^\alpha, \quad S_{0,i}(A)R^i_\alpha(A) = 0,$$

where ξ^α are arbitrary functions with Grassmann parities $\varepsilon(\xi^\alpha) \equiv \varepsilon_\alpha$, and $R^i_\alpha(A)$, $\varepsilon(R^i_\alpha(A)) = \varepsilon_i + \varepsilon_\alpha$ are generators of gauge transformations. For the Yang-Mills fields, using DeWitt's notation [75], the content of indices i and α is $i = (x, \mu, a)$ and $\alpha = (x, a)$ and, correspondingly, for the matter fields. The latter are assumed to transform homogeneously according to some – but not necessarily the same – representation of the gauge group with group generators $(X_a)^r_s$ obeying the same Lie bracket (2.1.2) as the generators in the adjoint representation, e.g.,

$$\delta\psi^r = (X_a)^r_s\psi^s, \qquad \delta\varphi^r = (X_a)^r_s\varphi^s, \quad \ldots\ .$$

Then the algebra of the gauge generators R^i_α has the following form:

$$R^i_{\alpha,j}(A)R^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}(A)R^j_\alpha(A) = -R^i_\gamma(A)F^\gamma{}_{\alpha\beta}, \tag{2.14.49}$$

where $F^\gamma{}_{\alpha\beta} = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}F^\gamma{}_{\beta\alpha}$ are the structure *constants* – in the case of general gauge theories, which will be considered starting with the next Chapter, they *may depend* upon the fields A^i. If, in addition, the generators R^i_α form a set of linear independent operators with respect to $\{\alpha\}$, then the algebra (2.14.49) allows for the application of the Faddeev-Popov quantization to the theory under consideration.

Let us introduce the extended configuration space of the fields as follows:

$$\begin{aligned}
&\phi^A = (A^i,\ B^\alpha,\ C^\alpha,\ \bar{C}^\alpha),\\
&\varepsilon(A^i) = \varepsilon_i,\ \ \varepsilon(B^\alpha) = \varepsilon_\alpha,\ \ \varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = \varepsilon_\alpha + 1,\\
&gh(A^i) = gh(B^\alpha) = 0, \quad gh(C^\alpha) = 1, \quad gh(\bar{C}^\alpha) = -1,
\end{aligned}$$

where B^α are Nakanishi-Lautrup auxiliary fields , C^α and $\bar{C}^\alpha$ are the Faddeev-Popov ghost and anti-ghost fields.

Let us define the total (effective) action of the theory according to the rule

$$S_{\rm eff}(\phi) = S_0(A) + \bar{C}^\alpha \chi_{\alpha,i}(A) R^i_\beta(A) C^\beta + \chi_\alpha(A) B^\alpha \tag{2.14.50}$$

where χ_α, $\varepsilon(\chi_\alpha) = \varepsilon_\alpha$, is some gauge functional lifting the degeneracy of the classical gauge invariant action $S_0(A)$. Then the generating functional of the Green functions can be represented in the form of a functional integral

$$Z(J) = \int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\Big(S_{\rm eff}(\phi) + J_A\phi^A\Big)\right\}. \tag{2.14.51}$$

If, in addition, the following conditions

$$(-1)^{\varepsilon_\beta} F^\beta{}_{\beta\alpha} = (-1)^{\varepsilon_i} \frac{\delta_l R^i_\alpha}{\delta A^i} = 0 \tag{2.14.52}$$

are fulfilled then it is possible also to establish the gauge independence of the vacuum functional $Z(0)$ and of the S-matrix (see below). (Here and elsewhere the subscript "l" denotes the *left derivative* with respect to a field.) For Yang-Mills theories considered above the relations (2.14.52) are valid due to the property of the antisymmetry of the structure constants f^{abc}.

The action (2.14.50) is invariant under the following BRST transformation

$$\delta_{\rm B} S_{\rm eff}(\phi) = 0,$$

with

$$\delta_{\rm B} A^i = R^i_\alpha(A) C^\alpha \lambda, \qquad \delta_{\rm B} C^\alpha = -\tfrac{1}{2}(-1)^{\varepsilon_\beta} F^\alpha{}_{\beta\gamma} C^\gamma C^\beta \lambda, \tag{2.14.53}$$

$$\delta_{\rm B} \bar{C}^\alpha = B^\alpha \lambda, \qquad \delta_{\rm B} B^\alpha = 0, \tag{2.14.54}$$

where λ is a constant Grassmann parameter ($\varepsilon(\lambda) = 1$). One easily verifies the property of nilpotency of the BRST-transformation as well as of the corresponding BRST operator.

(ii) Extended action and gauge fixing

As in the case of pure Yang-Mills theory it is useful to modify the action $S_{\rm eff}(\phi)$ in the following way:

$$S_{\rm ext}(\phi,\phi^*) = S_{\rm eff}(\phi) + A^*_i R^i_\alpha(A) C^\alpha - \tfrac{1}{2} C^*_\alpha F^\alpha{}_{\beta\gamma} C^\gamma C^\beta (-1)^{\varepsilon_\beta} + \bar{C}^*_\alpha B^\alpha \tag{2.14.55}$$

with the antifields $\phi^*_A = (A^*_i,\ B^*_\alpha,\ C^*_\alpha,\ \bar{C}^*_\alpha)$, $\varepsilon(\phi^*_A) = \varepsilon_A + 1$. By construction, that extended action (2.14.55) is invariant under the BRST transformations,

$$\delta_B S_{\rm ext}(\phi,\phi^*) = 0, \tag{2.14.56}$$

which, again, may be represented in the equivalent form of the Zinn-Justin equation

$$\frac{\delta S_{\rm ext}}{\delta\phi^A} \frac{\delta S_{\rm ext}}{\delta\phi^*_A} = 0. \tag{2.14.57}$$

Eqs. (2.14.56) and (2.14.57) follow from the gauge invariance of the initial action $S_0(A)$. Note also that one can express the BRST-transformations (2.14.53) and (2.14.54) by means of S_{ext} in a unique form:

$$\delta_B \phi^A = \frac{\delta S_{\text{ext}}}{\delta \phi^*_A}\lambda, \qquad \delta_B \phi^*_A = 0.$$

Again as in the case of pure Yang-Mills theories one obtains the Ward identities for the generating functionals of (connected) Green's functions and 1PI-vertex functions in the same manner with formally the same result.

Furthermore, using the extended action *it is possible to describe the gauge fixing in a unique way.* To do this let us consider the action

$$S(\phi,\phi^*) = S_0(A) + A^*_i R^i_\alpha(A) C^\alpha - \tfrac{1}{2} C^*_\alpha F^\alpha{}_{\beta\gamma} C^\gamma C^\beta (-1)^{\varepsilon_\beta} + \bar{C}^*_\alpha B^\alpha, \qquad (2.14.58)$$

i.e., replacing S_{eff} by $S_0(A)$ in Eq. (2.14.55) and omitting the gauge fixing and the ghost actions. It is obvious that this action also satisfies Eq. (2.14.57)

$$\frac{\delta S}{\delta \phi^A}\frac{\delta S}{\delta \phi^*_A} = 0 \qquad (2.14.59)$$

as well as the boundary condition

$$S|_{\phi^*=0} = S_0(A).$$

The BRST-transformations are also expressed through S

$$\delta_B \phi^A = \frac{\delta S}{\delta \phi^*_A}\lambda, \qquad \delta_B \phi^*_A = 0.$$

Now, let us introduce a functional $\Psi(\phi)$, the gauge fixing fermion, by the rule

$$\Psi(\phi) = \bar{C}^\alpha \chi_\alpha(A).$$

Then the actions S_{eff}, Eq. (2.14.50), and S_{ext}, Eq. (2.14.55), may be expressed by $S(\phi,\phi^*)$ as follows

$$S_{\text{eff}}(\phi) = S\left(\phi, \phi^* = \frac{\delta \Psi}{\delta \phi}\right), \qquad S_{\text{ext}}(\phi) = S\left(\phi, \phi^* + \frac{\delta \Psi}{\delta \phi}\right). \qquad (2.14.60)$$

We emphasize once again that the Eq. (2.14.59) is of a very general form, which does not contain explicit information about the initial gauge group. *All the information about the initial theory is contained, in fact, in the boundary condition (also including the field content).*

(iii) Gauge independence of the vacuum functional

It follows from the definition of the effective action (2.14.50) that the functional $Z(J)$, Eq. (2.14.51), depends on the gauge function $\chi(A)$ or, likewise, on $\Psi(\phi)$. Let us consider the *vacuum functionals* of the theory $Z_\chi \equiv Z(0)$ and $Z_{\chi+\delta\chi}$ corresponding to the gauges $\chi_\alpha(A)$ and $\chi_\alpha(A) + \delta\chi_\alpha(A)$, respectively. In the functional integral

$$Z_{\chi+\delta\chi} = \int D\phi \exp\left\{\frac{i}{\hbar}\left(S_{\text{eff}}(\phi) + \bar{C}^\alpha \delta\chi_{\alpha,i}(A) R^i_\beta(A) C^\beta + \delta\chi_\alpha(A) B^\alpha\right)\right\} \qquad (2.14.61)$$

we make a change of variables being given by Eqs. (2.14.53) and (2.14.54) with some functional $\Lambda = \Lambda(\phi)$ instead of the constant Grassmann odd variable λ. Of course, the effective action is invariant under such a change of variables. There appear contributions only from the integration measure, resulting in a corresponding Jacobian, and from the terms containing $\delta\chi$. Restricting to first order in $\Lambda(\phi)$ and $\delta\chi_\alpha(A)$ and rewriting the Jacobian according to $\mathrm{sDet}\, M = \exp(\mathrm{sTr} M)$, with $M^A{}_B \equiv \delta(\delta\phi^A)/\delta\phi^B$ we obtain

$$Z_{\chi+\delta\chi} = \int \mathcal{D}\phi \exp\left\{ \frac{i}{\hbar}\Big(S_{\rm eff}(\phi) + \bar{C}^\alpha \delta\chi_{\alpha,i}(A) R^i_\beta(A) C^\beta + \delta\chi_\alpha(A) B^\alpha + \right. \tag{2.14.62}$$

$$\left. + i\hbar \Lambda_{,i} R^i_\alpha(A) C^\alpha - i\hbar \frac{1}{2}(-1)^{\varepsilon_\alpha+\varepsilon_\beta} F^\alpha{}_{\beta\gamma} C^\gamma C^\beta \frac{\delta\Lambda}{\delta C^\alpha} + i\hbar \frac{\delta\Lambda}{\delta\bar{C}_\alpha} B^\alpha \Big) \right\}.$$

Choosing the functional $\Lambda(\phi)$ as

$$\Lambda = \frac{i}{\hbar}\bar{C}^\alpha \delta\chi_\alpha(A)\ ,$$

it follows from (2.14.62) that *the vacuum functional does not depend on the choice of the gauge*,

$$Z_{\chi+\delta\chi}(0) = Z_\chi(0)\ .$$

From this it is possible to prove gauge independence of the S–matrix [135] (see, Chapter **3**).

2.15 Unitarity and admissible gauge generators

Considering the FP procedure as the method of quantization for gauge theories, one usually says that this method can not be applied for theories with open gauge algebras (for the corresponding definition, see below), for the case of reducible theories. But we would like to mark that the FP-method is responsive to a choice of admissible generators of gauge transformations (see [154]). Indeed, let us consider the theory with the action

$$S(\varphi,\omega) = \int dx \Big(\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) - V(\varphi) \Big),$$

where φ, ω are real scalar fields. It is a gauge theory. Choosing the generators of gauge transformations of fields φ, ω in the form $R_\varphi = 0, R_\omega = \Box + \varphi^2$ ($\delta\varphi = 0,\ \delta\omega = (\Box + \varphi^2)\xi$) and gauge as $\chi = \omega = 0$ we obtain the effective action

$$S_{eff}(\varphi,\omega,\bar{C},C\lambda) = \int dx \Big(\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) - V(\varphi) + \omega\lambda + \bar{C}(\Box + \varphi^2)C \Big),$$

where $\bar{C}, C$ are ghost fields, and λ is an auxiliary field introducing the gauge. It is obvious that the unitarity of this theory is broken in the subspace of φ. If one chooses for this theory the gauge transformations in the form ($\delta\varphi = 0, \delta\omega = \xi$) and uses the same gauge-fixing, then the effective action is equal to

$$S_{eff}(\varphi,\omega,\bar{C},C\lambda) = \int dx \Big(\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) - V(\varphi) + \omega\lambda + \bar{C}C \Big),$$

so that the ghosts $\bar{C}, C$ and gauge field ω are not dynamical, and do not give any contribution to the dynamics of the fields φ. Therefore, in this case the S-matrix coincides with the S-matrix of real scalar field (1.7.35), and there is no unitarity problem.

Chapter 3

Batalin–Vilkovisky Method

In the middle of the 1970's, supergravity theories were discovered [91, 70, 92]. Direct application of the Faddeev-Popov answers (2.14.50), (2.14.51) leads in the case of these theories to an incorrect result; namely, the violation of the physical S-matrix unitarity. The reason lies in the structure of gauge transformations for these theories. In this case, the invariance transformations for the initial action *do not form* a gauge group. The arising structure coefficients may depend on the fields of the initial theory, and the gauge algebra of these transformations may be opened by terms proportional to the equations of motion. Moreover, attempts of covariant quantization of gauge theories with linearly-dependent generators of gauge transformations result in the understanding of the fact that it is impossible to use the Faddeev-Popov rules to construct a suitable quantum theory [193, 183, 121]. Therefore, the quantization of gauge theories requires taking into account many new aspects (in comparison with QED) such as open algebras, reducible generators and so on. It was realized how to quantize them using different types of ghosts, antighosts, ghosts for ghosts (Nielsen, Kallosh ghosts etc.) [76, 90, 162, 134, 163, 67, 93, 154].

A unique closed approach to the problem of covariant quantization summarized all these attempts was proposed by Batalin and Vilkovisky [40, 41]. The Batalin-Vilkovisky (BV) formalism gives the rules for the quantization of a general gauge theories.

3.1 General gauge theories

The starting point of the BV-method is a theory of fields $A^i, i = 1, 2, ..., n,\ \varepsilon(A^i) = \varepsilon_i$ for which the initial classical action $S_0(A)$ is assumed to have at least one stationary point $A_0 = \{A_0^i\}$

$$S_{0,i}(A)|_{A_0} = 0, \tag{3.1.1}$$

and to be regular in the neighborhood of A_0. Equation (3.1.1) defines a surface Σ in space of functions A^i. Invariance of the action $S_0(A)$ under the gauge transformations $\delta A^i = R^i_\alpha(A)\xi^\alpha$ in the neighborhood of the stationary point is assumed:

$$S_{0,i}(A)R^i_\alpha(A) = 0, \quad \alpha = 1, 2, ..., m, \quad 0 < m < n, \quad \varepsilon(\xi^\alpha) = \varepsilon_\alpha. \tag{3.1.2}$$

Here ξ^α are arbitrary functions of space-time coordinates , and $R^i_\alpha(A)$ ($\varepsilon(R^i_\alpha) = \varepsilon_i + \varepsilon_\alpha$) are generators of gauge transformations. We have also used DeWitt's condensed notations [74], when any index includes all particular ones (space - time, index of internal group, Lorentz index and so on). Summation over repeated indeces implies integration over continuos ones and usual summation over discrete ones.

As an example, for the Yang–Mills theory of fields A^a_μ, we have

$$A^i \equiv A^a_\mu(x), \quad R^i_\alpha(A) \equiv D^{ab}_\mu(A(x))\delta(x-y), \quad i=(x,\mu,a), \quad \alpha=(y,b),$$

and so on.

It follows from the identities (3.1.2) (the Noether identities) that, first, the equations of motion are not independent and, second, (some) propagators do not exist because the Hessian matrix $H_{ij} = S_{0,ij}$ of S_0 is degenerate at any point on the stationary surface Σ:

$$S_{0,i}(A)R^i_{\alpha,j}(A) + S_{0,ji}(A)R^i_\alpha(-1)^{\varepsilon_\alpha\varepsilon_j} = 0 \quad \Longrightarrow S_{0,ji}R^i_\alpha|_{A_0} = 0.$$

The generators R^i_α are on shell zero-eigenvalue vectors of the Hessian matrix $S_{0,ij}$. We assume fulfilment of so-called *regularity condition* [41, 42, 43] which implys that the on-shell degeneracy of the Hessian matrix is due to the *only* independent zero-eigenvalue vectors R^i_α. There are two key consequences of the regularity condition:

(i) If a function $F(A)$ of the fields A^i vanishes on-shell ($S_{0,i}=0$) then F must be a linear combination of the equations of motion

$$F(A)|_\Sigma = 0 \Longrightarrow F(A) = S_{0,i}(A)\lambda^i,$$

with some quantities λ^i which may be functions of A^i.

(ii) Any solution to the Noether identities (3.1.2) is a gauge transformation, up to terms proportional to the equations of motion

$$S_{0,i}(A)\lambda^i = 0 \Longleftrightarrow \lambda^i = R^i_\alpha(A)\lambda^\alpha + S_{0,j}(A)M^{ij}(A), \tag{3.1.3}$$

where M^{ij} satisfies the condition

$$M^{ij} = -(-1)^{\varepsilon_i\varepsilon_j}M^{ji}.$$

The second term $R^i_{triv} = S_{0,j}M^{ij}$ in (3.1.3) is known as a *trivial* gauge transformation of the initial action $S_0(A)$, vanishing at the extremals of $S_0(A)$: $R^i_{triv}|_\Sigma = 0$.

Let

$$\text{rank}R^i_\alpha|_\Sigma = r$$

be the rank of the gauge generators taken at the extremals.

If the condition $r=m$ holds, then the generators R^i_α are linearly independent and the theory under consideration belongs to the class of *irreducible* theories.

If $r<m$, then the generators R^i_α are linearly dependent. In that case the gauge theory belongs to the class of *reducible* theories. Linear dependence of R^i_α implies that the matrix R^i_α has at the extremals $S_{0,j}(A)=0$ zero-eigenvalue eigenvectors $Z^\alpha_{\alpha_1} = Z^\alpha_{\alpha_1}(A)$, such that

$$R^i_\alpha Z^\alpha_{\alpha_1} = S_{0,j}K^{ij}_{\alpha_1}, \quad \alpha_1 = 1,...,m_1 \tag{3.1.4}$$

and the number $\varepsilon_{\alpha_1}=0,1$ canbe found in such a way that $\varepsilon(Z^\alpha_{\alpha_1}) = \varepsilon_\alpha + \varepsilon_{\alpha_1}$. Matrices $K^{ij}_{\alpha_1}$ in (3.1.4) can be chosen to possess the property:

$$K^{ij}_{\alpha_1} = -(-1)^{\varepsilon_i\varepsilon_j}K^{ji}_{\alpha_1}.$$

Let the rank of the matrix $Z^{\alpha}_{\alpha_1}$ at the extremals be:

$$\text{rank} Z^{\alpha}_{\alpha_1}|_{\Sigma} = r_1.$$

If the condition $r_1 = m_1$ is satisfied, then the gauge theory is the *first-stage reducible* one. In general case $r_1 < m_1$, the set $Z^{\alpha}_{\alpha_1}$ is linearly dependent as itself, so that at the extremals $S_{0,i} = 0$ there exists the set of zero-eigenvalue eigenvectors $Z^{\alpha_1}_{\alpha_2} = Z^{\alpha_1}_{\alpha_2}(A)$

$$Z^{\alpha}_{\alpha_1} Z^{\alpha_1}_{\alpha_2} = S_{0,j} L^{\alpha j}_{\alpha_2}, \quad \alpha_2 = 1, ..., m_2 \tag{3.1.5}$$

and numbers $\varepsilon_{\alpha_2} = 0, 1$ such that $\varepsilon(Z^{\alpha_1}_{\alpha_2}) = \varepsilon_{\alpha_1} + \varepsilon_{\alpha_2}$.

Let, in its turn:

$$\text{rank} Z^{\alpha-1}_{\alpha_2}|_{\Sigma} = r_2.$$

If $r_2 = m_2$, them we deal, by the definition, with the *second-stage reducible* gauge theory. In the general case the set $Z^{\alpha-1}_{\alpha_2}$ can be redundant, i.e., $r_2 < m_2$ and so on. In such a way the sequence of reducibility equations arises:

$$Z^{\alpha_{s-2}}_{\alpha_{s-1}} Z^{\alpha_{s-1}}_{\alpha_s} = S_{0,j} L^{\alpha_{s-2} j}_{\alpha_s}, \quad \alpha_s = 1, ..., m_s; s = 1, .., L, \tag{3.1.6}$$

where the following notations are introduced:

$$\begin{aligned} & Z^{\alpha-1}_{\alpha_0} \equiv R^i_{\alpha}, \quad L^{\alpha-1 j}_{\alpha_0} \equiv K^{ij}_{\alpha}, \\ & \varepsilon(Z^{\alpha_{s-1}}_{\alpha_s}) = \varepsilon_{\alpha_{s-1}} + \varepsilon_{\alpha_{\alpha_s}}, \\ & \text{rank} Z^{\alpha_{s-1}}_{\alpha_s} \equiv r_s. \end{aligned} \tag{3.1.7}$$

The stage L of reducibility is defined by the last value s for which $r_s = m_s$.

It should be noted here that for the given gauge theory the gauge generators R^i_{α} as well as the zero-eigen eigenvectors $Z^{\alpha_{s-1}}_{\alpha_s}$ are defined nonuniquely. Characteristic arbitrariness in their definition can be described by the following relations:

$$\begin{aligned} \bar{R}^i_{\alpha} &= R^i_{\beta} X^{\beta}_{\alpha} + S_{0,j} Y^{ij}_{\alpha}, \quad Y^{ij}_{\alpha} = -(-1)^{\varepsilon_i \varepsilon_j} Y^{ji}_{\alpha}, \\ \bar{Z}^{\alpha_{s-1}}_{\alpha_s} &= Z^{\alpha_{s-1}}_{\beta_s} D^{\beta_s}_{\alpha_s} + S_{0,j} E^{\alpha_{s-1} j}_{\alpha_s}, \quad s = 1, ..., L, \end{aligned}$$

where the matrices X^{β}_{α}, $D^{\beta_s}_{\alpha_s}$ are inversible.

The set of gauge generators $\{R^i_{\alpha}\}$ (3.1.2), eigenvectors $\{Z^{\alpha_{s-1}}_{\alpha_s}\}$ (3.1.6) and structure functions $\{L^{\alpha_{s-2} j}_{\alpha_s}\}$ (3.1.6) defines the structure of gauge algebra on the first level.

The structure of gauge algebra on the second level can be found by studying the commutator of gauge transformations and some consequences from the relations (3.1.2) and (3.1.6). We assume that the set $\{R^i_{\alpha}(A)\}$ is complete. Consider the commutator of two gauge transformations $[\delta_1, \delta_2] A^i = \delta_1(\delta_2 A^i) - \delta_2(\delta_1 A^i)$ with gauge parameters ξ^{α}_1, ξ^{β}_2. It leads to

$$[\delta_1, \delta_2] A^i = \left(R^i_{\alpha,j} R^j_{\beta} - (-1)^{\varepsilon_{\alpha} \varepsilon_{\beta}} R^i_{\beta,j} R^j_{\alpha} \right) \xi^{\beta}_1 \xi^{\alpha}_2.$$

Since this commutator is also a gauge symmetry of action we have after factoring out the gauge parameters ξ^{α}_1, ξ^{β}_2 the Noether identities

$$S_{0,i} \left(R^i_{\alpha,j} R^j_{\beta} - (-1)^{\varepsilon_{\alpha} \varepsilon_{\beta}} R^i_{\beta,j} R^j_{\alpha} \right) = 0.$$

Therefore as a consequence of the condition of completeness, one can prove that the algebra of generators has the following general form ([203, 42, 43]):

$$R^i_{\alpha,j}(A)R^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}(A)R^j_\alpha(A) = -R^i_\gamma(A)F^\gamma_{\alpha\beta}(A) - S_{0,j}(A)M^{ij}_{\alpha\beta}(A), \quad (3.1.8)$$

where $F^\gamma_{\alpha\beta}(A)$ are structure functions depending, in general, on the fields A^i with the following properties of symmetry $F^\gamma_{\alpha\beta}(A) = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}F^\gamma_{\beta\alpha}(A)$ and $M^{ij}_{\alpha\beta}(A)$ satisfies the conditions $M^{ij}_{\alpha\beta}(A) = -(-1)^{\varepsilon_i\varepsilon_j}M^{ji}_{\alpha\beta}(A) = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}M^{ij}_{\beta\alpha}(A)$.

If $M^{ij}_{\alpha\beta}(A) = 0$, then the theory is called a gauge theory with a *closed* gauge algebra. If $M^{ij}_{\alpha\beta}(A) \neq 0$, then the gauge algebra is called *open*. In this case due to the symmetry properties of $M^{ij}_{\alpha\beta}(A)$, the quantities $R^i_{\alpha\beta,triv}(A) = S_{0,j}(A)M^{ij}_{\alpha\beta}(A)$ are symmetry (trivial) generators of the initial action $S_0(A)$, vanishing at the extremals of $S_0(A)$:

$$R^i_{\alpha\beta,triv}(A)|_{S_{0,i}=0} = 0,$$

but they are not connected with an additional degeneration of $S_0(A)$ because rank of the Hessian matrix, describing of degeneration of initial action, is defined at the extremals $S_{0,i} = 0$.

If $M^{ij}_{\alpha\beta}(A) = 0$, and $F^\gamma_{\alpha\beta}$ does not depend on the fields, the gauge transformations form a gauge group and (3.1.8) reduces to (2.14.49) and define a *Lie algebra* (for details, see Appendix **A**).

For irreducible theories the structure of gauge algebra on the second level is defiend by the set of structure functions $\{F^\gamma_{\alpha\beta}\}$ and matrices $\{M^{ij}_{\alpha\beta}\}$ in Eq. (3.1.8). For reducible theories the existence of relations among the $Z^{\alpha_{s-1}}_{\alpha_s}$ (3.1.6) leads to the appearance of new structure functions. Let us demonstrate this point for a first-stage reducible gauge theory. To this end let us multiply the relation (3.1.8) by the eigenvector $Z^\beta_{\alpha_1}$. We obtain

$$\left(R^i_{\alpha,j}R^j_\beta - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}R^j_\alpha + R^i_\gamma F^\gamma_{\alpha\beta} + S_{0,j}M^{ij}_{\alpha\beta}\right)Z^\beta_{\alpha_1} = 0. \quad (3.1.9)$$

First, note that relations (3.1.4) allows us to express $R^j_\beta Z^\beta_{\alpha_1}$ as a term proportional to the equations of motion. Second, by differentiating Eqs. (3.1.4) and (3.1.2) with respect to A one obtains that

$$R^i_{\beta,j}Z^\beta_{\alpha_1}(-1)^{\varepsilon_j(\varepsilon_\beta+\varepsilon_{\alpha_1})} + R^i_\beta Z^\beta_{\alpha_1,j} = S_{0,jl}K^{li}_{\alpha_1}(-1)^{\varepsilon_j(\varepsilon_i+\varepsilon_{\alpha_1})} + S_{0,l}K^{il}_{\alpha_1,j}, \quad (3.1.10)$$

$$S_{0,ji}R^j_\alpha(-1)^{\varepsilon_l\varepsilon_\alpha} + S_{0,i}R^i_{\alpha,j} = 0, \quad (3.1.11)$$

Then multiplying Eqs. (3.1.10) by R^j_α, using the Noether identities (3.1.2) and relations (3.1.11), we find

$$-(-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}R^j_\alpha Z^\beta_{\alpha_1} = (-1)^{\varepsilon_\alpha\varepsilon_{\alpha_1}}R^i_\beta Z^\beta_{\alpha_1,j}R^j_\alpha + S_{0,j}(R^j_{\alpha,l}K^{il}_{\alpha_1}(-1)^{\varepsilon_\alpha\varepsilon_i} - K^{ij}_{\alpha_1,l}R^l_\alpha(-1)^{\varepsilon_\alpha\varepsilon_{\alpha_1}}).$$

Returning with this result into (3.1.9) one can obtained the relations

$$R^i_\beta((-1)^{\varepsilon_\alpha\varepsilon_{\alpha_1}}Z^\beta_{\alpha_1,j}R^j_\alpha - F^\beta_{\alpha\gamma}Z^\gamma_{\alpha_1}) = S_{0,j}Y^{ij}_{\alpha_1\alpha}$$

where all terms proportional to the equation of motion have been collected into $Y^{ij}_{\alpha\alpha_1}$. Taking into account the completeness of the set of eigenvectors $Z^\alpha_{\alpha_1}$ the general solution to this equation

$$(-1)^{\varepsilon_\alpha\varepsilon_{\alpha_1}}Z^\beta_{\alpha_1,j}R^j_\alpha - F^\beta_{\alpha\gamma}Z^\gamma_{\alpha_1} = -Z^\beta_{\beta_1}P^{\beta_1}_{\alpha_1\alpha} - S_{0,j}Q^{\beta j}_{\alpha_1\alpha} \quad (3.1.12)$$

defines a new gauge-structure relation similar to Eq. (3.1.8). Therefore two new structure functions $P^{\beta_1}_{\alpha\alpha_1}$ and $Q^{\beta j}_{\alpha\alpha_1}$ arise to complete definition of the structure of gauge algebra for the first-stage reducible theory on the second level.

To define the structure of gauge algebra on the third level one has to consider the Jacobi identity for gauge transformations and some consequences from gauge-structure relations of previous levels. Thus for irreducible theories one has to consider the Jacobi identity for commutators of gauge transformations

$$[\delta_1,[\delta_2,\delta_3]]A^i + cycl.perm.(1,2,3) = 0,$$

and to find

$$(R^i_\gamma D^\gamma_{\alpha\beta\delta} + S_{0,k} Z^{ik}_{\alpha\beta\delta})\xi^\delta_1\xi^\beta_2\xi^\alpha_3 + cycl.perm.(1,2,3) = 0, \tag{3.1.13}$$

where we have defined

$$\begin{aligned} D^\gamma_{\alpha\beta\delta} &= (-1)^{\varepsilon_\alpha\varepsilon_\delta}(F^\gamma_{\alpha\sigma}F^\sigma_{\beta\delta} + F^\gamma_{\alpha\beta,i}R^i_\delta) + cycl.perm.(\alpha,\beta,\delta), \\ Z^{ik}_{\alpha\beta\delta} &= (-1)^{\varepsilon_\alpha\varepsilon_\delta}(M^{ik}_{\alpha\sigma}F^\sigma_{\beta\delta} + M^{ik}_{\alpha\beta,j}R^j_\delta - (-1)^{\varepsilon_i\varepsilon_\alpha}R^k_{\alpha,j}M^{ij}_{\beta\delta} + \\ &\quad +(-1)^{\varepsilon_k(\varepsilon_\alpha+\varepsilon_i)}R^i_{\alpha,j}M^{kj}_{\beta\delta}) + cycl.perm.(\alpha,\beta,\delta) \end{aligned}$$

with the following graded-antisymmetric properties

$$\begin{aligned} D^\gamma_{\alpha\beta\delta} &= -(-1)^{\varepsilon_{\alpha\beta\delta}}D^\gamma_{\beta\alpha\delta} = -(-1)^{\varepsilon_{\alpha\beta\delta}}D^\gamma_{\alpha\delta\beta}, \\ Z^{ik}_{\alpha\beta\delta} &= -(-1)^{\varepsilon_k\varepsilon_i}Z^{ki}_{\alpha\beta\delta} = -(-1)^{\varepsilon_{\alpha\beta\delta}}Z^{ik}_{\beta\alpha\delta} = -(-1)^{\varepsilon_{\alpha\beta\delta}}Z^{ik}_{\alpha\delta\beta}. \end{aligned}$$

Because of the linear independence of the generators R^i_α and their completeness Eq.(3.1.13) has the following solution

$$D^\gamma_{\alpha\beta\delta} = S_{0,k}Q^{\gamma k}_{\alpha\beta\delta} \tag{3.1.14}$$

with the properties of graded antisymmetry

$$\begin{aligned} Q^{\gamma k}_{\alpha\beta\delta} &= -(-1)^{\varepsilon_{\alpha\beta\delta}}Q^{\gamma k}_{\beta\alpha\delta} = -(-1)^{\varepsilon_{\alpha\beta\delta}}Q^{\gamma k}_{\alpha\delta\beta}, \\ &\varepsilon_{\alpha\beta\delta} \equiv \varepsilon_\alpha\varepsilon_\beta + \varepsilon_\alpha\varepsilon_\gamma + \varepsilon_\beta\varepsilon_\gamma. \end{aligned}$$

Using this solution Eq. (3.1.13) can be presented in the form

$$S_{0,k}\left(Z^{ik}_{\alpha\beta\delta} + (-1)^{\varepsilon_k(\varepsilon_i+\varepsilon_\gamma)}R^i_\gamma Q^{\gamma k}_{\alpha\beta\delta}\right)\xi^\delta_1\xi^\beta_2\xi^\alpha_3 + cycl.perm.(1,2,3) = 0.$$

Due to the completeness of gauge generators R^i_α the general solution of this equation is of the form

$$Z^{ik}_{\alpha\beta\delta} + (-1)^{\varepsilon_k(\varepsilon_i+\varepsilon_\gamma)}R^i_\gamma Q^{\gamma k}_{\alpha\beta\delta} - (-1)^{\varepsilon_k\varepsilon_\gamma}R^k_\gamma Q^{\gamma i}_{\alpha\beta\delta} = S_{0,j}M^{ikj}_{\alpha\beta\delta}, \tag{3.1.15}$$

where $M^{ikj}_{\alpha\beta\delta}$ obeys graded antisymmetry in i,j,k and α,β,δ

$$\begin{aligned} M^{ikj}_{\alpha\beta\delta} &= -(-1)^{\varepsilon_k\varepsilon_i}M^{kij}_{\alpha\beta\delta} = -(-1)^{\varepsilon_k\varepsilon_j}M^{ijk}_{\alpha\beta\delta}, \\ M^{ikj}_{\alpha\beta\delta} &= -(-1)^{\varepsilon_{\alpha\beta\delta}}M^{ikj}_{\beta\alpha\delta} = -(-1)^{\varepsilon_{\alpha\beta\delta}}M^{ikj}_{\alpha\delta\beta}. \end{aligned}$$

For irreducible theories the functions $Q^{\gamma k}_{\alpha\beta\delta}$ and $M^{ikj}_{\alpha\beta\delta}$ define the structure of gauge algebra on the third level. In its turn Eq. (3.1.15) can be considered as a new gauge-structure relations on this level. In case of reducible theories new structure functions arise additionaly on the third level. Here we are going to demonstrate this fact for a first-stage reducible gauge theory. The eigenvectors $Z^{\alpha_1}_{\alpha}$ lead to modification of the solution of the Jacobi identity (3.1.13). Instead of Eq. (3.1.14), we have

$$D^{\gamma}_{\alpha\beta\delta} + Z^{\gamma}_{\sigma_1} F^{\sigma_1}_{\alpha\beta\delta} = S_{0,k} Q^{\gamma k}_{\alpha\beta\delta} \tag{3.1.16}$$

and therefore the Jacobi identity can be rewritten in the form

$$S_{0,k}\left(Z^{ik}_{\alpha\beta\delta} + (-1)^{\varepsilon_k(\varepsilon_i+\varepsilon_\gamma)} R^i_\gamma Q^{\gamma k}_{\alpha\beta\delta} + K^{ik}_{\alpha_1} F^{\alpha_1}_{\alpha\beta\delta}\right)\xi^\delta_1 \xi^\beta_2 \xi^\alpha_3 + cycl.perm.(1,2,3) = 0$$

with $K^{ij}_{\alpha_1}$ defined in (3.1.4). The general solution is

$$Z^{ik}_{\alpha\beta\delta} + (-1)^{\varepsilon_k(\varepsilon_i+\varepsilon_\gamma)} R^i_\gamma Q^{\gamma k}_{\alpha\beta\delta} - (-1)^{\varepsilon_k\varepsilon_\gamma} R^k_\gamma Q^{\gamma i}_{\alpha\beta\delta} + K^{ik}_{\alpha_1} F^{\alpha_1}_{\alpha\beta\delta} = S_{0,j} M^{ikj}_{\alpha\beta\delta}. \tag{3.1.17}$$

Eq. (3.1.17) can be considered as a new gauge-structure relation. Functions $Q^{\gamma k}_{\alpha\beta\delta}$, $M^{ikj}_{\alpha\beta\delta}$ and $F^{\alpha_1}_{\alpha\beta\delta}$ define for the first-stage reducible theory the structure of gauge algebra on the third level. And so on. In general the structure of gauge algebra looks like a set of infinite number of structure functions which define infinite number of gauge-structure relations. It is remarkable fact that all these relations can be collected within the BV-method in a solution of *classical master equation*.

The gauge theories whose generators satisfy Eq.(3.1.8) are called *general gauge theories*.

Example: Yang-Mills theory

Let us consider some examples of gauge theories from the point of view of general definitions (3.1.4), (3.1.5), (3.1.6), (3.1.8).

For Yang–Mills theory we have the set of linear independent generators $R^i_\alpha = D^{ab}_\mu$ and the gauge algebra (3.1.8) with

$$M^{ij}_{\alpha\beta}(A) = 0, \quad F^{\gamma}_{\alpha\beta} \equiv f^{abc}\delta(x-y)\delta(y-z)\delta(x-z).$$

By definition the Yang-Mills theory belongs to the class of irreducible one with closed gauge algebra.

Example: W_3 gravity

Next example is model of W_3 gravity as an example of an irreducible theory with an open algebra and structure functions $F^{\gamma}_{\alpha\beta}$ dependind on fields. The classical action for W_3 gravity is [128, 199]

$$S_0(\phi,h,B) = \int d^2x\left[\frac{1}{2}\partial\phi\bar{\partial}\phi - \frac{1}{2}h(\partial\phi)^2 - \frac{1}{3}B(\partial\phi)^3\right]. \tag{3.1.18}$$

The fields $A^i = (\phi,h,B)$ are bosonic ones defined in a two-dimensional space with coordinates, $x = (z,\bar{z})$, so that $\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$.

The equations of motion read

$$\begin{aligned} \frac{\delta S_0}{\delta\phi} &= -\partial\bar{\partial}\phi + \partial h\partial\phi + \partial^2\phi h + (\partial\phi)^2\partial B + 2\partial\phi\partial^2\phi B,\\ \frac{\delta S_0}{\delta h} &= -\frac{1}{2}(\partial\phi)^2, \quad \frac{\delta S_0}{\delta B} = -\frac{1}{3}(\partial\phi)^3. \end{aligned} \tag{3.1.19}$$

The action (3.1.18) is invariant under the gauge transformations

$$\begin{aligned}\delta\phi &= (\partial\phi)\epsilon + (\partial\phi)^2\lambda,\\ \delta h &= \bar{\partial}\epsilon - h\partial\epsilon + (\partial\phi)^2((\partial B)\lambda - B\partial\lambda),\\ \delta B &= (\partial B)\epsilon - 2B\partial\epsilon + \bar{\partial}\lambda - h\partial\lambda + 2(\partial h)\lambda \end{aligned} \qquad (3.1.20)$$

with the bosonic parametres $\xi^\alpha = (\epsilon, \lambda)$ and the following identification of gauge generators R^i_α:

$$\begin{aligned} R^\phi_\alpha &= \left(\partial\phi, \quad (\partial\phi)^2\right),\\ R^h_\alpha &= \left(\bar{\partial} - h\partial + \partial h, \quad (\partial\phi)^2((\partial B) - B\partial)\right),\\ R^B_\alpha &= \left(\partial B - 2B\partial, \quad \bar{\partial} - h\partial + 2\partial h\right). \end{aligned} \qquad (3.1.21)$$

The generators (3.1.21) are linearly independent.

Alrebra of gauge transformations (3.1.20) has the form

$$\begin{aligned} [\delta_1,\delta_2]\phi &= -\partial\phi\epsilon_{(1,2)} - (\partial\phi)^2(\epsilon\lambda)_{(1,2)} + (\partial\phi)^2(\epsilon\lambda)_{(2,1)} - 2(\partial\phi)^3\lambda_{(1,2)},\\ [\delta_1,\delta_2]h &= -(\bar{\partial} - h\partial + \partial h)\epsilon_{(1,2)} - (\partial\phi)^2((\partial B) - B\partial)(\epsilon\lambda)_{(1,2)}\\ &\quad +(\partial\phi)^2((\partial B) - B\partial)(\epsilon\lambda)_{(2,1)}\\ &\quad -(\partial\phi)^2[\bar{\partial} - h\partial + 3\partial h + 2\partial\phi\partial B + 4\partial^2\phi B]\lambda_{(1,2)},\\ [\delta_1,\delta_2]B &= -(\partial B - 2B\partial)\epsilon_{(1,2)} - (\bar{\partial} - h\partial + 2\partial h)(\epsilon\lambda)_{(1,2)}\\ &\quad +(\bar{\partial} - h\partial + 2\partial h)(\epsilon\lambda)_{(2,1)}\\ &\quad -[(\partial\phi)^2\partial B - 4\partial\phi\partial^2\phi B - 2(\partial\phi)^2 B\partial]\lambda_{(1,2)} \end{aligned} \qquad (3.1.22)$$

where we have used the notations

$$\epsilon_{(1,2)} = (\epsilon_1\partial\epsilon_2 - (\partial\epsilon_1)\epsilon_2), \quad (\epsilon\lambda)_{(1,2)} = (\epsilon_1\partial\lambda_2 - 2(\partial\epsilon_1)\lambda_2), \quad \lambda_{(1,2)} = (\lambda_1\partial\lambda_2 - (\partial\lambda_1)\lambda_2).$$

Taking into account general structure of gauge algebra (3.1.8) and definitions of gauge generators (3.1.21) for W_3 model, it follows from (3.1.22) possible definitions of structure functions $F^1_{11}, F^2_{21}, F^2_{12}$:

$$\epsilon_{(1,2)} = F^1_{11}\epsilon_1\epsilon_2, \quad (\epsilon\lambda)_{(1,2)} = F^2_{21}\epsilon_1\lambda_2, \quad (\epsilon\lambda)_{(2,1)} = -F^2_{12}\epsilon_2\lambda_1.$$

or, equivalently,

$$\begin{aligned} F^1_{11} &= \delta(x-y_1)\partial_x\delta(x-y_2) - \delta(x-y_2)\partial_x\delta(x-y_1),\\ F^2_{21} &= \delta(x-y_1)\partial_x\delta(x-y_2) - 2\delta(x-y_2)\partial_x\delta(x-y_1),\\ F^2_{12} &= -(\delta(x-y_2)\partial_x\delta(x-y_1) - 2\delta(x-y_1)\partial_x\delta(x-y_2)). \end{aligned}$$

These structure functions do not depend on fields. Then from Eq. (3.1.22) for $[\delta_1,\delta_2]\phi$ we can suggest the following Ansatz for the remainder

$$2(\partial\phi)^3\lambda_{(12)} = R^\phi_1 F^1_{22}\lambda_1\lambda_2 + R^\phi_2 F^2_{22}\lambda_1\lambda_2 + \frac{\delta S_0}{\delta h} M^{\phi h}_{22}\lambda_1\lambda_2 + \frac{\delta S_0}{\delta B} M^{\phi B}_{22}\lambda_1\lambda_2.$$

From (3.1.19) and (3.1.21) we can parametrize $F^1_{22}, F^2_{22}, M^{\phi h}_{22}, M^{\phi B}$

$$\begin{aligned} F^1_{22}\lambda_1\lambda_2 &= \alpha_1(\partial\phi)^2\lambda_{(12)}, \quad F^2_{22}\lambda_1\lambda_2 = \alpha_2(\partial\phi)\lambda_{(12)},\\ M^{\phi h}_{22}\lambda_1\lambda_2 &= 2\beta_1(\partial\phi)\lambda_{(12)}, \quad M^{\phi B}_{22}\lambda_1\lambda_2 = 3\beta_2\lambda_{(12)} \end{aligned}$$

or

$$
\begin{aligned}
F^1_{22} &= \alpha_1(\partial\phi)^2\Big(\delta(x-y_1)\partial_x\delta(x-y_2)-\delta(x-y_2)\partial_x\delta(x-y_1)\Big),\\
F^2_{22} &= \alpha_2(\partial\phi)\Big(\delta(x-y_1)\partial_x\delta(x-y_2)-\delta(x-y_2)\partial_x\delta(x-y_1)\Big),\\
M^{\phi h}_{22} &= 2\beta_1(\partial\phi)\delta(x-y)\Big(\delta(y-y_2)\partial_y\delta(y-y_1)-\delta(y-y_1)\partial_y\delta(x-y_2)\Big),\\
M^{\phi B}_{22} &= 3\beta_2\delta(x-y)\Big(\delta(y-y_2)\partial_y\delta(y-y_1)-\delta(y-y_1)\partial_y\delta(x-y_2)\Big).
\end{aligned}
\tag{3.1.23}
$$

Here $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constant parameters which satisfy the equation

$$\alpha_1+\alpha_2-\beta_1-\beta_2=2.$$

Returning with these results to the remainders in $[\delta_1,\delta_2]h$ and $[\delta_1,\delta_2]B$ we can put that

$$
\begin{aligned}
(\partial\phi)^2[\bar\partial-h\partial+3\partial h+2\partial\phi\partial B+4\partial^2\phi\ B]\lambda_{(12)} &= \Big(R^h_1F^1_{22}+R^h_2F^2_{22}+\\
&\quad+\frac{\delta S_0}{\delta\phi}M^{h\phi}_{22}+\frac{\delta S_0}{\delta B}M^{hB}_{22}\Big)\lambda_1\lambda_2,\\
[(\partial\phi)^2\partial B-4\partial\phi\partial^2\phi B-2(\partial\phi)^2B\partial]\lambda_{(12)} &= \Big(R^B_1F^1_{22}+R^B_2F^2_{22}+\\
&\quad+\frac{\delta S_0}{\delta h}M^{Bh}_{22}+\frac{\delta S_0}{\delta\phi}M^{B\phi}_{22}\Big)\lambda_1\lambda_2.
\end{aligned}
$$

Using representation for M^{hB}_{22}

$$M^{hB}_{22}\lambda_1\lambda_2=M\lambda_{(12)} \tag{3.1.24}$$

with some operator M, definitions (3.1.23) and Eq. (3.1.24) we find the following relations to define explicitly the gauge algebra for the W_3 gravity

$$\alpha_1=1,\quad \alpha_2=0,\quad \beta_1+\beta_2=-1,\quad (\partial\phi)^2M=6\beta_2\frac{\delta S_0}{\delta\phi}. \tag{3.1.25}$$

If $\beta_2\neq 0$ then we have a realization of the algebra with non-analitical (with respect to field ϕ) matrix M^{hB}_{22}. Under requirement of analiticity ($\beta_2=0$) we find (within Ansatz suggested) simple realization of the gauge algebra of W_3 gravity with gauge generators R^i_α (3.1.21), non-vanishing structure functions $F^\gamma_{\alpha\beta}$

$$
\begin{aligned}
F^1_{11} &= \delta(x-y_1)\partial_x\delta(x-y_2)-\delta(x-y_2)\partial_x\delta(x-y_1),\\
F^2_{21} &= \delta(x-y_1)\partial_x\delta(x-y_2)-2\delta(x-y_2)\partial_x\delta(x-y_1),\\
F^1_{22} &= (\partial\phi)^2\Big(\delta(x-y_1)\partial_x\delta(x-y_2)-\delta(x-y_2)\partial_x\delta(x-y_1)\Big)
\end{aligned}
\tag{3.1.26}
$$

and non-vanishing matrices $M^{ij}_{\alpha\beta}$

$$M^{\phi h}_{22} = -2(\partial\phi)\delta(x-y)\Big(\delta(y-y_2)\partial_y\delta(y-y_1)-\delta(y-y_1)\partial_y\delta(y-y_2)\Big) \tag{3.1.27}$$

depending on the field ϕ. Namely this realization of gauge algebra has been used in [71, 198] to construct solution to the classical master equation.

Notice in general for given set of generators $\{R^i_\alpha\}$ gauge structure functions of higher levels are defined non-uniquely. The best (economic) way to study this point is connected with consideration of different solutions to the classical master equation corresponding to boundary conditions with both fixed classical action $S_0(A)$ and set of gauge generators $\{R^i_\alpha\}$. Later we will discuss these peculiarities using as an example of the W_3 gravity.

Example: Freedman-Townsend model

Let us consider the Freedman-Townsend model as an example of an reducible theory in $d = 4$. The theory of a non-abelian antisymmetric field $B^p_{\mu\nu}$, suggested by Freedman and Townsend [90], is described (in the first order formalism) by the action

$$S_0(A^p_\mu, B^p_{\mu\nu}) = \int d^4x \left(-\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} F^p_{\mu\nu} B^p_{\rho\sigma} + \frac{1}{2} A^p_\mu A^{p\mu} \right), \tag{3.1.28}$$

where A^p_μ is a vector field with the strength $F^p_{\mu\nu} = \partial_\mu A^p_\nu - \partial_\nu A^p_\mu + f^{pqr} A^q_\mu A^r_\nu$ and the coupling constant being absorbed into the structure coefficients f^{pqr}; the Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$ is normalized as $\varepsilon^{0123} = 1$. Eliminating the auxiliary gauge field A^p_μ through its field equations leads to the more complicated action of the second order formalism.

The action (3.1.28) is invariant under the gauge transformations

$$\delta A^p_\mu = 0, \quad \delta B^p_{\mu\nu} = D^{pq}_\mu \xi^q_\nu - D^{pq}_\nu \xi^q_\mu \equiv R^{pq}_{\mu\nu\alpha} \xi^{q\alpha}, \tag{3.1.29}$$

where ξ^p_μ are arbitrary parameters, and D^{pq}_μ is the covariant derivative with potential A^p_μ ($D^{pq}_\mu = \delta^{pq}\partial_\mu + f^{prq} A^r_\mu$).

The gauge transformations (3.1.29) form an abelian algebra (in Eq. (3.1.8) $F^\gamma_{\alpha\beta} = 0$, $M^{ij}_{\alpha\beta} = 0$) with the generators $R^{pq}_{\mu\nu\alpha}$ possessing at the extremals of the action (3.1.28) the zero-eigenvectors $Z^{pq}_\mu \equiv D^{pq}_\mu$ (see (3.1.4))

$$R^{pr}_{\mu\nu\alpha} Z^{rq\alpha} = \varepsilon_{\mu\nu\alpha\beta} f^{prq} \frac{\delta S_0}{\delta B^r_{\alpha\beta}}, \quad K^{ij}_{\alpha_1} \equiv \varepsilon_{\mu\nu\alpha\beta} f^{pqr}, \tag{3.1.30}$$
$$i = (p, \mu, \nu), \; j = (q, \alpha, \beta), \; \alpha_1 = r,$$

which, in their turn, are linearly independent. According to the accepted terminology, the model (3.1.28), (3.1.29) and (3.1.30) is an abelian gauge theory of first stage reducibility.

3.2 Rules of BV quantization

The procedure of the BV-quantization for general gauge theories in question involves the following steps.

Configuration space

The total configuration space ϕ^A is introduced. For irreducible theories the space ϕ^A includes the ghost and antighost fields C^α and $\bar{C}^\alpha$ and the auxiliary (Nakanishi-Lautrup) fields B^α

$$\phi^A = (A^i, \; B^\alpha, \; C^\alpha, \; \bar{C}^\alpha), \quad \varepsilon(\phi^A) = \varepsilon_A, \tag{3.2.31}$$

with the following distribution of the Grassmann parity and ghost number

$$\varepsilon(A^i) = \varepsilon_i, \quad \varepsilon(B^\alpha) = \varepsilon_\alpha, \quad \varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = \varepsilon_\alpha + 1,$$
$$gh(A^i) = gh(B^\alpha) = 0, \quad gh(C^\alpha) = 1, \quad gh(\bar{C}^\alpha) = -1.$$

We see that as in the case of Yang-Mills type theories, for irreducible gauge theories in the BV-formalism the total configuration space is constructed by extending the fields A^i with the set of Nakanishi–Lautrup fields, ghost and antighost fields, with respect to the number of the gauge functions $\{\xi^\alpha\}$. For reducible theories the space ϕ^A has more complicated structure [41] and contains main chains of the ghost $C_s^{\alpha_s}$, antighost $\bar{C}_s^{\alpha_s}$ and auxiliary $B_s^{\alpha_s}$ fields as well as pyramids of the ghost for ghost $C_{s(n_s)}^{\alpha_s}$ and auxiliary $B_{s(n_s)}^{\alpha_s}$ fields ($C_0^{\alpha_0} \equiv C^\alpha$, $\bar{C}_0^{\alpha_0} \equiv \bar{C}^\alpha$ $B_0^{\alpha_0} \equiv B^\alpha$ in (3.2.31))

$$\phi^A = \Big(A^i;\ B_s^{\alpha_s},\ C_s^{\alpha_s},\ \bar{C}_s^{\alpha_s}, s = 0, 1, ..., L;\ B_{s(n_s)}^{\alpha_s},\ C_{s(n_s)}^{\alpha_s}, s = 1, ..., L, n_s = 1, ..., s\Big) \quad (3.2.32)$$

with the properties

$$\begin{aligned}
\varepsilon(A^i) &= \varepsilon_i,\\
\varepsilon(B_s^{\alpha_s}) &= (\varepsilon_\alpha + s) \bmod 2, \quad s = 0, 1, ..., L,\\
\varepsilon(B_{s(n_s)}^{\alpha_s}) &= (\varepsilon_{\alpha_s} + s) \bmod 2, s = 1, ..., L,\ n_s = 1, ..., s,\\
\varepsilon(C_s^{\alpha_s}) &= \varepsilon(\bar{C}_s^{\alpha_s}) = (\varepsilon_{\alpha_s} + s + 1) \bmod 2, s = 0, 1, ..., L,\\
\varepsilon(C_{s(n_s)}^{\alpha_s}) &= (\varepsilon_{\alpha_s} + s + 1) \bmod 2, \quad s = 1, ..., L,\ n_s = 1, ..., s,\\
gh(A^i) &= 0,\\
gh(B_s^{\alpha_s}) &= -s, \quad s = 0, 1, ..., L;\\
gh(B_{s(n_s)}^{\alpha_s}) &= s - 2(n_s - 1), \quad s = 1, ..., L,\ n_s = 1, ..., s;\\
gh(C_s^{\alpha_s}) &= -gh(\bar{C}_s^{\alpha_s}) = (s + 1),\ s = 0, 1, ..., L\\
gh(C_{s(n_s)}^{\alpha_s}) &= s + 1 - 2n_s,\ s = 1, ..., L,\ n_s = 1, ..., s.
\end{aligned} \quad (3.2.33)$$

In comparison with original proposal of Ref. [41] we have slightly (for simplicity and uniformity) changed notation of auxiliary fields and pyramids of fields. In particular, $\pi_{s\alpha_s} \equiv B_s^{\alpha_s}$. As an example for a second-stage reducible theory the following identification for the pyramids of fields exists:

$$\begin{aligned}
C_1^{'\alpha_1} &\equiv C_{1(1)}^{\alpha_1}, \quad C_2^{'\alpha_2} \equiv C_{2(1)}^{\alpha_2}, \quad \bar{C}_{2\alpha_2}^{''} \equiv C_{2(2)}^{\alpha_2},\\
\pi_1^{'\alpha_1} &\equiv B_{1(1)}^{\alpha_1}, \quad \pi_2^{'\alpha_2} \equiv B_{2(1)}^{\alpha_2}, \quad \pi_{2\alpha_2}^{''} \equiv B_{2(2)}^{\alpha_2}.
\end{aligned}$$

Antifields

To each field ϕ^A (of the total configuration space) one introduces corresponding antifield ϕ_A^*

$$\begin{aligned}
\phi_A^* = &\Big(A_i^*,\ B_{s\alpha_s}^*,\ C_{s\alpha_s}^*,\ \bar{C}_{s\alpha_s}^*, s = 0, 1, ..., L;\ B_{s(n_s)\alpha_s}^*,\\
&C_{s(n_s)\alpha_s}^*, s = 1, ..., L, n_s = 1, ..., s\Big)
\end{aligned} \quad (3.2.34)$$

The statistics of ϕ_A^* is opposite to the statistics of the corresponding fields ϕ^A

$$\varepsilon(\phi_A^*) = \varepsilon_A + 1$$

and ghost numbers of fields and corresponding antifields are connected by the rule

$$gh(\phi_A^*) = -1 - gh(\phi^A).$$

Antibracket

On the space of the fields ϕ^A and antifields ϕ^*_A one defines an odd symplectic structure (,) called the antibracket

$$(F,G) \equiv \frac{\delta F}{\delta \phi^A}\,\frac{\delta G}{\delta \phi^*_A} - (F \leftrightarrow G)\,(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}. \tag{3.2.35}$$

The derivatives with respect to fields are understood as right ones and those with respect to sources as left ones (see, **Appendix C**). One can easely verify that the following properties of the antibracket follow from the definition (3.2.35)
(1) Grassmann parity

$$\varepsilon((F,G)) = \varepsilon(F) + \varepsilon(G) + 1 = \varepsilon((G,F)) \tag{3.2.36}$$

(2) Generalized antisymmetry

$$(F,G) = -(G,F)(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}, \tag{3.2.37}$$

(3) Leibniz rule

$$(F,GH) = (F,G)H + (F,H)G(-1)^{\varepsilon(G)\varepsilon(H)}, \tag{3.2.38}$$

(4) Generalized Jacobi identity

$$((F,G),H)(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \mathsf{cycle}(F,G,H) \equiv 0. \tag{3.2.39}$$

One can readily verify that the antibracket (3.2.35) is invariant under the *anticanonical* transformation of variables ϕ, ϕ^* with the generating functional $X = X(\phi,\phi^*),\ \varepsilon(X) = 1$:

$$\phi'^A = \frac{\delta X(\phi,\phi^{*'})}{\delta \phi^{*'}_A}, \quad \phi^*_A = \frac{\delta X(\phi,\phi^{*'})}{\delta \phi^A}. \tag{3.2.40}$$

This property of the odd symplectic structure (3.2.35) on the space of $\phi,\ \phi^*$ is a conterpart to the invariance property of the even simplectic structure (the Poisson bracket) under a canonical transformation of canonical variables $(p,\ q)$ (for further discussions of non-trivial relations between the Poisson bracket and the antibracket, see [18, 30]). For the first time, the importance of anticanonical transformations (3.2.40) in the formulation of the BV-method was realized in [202] (for further discussions, see [42, 153, 194, 200, 109]).

Δ -operator

The nilpotent generating operator Δ is introduced,

$$\Delta = (-1)^{\varepsilon_A}\frac{\delta_l}{\delta \phi^A}\,\frac{\delta}{\delta \phi^*_A}, \quad \Delta^2 = 0 \quad \varepsilon(\Delta) = 1.$$

The operator (3.2.41) is not well-defined on local functionals because for any local functional S $\Delta S \sim \delta(0)$, and one is faced with the so-called "problem of $\delta(0)$". The usual way 'to solve' this problem is to use the dimensional regularization [155] when the corresponding singularity $\sim \delta(0)$ is equal to zero. Quite recently, a new calculus for local variational differential operators in local quantum field theory has been proposed by Shahverdiev, Tyutin and Voronov [179], where $\delta(0)$ does not arise at all. We will always suppose that our all formal manipulations with operators like Δ can be supported by suitable regularization scheme. Note that acting by Δ on the product of two functionals F and G can reproduce the antibracket:

$$\Delta[F \cdot G] = (\Delta F)\cdot G + F \cdot (\Delta G)(-1)^{\varepsilon(F)} + (F,G)(-1)^{\varepsilon(F)}.$$

3.3 Quantum master equation

The quantum master equation (QME) is defined as

$$\frac{1}{2}(S,S) = i\hbar\Delta S \tag{3.3.41}$$

or, equivalently,

$$\Delta \exp\left\{\frac{i}{\hbar}S\right\} = 0, \tag{3.3.42}$$

where $S = S(\phi, \phi^*)$ is a bosonic functional satisfying the boundary condition

$$S|_{\phi^*=\hbar=0} = S_0(A). \tag{3.3.43}$$

The bosonic functional S is the basic object of the BV-quantization. Note, the classical part ($\hbar = 0$) of QME (3.3.41) formally coinsides with the Zinn-Justin equation (2.14.57).

3.4 Generating functional of Green's functions

The generating functional of Green's functions $Z(J)$ is defined as

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{eff}(\phi) + J_A\phi^A]\right\},$$
$$S_{eff}(\phi) = S(\phi, \phi^* = \delta\Psi/\delta\phi). \tag{3.4.44}$$

Here, $\Psi = \Psi(\phi)$ is a fermionic gauge functional, and J_A ($\varepsilon(J_A) = \varepsilon_A$) are the usual external sources to the fields ϕ^A.

Note [202], that the gauge-fixing procedure (3.4.44) in the BV-quantization can be described in terms of anticanonical transformation of the variables ϕ, ϕ^* (3.2.40) in $S(\phi, \phi^*)$ with the generating functional X

$$X(\phi, \phi^*) = \phi^*_A\phi^A + \Psi(\phi).$$

3.5 BRST symmetry

To discuss some features of the BV-quantization, it is convenient to rewrite the expression for the generating functional $Z(J)$ in the equivalent form

$$\begin{aligned} Z(J) &= \int d\phi d\phi^* \delta(\phi^* - \delta\Psi/\delta\phi) \exp\left\{\frac{i}{\hbar}[S(\phi, \phi^*) + J_A\phi^A]\right\} \\ &= \int d\phi d\phi^* d\lambda \exp\left\{\frac{i}{\hbar}\left[S(\phi, \phi^*) + (\phi^*_A - \delta\Psi/\delta\phi^A)\lambda^A + J_A\phi^A\right]\right\} \end{aligned} \tag{3.5.45}$$

where we have introduced the auxiliary (Nakanishi-Lautrup) fields λ^A, $\varepsilon(\lambda^A) = \varepsilon_A + 1$.

Note, first of all, that the integrand in (3.5.45) for $J_A = 0$ is invariant under the following global transformations:

$$\delta\phi^A = \lambda^A\mu, \quad \delta\phi^*_A = \mu\frac{\delta S}{\delta\phi^A}, \quad \delta\lambda^A = 0.$$

It is very important to realize that the existence of this symmetry is the consequence of the fact that the bosonic functional S satisfies the generating equation (3.3.41). These transformations represent the BRST-transformations in the space of variables ϕ, ϕ^*, λ.

3.6 Gauge-independence of the S-matrix

The symmetry of the vacuum functional $Z(0)$ under the BRST transformations permits establishing the independence of the S matrix from the choice of gauge in the BV-quantization. Indeed, suppose $Z_\Psi \equiv Z(0)$. We shall change the gauge $\Psi \to \Psi + \delta\Psi$. In the functional integral for $Z_{\Psi+\delta\Psi}$ we make the change of variables, choosing for μ :

$$\mu = -\frac{i}{\hbar}\delta\Psi.$$

After simple algebraic calculations we find that

$$Z_{\Psi+\delta\Psi} = Z_\Psi. \tag{3.6.46}$$

Here we need to refer to the equivalence theorem proved by Kallosh and Tyutin [135]. According to this theorem if one has two theories with generating functionals of Green's functions $Z(J)$ and $Z^{'}(J)$ of the form

$$\begin{aligned} Z(J) &= \int d\phi \exp\left\{\frac{i}{\hbar}[S(\phi) + J_A\phi^A]\right\}, \\ Z^{'}(J) &= \int d\phi \exp\left\{\frac{i}{\hbar}[S(\phi) + J_A(\phi^A + f^A(\phi))]\right\} \end{aligned}$$

with some functions $f^A(\phi)$ being the regular functions with respect to ϕ, then one can claim that the S-matrices for these theories coincide. Equality (3.6.46) means the gauge independence of the vacuum functional within the BV-method. Due to the equivalence theorem the same is valid for the S-matrix.

3.7 Ward identity

Now, we shall proceed with the derivation of the Ward identity, which is a consequence of the BRST-symmetry. To do this, consider the extended generating functional of the Green functions

$$\mathcal{Z}(J, \phi^*) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{ext}(\phi, \phi^*) + J_A\phi^A]\right\}, \tag{3.7.47}$$

where

$$S_{ext}(\phi, \phi^*) = S(\phi, \phi^* + \delta\Psi/\delta\phi). \tag{3.7.48}$$

From the above definition it follows that

$$\mathcal{Z}(J, \phi^*)|_{\phi^*=0} = Z(J),$$

where $Z(J)$ has been defined in (3.7.47).

Note, first of all, that the action S_{ext} (3.7.48) satisfy the QME (3.3.42). Indeed, the equality holds

$$\exp\left\{\frac{i}{\hbar}S_{\text{ext}}(\phi,\ \phi^*)\right\} = \exp\{[\Psi,\ \Delta]_+\}\exp\left\{\frac{i}{\hbar}S(\phi,\ \phi^*)\right\} \tag{3.7.49}$$

where $S(\phi,\ \phi^*)$ is a solution of the master equation (3.3.42) and $\Psi = \Psi(\phi)$. Then

$$[\Psi,\ \Delta]_+ = \frac{\delta\Psi}{\delta\phi^A}\frac{\delta}{\delta\phi^*_A} \qquad (3.7.50)$$

and the operator $\exp\{[\Psi,\ \Delta]_+\}$ acts as the translation operator with respect to ϕ^*_A. Note that

$$[\Delta,\ [\Psi,\ \Delta]_+]_- = 0, \qquad (3.7.51)$$

and therefore

$$\Delta \exp\left\{\frac{i}{\hbar}S_{\text{ext}}\right\} = 0, \qquad (3.7.52)$$

where we have used the notation (3.7.48).

Taking into account that S_{ext} satisfies the QME (3.3.42) and the fact that the integration in (3.7.47) is performed over ϕ, we have the evident relations

$$\begin{aligned} 0 &= \int d\phi \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\Delta\exp\left\{\frac{i}{\hbar}S_{\text{ext}}(\phi,\phi^*)\right\} \\ &= (-1)^{\varepsilon_A}\frac{\delta}{\delta\phi^*_A}\int d\phi \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\frac{\delta_l}{\delta\phi^A}\exp\left\{\frac{i}{\hbar}S_{\text{ext}}(\phi,\phi^*)\right\}. \end{aligned}$$

Integrating by parts in the last integral, one finds that the theory in question satisfies the equality

$$J_A\frac{\delta\mathcal{Z}}{\delta\phi^*_A} = 0. \qquad (3.7.53)$$

This is the Ward identity written for the extended generating functional of Green's functions.

Introducing the generating functional of connected Green's functions $\mathcal{W} = \mathcal{W}(J,\phi^*)$ ($\mathcal{Z} = \exp\{(i/\hbar)\mathcal{W}\}$), the identity (3.7.53) can be presented in the form

$$J_A\frac{\delta\mathcal{W}}{\delta\phi^*_A} = 0. \qquad (3.7.54)$$

Let us introduce, in a standard manner, through the Legendre transformation of W, the generating functional of the vertex functions $\Gamma = \Gamma(\phi,\ \phi^*)$

$$\Gamma(\phi,\ \phi^*) = \mathcal{W}(J,\phi^*) - J_A\phi^A, \quad \phi^A = \frac{\delta\mathcal{W}}{\delta J_A}, \quad \frac{\delta\Gamma}{\delta\phi^A} = -J_A.$$

Rewriting the Ward identity (3.7.54) for the generating functional of the vertex functions, we obtain the unique form

$$(\Gamma,\Gamma) = 0. \qquad (3.7.55)$$

Sometimes it is useful to present the Ward identities (3.7.53), (3.7.54), (3.7.55) in an equivalent form. To do this, let us introduce the odd nilpotent operator V:

$$V = J_A\frac{\delta}{\delta\phi^*_A}, \quad V^2 = 0. \qquad (3.7.56)$$

Then we obtain the following representation of (3.7.53), (3.7.54)

$$V\mathcal{Z} = 0, \quad V\mathcal{W} = 0.$$

The Ward identity for Γ can be presented in the form

$$\mathcal{B}(\Gamma)\cdot\Gamma = 0,$$

where we have used the notation $\mathcal{B}(\Gamma)$ for the so-called *Slavnov-Taylor operator*:

$$\mathcal{B}(\Gamma) = \frac{\delta\Gamma}{\delta\phi^A}\frac{\delta}{\delta\phi^*_A} - (-1)^{\varepsilon_A}\frac{\delta\Gamma}{\delta\phi^*_A}\frac{\delta_l}{\delta\phi^A} = (\Gamma,\ \cdot) \tag{3.7.57}$$

The operator $\mathcal{B}(\Gamma)$ (3.7.57) obeys the property of nilpotency $\mathcal{B}(\Gamma)^2 = 0$ due to (3.7.55) and can be considered as the Legendre transformation of V.

Among the issues related to the method in question, we shall consider only the problems of gauge dependence of Green's functions, the existence theorem for generating equation, and renormalizability.

3.8 Gauge dependence of Green's functions

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It is well-known that Green's functions in gauge theories depend on the choice of gauge [132, 74, 161, 117, 95, 55, 151, 152, 192, 150, 14]. From the gauge-independence of the S-matrix (see 3.6.46) it follows that the gauge dependence of Green's functions in gauge theories must be of a special character. To study the character of this dependence, let us consider an infinitesimal variation of the gauge functional $\Psi(\phi) \to \Psi(\phi) + \delta\Psi(\phi)$. Then the variation of $\exp\{(i/\hbar)S_{\rm ext}\}$ reads

$$\delta\biggl(\exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\}\biggr) = [\delta\Psi,\ \Delta]_+\ \exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\} = \Delta\ \delta\Psi\ \exp\left\{\frac{i}{\hbar}S_{\rm ext}\right\} \tag{3.8.58}$$

because in the case, when Ψ and $\delta\Psi$ depend on the variables ϕ only, the operator $[\delta\Psi,\ \Delta]_+$ commutes with $[\Psi,\ \Delta]_+$.

Next, the corresponding variation of the functional $\mathcal{Z}(J,\ \phi^*)$ has the form

$$\begin{aligned}
\delta\mathcal{Z}(J,\ \phi^*) &= \int d\phi\ \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\Delta\ \delta\Psi\ \exp\left\{\frac{i}{\hbar}S_{\rm ext}(\phi,\ \phi^*)\right\} \\
&= (-1)^{\varepsilon_A}\frac{\delta}{\delta\phi^*_A}\int d\phi\ \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\frac{\delta_l}{\delta\phi^A}\ \delta\Psi\ \exp\left\{\frac{i}{\hbar}S_{\rm ext}(\phi,\ \phi^*)\right\} \\
&= -\frac{i}{\hbar}\frac{\delta}{\delta\phi^*_A}J_A\int d\phi\ \delta\Psi\ \exp\left\{\frac{i}{\hbar}\Bigl[S_{\rm ext}(\phi,\ \phi^*) + J_A\phi^A\Bigr]\right\}.
\end{aligned}$$

Therefore

$$\delta\mathcal{Z} = -\frac{i}{\hbar}J_A\frac{\delta}{\delta\phi^*_A}\delta\widehat{\Psi}\mathcal{Z}(J,\ \phi^*) = -\frac{i}{\hbar}V\delta\widehat{\Psi}\mathcal{Z}(J,\ \phi^*), \tag{3.8.59}$$

where we have introduced the operator $\delta\widehat{\Psi}$ according to

$$\delta\widehat{\Psi} \equiv \delta\Psi\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right).$$

and have used the definition (3.7.56) In terms of the generating functional $\mathcal{W} = \mathcal{W}(J,\ \phi^*)$ of connected Green's functions

$$\delta\mathcal{Z} = \frac{i}{\hbar}\,\delta\mathcal{W}\exp\left\{\frac{i}{\hbar}\mathcal{W}\right\},$$

and we have

$$\delta\mathcal{W} = -J_A\frac{\delta\langle\delta\widehat{\Psi}\rangle}{\delta\phi^*_A} = -V\langle\delta\widehat{\Psi}\rangle. \tag{3.8.60}$$

Here, we have taken into account the Ward identity for $\mathcal{W}$ (3.7.54) and have used the notation $\langle\delta\widehat{\Psi}\rangle$ for vacuum expectation of the operator $\delta\widehat{\Psi}$

$$\langle\delta\widehat{\Psi}\rangle = \delta\Psi\left(\frac{\delta\mathcal{W}}{\delta J} + \frac{\hbar}{i}\frac{\delta}{\delta J}\right).$$

The variation of the generating functional of vertex functions $\Gamma = \Gamma(\phi,\ \phi^*)$ obtains

$$\delta\Gamma = \frac{\delta\Gamma}{\delta\phi^A}\left(\frac{\delta\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^*_A} + \frac{\delta\phi^B}{\delta\phi^*_A}\frac{\delta_l\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^B}\right),$$

where we have used the equality

$$\left.\frac{\delta}{\delta\phi^*_A}\right|_J = \left.\frac{\delta}{\delta\phi^*_A}\right|_\phi + \left.\frac{\delta\phi^B}{\delta\phi^*_A}\right|_J\left.\frac{\delta_l}{\delta\phi^B}\right|_{\phi^*}$$

and also introduced the notations

$$\langle\langle\delta\widehat{\Psi}\rangle\rangle = \delta\Psi\left(\phi^A + i\hbar(G^{''-1})^{AB}\frac{\delta_l}{\delta\phi^B}\right),$$

$$(G^{''})_{AB} = \frac{\delta_l}{\delta\phi^A}\left(\frac{\delta\Gamma}{\delta\phi^B}\right), \quad (G^{''-1})^{AB}G_{BC} = \delta^A_C.$$

We can see that, at the extremals, the functional Γ does not depend on the gauge

$$\left.\delta\Gamma\right|_{\frac{\delta\Gamma}{\delta\phi}=0} = 0. \tag{3.8.61}$$

There are other points connected with this fact. Consider the equalities

$$J_A\frac{\delta\phi^B}{\delta\phi^*_A} = J_A\frac{\delta}{\delta\phi^*_A}\left(\frac{\delta\mathcal{W}}{\delta J_B}\right),$$

$$\frac{\delta}{\delta J_B}\left(J_A\frac{\delta\mathcal{W}}{\delta\phi^*_A}\right) = 0 = \frac{\delta\mathcal{W}}{\delta\phi^*_B} + (-1)^{\varepsilon_B}J_A\frac{\delta}{\delta\phi^*_A}\left(\frac{\delta\mathcal{W}}{\delta J_B}\right).$$

Therefore,

$$\begin{aligned}\delta\Gamma &= \frac{\delta\Gamma}{\delta\phi^A}\frac{\delta\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^*_A} + (-1)^{\varepsilon_B}\frac{\delta\Gamma}{\delta\phi^*_B}\frac{\delta_l\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^B}\\ &= \frac{\delta\Gamma}{\delta\phi^A}\frac{\delta\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^*_A} - \frac{\delta_l\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^B}\frac{\delta\Gamma}{\delta\phi^*_B}.\end{aligned}$$

The last equation has the form

$$\delta\Gamma = (\Gamma,\ \langle\langle\delta\widehat{\Psi}\rangle\rangle) = \mathcal{B}(\Gamma)\cdot\langle\langle\delta\widehat{\Psi}\rangle\rangle. \tag{3.8.62}$$

We can see that the variation of the functional Γ under a small change of gauge may be expressed in the form of anticanonical transformation (3.2.40) of the fields and antifields with the generating function $X = X(\phi,\ \phi^*) = \phi^*_A\phi^A + \langle\langle\delta\widehat{\Psi}\rangle\rangle$

$$\phi'^A = \phi^A + \frac{\delta\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^*_A}, \quad \phi^{*'}_A = \phi^*_A - \frac{\delta\langle\langle\delta\widehat{\Psi}\rangle\rangle}{\delta\phi^A}.$$

3.9 Gauge fixing procedure

Note that there exists a freedom in the choice of a gauge fixing procedure applied to obtain a well-defined generating functional of Green's functions in the BV-method. To do this, let us consider the following vacuum functional [20]

$$Z(0) \equiv Z_X = \int d\phi d\phi^* \exp\left\{\frac{i}{\hbar}(S(\phi,\phi^*) + X(\phi,\phi^*))\right\}$$

where both boson functionals S and X satisfy the QME of the BV-scheme

$$\Delta\exp\left\{\frac{i}{\hbar}S(\phi,\phi^*)\right\} = 0, \quad \Delta\exp\left\{\frac{i}{\hbar}X(\phi,\phi^*)\right\} = 0.$$

In [20], it was shown that $Z_X = T_{X'}$ for any X' satisfyiing the QME. Indeed, taking into account that two solutions of the QME can be presented in the form of a maximal deformation [37, 38]

$$\exp\left\{\frac{i}{\hbar}X'\right\} = \exp[\Delta,\Psi]_+\exp\left\{\frac{i}{\hbar}X\right\}$$

with a fermionic functional Ψ, we have for an infinitesimal transformation

$$\delta\left(\exp\left\{\frac{i}{\hbar}X'\right\}\right) = [\Delta,\Psi]_+\exp\left\{\frac{i}{\hbar}X\right\}.$$

Then,

$$\begin{aligned} Z_{X'} - Z_X &= \int d\phi d\phi^* \exp\left\{\frac{i}{\hbar}S\right\}[\Delta,\Psi]_+\exp\left\{\frac{i}{\hbar}X\right\} \\ &= \int d\phi d\phi^*\left[\exp\left\{\frac{i}{\hbar}S\right\}\Delta\Psi\exp\left\{\frac{i}{\hbar}X\right\} + \exp\left\{\frac{i}{\hbar}S\right\}\Psi\Delta\exp\left\{\frac{i}{\hbar}X\right\}\right] \\ &= \int d\phi d\phi^* \exp\left\{\frac{i}{\hbar}S\right\}\Delta\Psi\exp\left\{\frac{i}{\hbar}X\right\}. \end{aligned}$$

Integrating by parts twice in the last functional integral, we obtain

$$Z_{X'} - Z_X = \int d\phi d\phi^*\left(\Delta\exp\left\{\frac{i}{\hbar}S\right\}\right)\Psi\exp\left\{\frac{i}{\hbar}X\right\} = 0.$$

One can also find in [97] an alternative approach to generalize gauge fixing procedure within BV-method.

3.10 Existence theorem

.

The question of existence of solution of Eq. (3.3.41) satisfying the boundary condition (3.3.43) is principal in the construction of BV-formalism [203, 42, 43]. We restrict ourselfs to the proof of the existence of solutions of the equation

$$(S, S) = 0. \tag{3.10.63}$$

with the boundary condition (3.3.43). Note that, for local S, ΔS is proportional to $\delta(0)$, and, using the dimensional regularization $[\delta(0) = 0]$, Eq. (3.3.41) becomes (3.10.63).

The solution of Eq. (3.10.63) will be sought in the special form

$$S(\phi, \phi^*) = S(\phi_{min}, \phi^*_{min}) + \bar{C}^*_{s\alpha_s} B^{\alpha_s}_s + C^*_{s(n_s)\alpha_s} B^{\alpha_s}_{s(n_s)}$$

where the *minimal* set of ϕ^A and ϕ^* is defined as

$$\phi^A_{min} = (A^i, C^{\alpha_s}_s,\ s = 0, 1, ..., L,) \quad \phi^*_{min\ A} = (A^*_i, C^*_{s\alpha_s},\ s = 0, 1, ..., L). \tag{3.10.64}$$

In the minimal sector (3.10.65), the solution $S_{min} = S(\phi_{min}, \phi^*_{min})$ will be sought in the form of a power series of fields C^α

$$S_{min} = S_0(A) + \sum_{n=1} S_n, \quad S_n \sim (C)^n$$

with $\varepsilon(S_n) = 0,\ \ gh(S_n) = 0.$

In what follows we consider (for simplicity) proof the existance theorem for an irreducible theory when the minimal configuration space of fields and antifields has the form

$$\phi^A_{min} = (A^i, C^\alpha), \quad \phi^*_{min\ A} = (A^*_i, C^*_\alpha). \tag{3.10.65}$$

Let us consider the first approximation S_1. The most general form of the functional S_1 meeting the above-mentioned requirements is

$$S_1 = A^*_i \Lambda^i_\alpha C^\alpha$$

where Λ^i_α are some unknown matrices depending on the fields A^i. Next, we require that the functional $S_0(A) + S_1$ satisfies Eq. (3.10.63) to first order. This leads to the following equation for Λ^i_α:

$$S_{0,i} \Lambda^i_\alpha C^\alpha = 0. \tag{3.10.66}$$

From Eq. (3.10.66) it follows that Λ^i_α can be identified with the generators of the gauge transformations

$$\Lambda^i_\alpha = R^i_\alpha. \tag{3.10.67}$$

Suppose now that we have constructed the functional $S^{[n]}_{min}$, where

$$S^{[n]}_{min} = S_0(A) + \sum_{k=1}^{n} S_k,$$

which satisfies (3.10.63) up to nth order:

$$\left(S_{min}^{[n]}, S_{min}^{[n]}\right)_k = 0, \quad k = 1, 2, ..., n. \tag{3.10.68}$$

In Eq. (3.10.68) and hereafter $(\ , \ )_k$ denotes the kth order in powers of fields C^α. For the $(n+1)$th approximation S_{n+1} of S_{min}, we have

$$WS_{n+1} = F_{n+1}. \tag{3.10.69}$$

The operator W in Eq. (3.10.69) is nilpotent and is given by

$$W = S_{0,i}\frac{\delta}{\delta A_i^*} + A_i^* R_\alpha^i \frac{\delta}{\delta C_\alpha^*}, \quad W^2 = 0. \tag{3.10.70}$$

The operator W (3.10.70) can be considered as lower approximation in power series of fields C^α to the Slavnov-Taylor operator $\mathcal{B}(\Gamma)$ (3.7.57). The functionals F_{n+1} in Eqs. (3.10.69) are constructed from $S_k, k < n$, by the rule

$$F_{n+1} = -\frac{1}{2}\left(S_{min}^{[n]}, S_{min}^{[n]}\right)_{n+1}.$$

From Eq. (3.10.70) it follows that for Eq. (3.10.69) to be compatible it is necessary that the relation

$$WF_{n+1} = 0 \tag{3.10.71}$$

holds. It is not difficult to prove that the relation (3.10.71) does hold. To this end one needs to consider the identity $(S_{min}, (S_{min}, S_{min})) \equiv 0$ in the $(n+1)$th approximation. We take into account that by virtue of Eqs.(3.10.68) and the lowest approximation for the expression (S_{min}, S_{min}) is $(n+1)$th order, which is equal to $WS_{n+1} - F_{n+1}$. Then in the $(n+1)$th approximation the identity $(S_{min}, (S_{min}, S_{min})) \equiv 0$ becomes

$$W(WS_{n+1} - F_{n+1}) = 0,$$

and therefore the relation (3.10.71) holds.

Further proof of the existence theorem rests on the following lemma.

Lemma: Any regular solution of the equation

$$WX = 0 \tag{3.10.72}$$

vanishing for $S_{0,i} = \phi^*_{min\ A} = 0$ has the form

$$X = WY$$

with some functional Y. In other words, cohomologies of W on space of solutions (3.10.72) vanishing for $S_{0,i} = \phi^*_{min\ A} = 0$ are trivial.

Proof: The proof is based on the possibility of reducing the operator W to the 'standard' form, i.e., to that of the operators $G_i\delta/\delta P_i$, where both the set of G_i and P_i are functionally independent.

The reduction of the operator W to the standard form is realized in several steps. First, from the initial variables A^i we go over, using a nonsingular change, to the variables A'^i:

$$A^i = A^i(A') \leftrightarrow A'^i = A'^i(A) = (\varphi^m, \eta^\alpha), \quad i = (m, \alpha). \tag{3.10.73}$$

Here, the initial classical action does not depend on the gauge fields η^α explicitly:

$$S_0(A) = S_0(A(A')) = S_0^{'}(A^{'}) = S_0^{'}(\varphi) \tag{3.10.74}$$

Given this, the gauge invariance condition (3.1.2) becomes

$$S_{0,i}(A)R^i_\alpha(A) = S^{'}_{0,i}(A')N^i_j R^j_\alpha(A(A')) = S^{'}_{0,i}(A')R^{'i}_\alpha(A') = 0, \tag{3.10.75}$$

where

$$R^{'i}_\alpha(A') = N^i_j R^j_\alpha(A(A')), \quad N^i_j(A) = \frac{\delta A^{'i}(A)}{\delta A^j}.$$

With allowance made for Eq. (3.10.74), the identity (3.10.75) can now be rewritten as

$$S^{'}_{0,i}R^{'i}_\alpha(A') = S^{'}_{0,m}R^{'m}_\alpha(A') = 0. \tag{3.10.76}$$

From Eq. (3.10.76) we conclude that $R'^m_\alpha(A')$ can be only trivial generators for the action $S^{'}_0(\varphi)$:

$$R^{'m}_\alpha(A') = S^{'}_{0,n}\Lambda^{mn}_\alpha(A'), \quad \Lambda^{mn}_\alpha = -(-1)^{\varepsilon_m\varepsilon_n}\Lambda^{nm}_\alpha.$$

The generators $R^{'i}_\alpha(A')$ can be represented in the form

$$R^{'i}_\alpha = (S^{'}_{0,n}\Lambda^{mn}_\alpha, \bar{R}^\beta_\alpha),$$

where $\bar{R}^\beta_\alpha$ is a nondegenerate matrix.

In addition to the changes (3.10.73), we also make the following antifield transformations:

$$A^{*'}_i = A^*_j(N^{-1})^j_i, \quad C^{*'}_\alpha = C^*_\beta(\bar{R}^{-1})^\beta_\alpha, \tag{3.10.77}$$

where we have introduced the notation:

$$(N^{-1})^i_j(A) \equiv \frac{\delta A^i(A')}{\delta A'^j}, \quad (N^{-1})^i_j N^j_k = \delta^i_k.$$

As a result of the changes (3.10.73) and (3.10.77) the operator $W \to W'$,

$$W' = \mathcal{J}_m \frac{\delta}{\delta A^{*'}_m} + (A^{*'}_\beta + A^{*'}_n \mathcal{J}_m \Lambda^{mn}_\alpha (\bar{R}^{-1})^\alpha_\beta)\frac{\delta}{\delta C^{*'}_\beta}, \quad \mathcal{J}_m \equiv S^{'}_{0,m} \tag{3.10.78}$$

In the operator W' (3.10.78), we make the change of variables

$$A^{*''}_m = A^{*'}_m, \quad A^{*''}_\alpha = A^{*'}_\alpha + A^{*'}_n \mathcal{J}_m \Lambda^{mn}_\beta (\bar{R}^{-1})^\beta_\alpha, \quad C^{*''}_\alpha = C^{*'}_\alpha, \tag{3.10.79}$$

and $W' \to W''$, where

$$W'' = \mathcal{J}_m \frac{\delta}{\delta A^{*''}_m} + A^{*''}_\alpha \frac{\delta}{\delta C^{*''}_\alpha}. \tag{3.10.80}$$

The operator W'' is already of the 'standard' form. We shall now construct an operator Q'' such that

$$W''Q'' + Q''W'' = N'', \quad (Q'')^2 = 0. \tag{3.10.81}$$

The solution of Eqs. (3.10.81) does exist. For example, for the operator Q'' one can choose

$$Q'' = A^{*''}_m \frac{\delta}{\delta \mathcal{J}_m} + C^{*''}_\alpha \frac{\delta}{\delta A^{*''}_\alpha}. \tag{3.10.82}$$

Then for operator N'' in Eqs. (3.10.80) we deduce

$$N'' = \mathcal{J}_n \frac{\delta}{\delta \mathcal{J}_n} A^{*''}_m \frac{\delta}{\delta A^{*''}_m} + A^{*''}_\alpha \frac{\delta}{\delta A^{*''}_\alpha} C^{*''}_\alpha \frac{\delta}{\delta C^{*''}_\alpha}. \tag{3.10.83}$$

By direct verification, we make sure that the equalities

$$W''N'' = N''W'', \quad Q''N'' = N''Q'' \tag{3.10.84}$$

do hold. In Eqs. (3.10.80) - (3.10.84) we now make transformations inverse to (3.10.73), (3.10.77) and (3.10.79), obtaining

$$WQ + QW = N, \quad Q^2 = 0, \quad WN = NW, \quad QN = NQ, \tag{3.10.85}$$

where the operator W is given by the expression (3.10.70) and the operators Q and N have the form

$$Q = A^*_i P^i_j \frac{\delta}{\delta \mathcal{S}_{0,j}} + C^*_\alpha L^\alpha_i \frac{\delta}{\delta A^*_i}, \quad N = S_{0,i} P^i_j \frac{\delta}{\delta S_{0,j}} + \phi^*_{Amin} \frac{\delta}{\delta \phi^*_{Amin}}, \tag{3.10.86}$$

In Eqs. (3.10.86) we have used the notation

$$P^i_j \equiv (N^{-1})^i_m N^m_j, \quad L^\alpha_i = (\bar{R}^{-1})^\alpha_\beta N^\beta_i$$

with the following properties

$$P^i_l P^l_j = P^i_j, \quad L^\alpha_j P^j_i = 0.$$

Now let us consider the solution of Eq. (3.10.72). We shall act upon Eq. (3.10.72) from the left by the operator Q (3.10.86) and take into account Eqs. (3.10.85). Then with allowance made for the fact that on the solutions $N > 0$, we have

$$X = W(N^{-1}\Gamma X),$$

which proves the validity of Lemma concerning solutions of Eq. (3.10.72).

We now return to the solution of Eqs. (3.10.69). Since $gh(F_{n+1}) = 1$ and $n > 0$, it follows that $F_{n+1} = 0$ for $S_{0,i} = \phi^*_{min\ A} = 0$, and therefore, by virtue of the Lemma, the solution of (3.10.71) can be represented in the form

$$F_{n+1} = WX_{n+1}.$$

Choosing $S_{n+1} = X_{n+1}$, we find that Eq. (3.10.63) is already satisfied to within $(n+1)$th-order terms. Then by induction we conclude the proof of existence of solutions of Eq. (3.10.63). Note that for S_{n+1} we could take the functional

$$S_{n+1} = X_{n+1} + WY_{n+1}. \tag{3.10.87}$$

and, as before, Eq. (3.10.63) would be satisfied to within terms of $(n+1)th$ order. On the basis of the Lemma, it is not difficult to show, given conditions (3.3.43) and (3.10.67), that the arbitrariness (3.10.87) in the choice of the $(n+1)$th approximation is unique.

Solution to CME for W_3-gravity

Often for practice it is sufficient to know solutions to the classical master equation up to second order in ghost fields C. For irreducible gauge theories one obtains

$$\begin{aligned} S(\phi,\ \phi^*) &= S_0(A) + A^*_i R^i_\alpha C^\alpha - \frac{1}{2} C^*_\gamma F^\gamma_{\alpha\beta} C^\beta C^\alpha (-1)^{\varepsilon_\alpha} \\ &+ \frac{1}{4} A^*_i A^*_j M^{ij}_{\alpha\beta} C^\beta C^\alpha (-1)^{\varepsilon_\alpha + \varepsilon_j} + \bar{C}^*_\alpha B^\alpha \end{aligned} \tag{3.10.88}$$

where $F^\gamma_{\alpha\beta}$ and $M^{ij}_{\alpha\beta}$ are the structure functions of gauge algebra on the second level (3.1.8).

The action for Yang-Mills type theories (2.14.49), (**??**) exactly belongs to this class (3.10.88) of solutions to the classical master equation with $M^{ij}_{\alpha\beta} = 0$.

Closed solution in the form (3.10.88) can be also constructed [129, 199, 71, 198] for the W_3 gravity (3.1.18) with non-trivial structure functions $F^\gamma_{\alpha\beta}$ (3.1.26), $M^{ij}_{\alpha\beta}$ (3.1.27) $(A^i \equiv (\phi, h, B),\ C^\alpha \equiv (c, l),\ B^\alpha \equiv (u, v))$:

$$S = S_0 + S_1 + \int d^2x \left[c^* (\partial c\, c + \partial l\, l\, (\partial\phi)^2) + l^* (\partial l\, c + 2\partial c\, l) + 2\phi^* h^* \partial l\, l\, \partial\phi \right] \tag{3.10.89}$$

where the initial classical action S_0 is defined in (3.1.18), the action S_1 defined by the set of gauge generators of the model is the first order contribution to the classical master equation

$$\begin{aligned} S_1 &= \int d^2x [\phi^* (\partial\phi\, c + (\partial\phi)^2 l) + \\ &+ h^* (\bar{\partial} c - h\, \partial c + \partial h\, c + (\partial\phi)^2 (\partial B\, l - B\, \partial l)) + \\ &+ B^* (\partial B\, c - 2B\, \partial c + \bar{\partial} l - h\, \partial l + 2\partial h\, l)], \end{aligned} \tag{3.10.90}$$

and we omitted trivial contributions of the form $\bar{C}^*_\alpha B^\alpha$ (see (3.10.88)).

It was pointed out in [199] existence of arbitraryness in choosing of gauge structure functions $F^\gamma_{\alpha\beta}$, $M^{ij}_{\alpha\beta}$ for the W_3 gravity. It was shown by constructing of the action

$$\begin{aligned} S = S_0 + S_1 \quad &+ \int d^2x [c^* (\partial c\, c + (1-\alpha) \partial l\, l\, (\partial\phi)^2) + l^* (\partial l\, c + 2\partial c\, l) + \\ &+ 2\alpha\, h^* (\bar{\partial} h^* - \partial h^*\, h) \partial l\, l - 2\alpha h^* (3B^*\, \partial B + 2B\, \partial B^*) \partial l\, l + \\ &+ 2(1+\alpha) \phi^* h^* \partial l\, l\, \partial\phi]. \end{aligned} \tag{3.10.91}$$

with the help of anticanonical transformations (3.2.40) in action (3.10.89) when generating functional of these transformations was choosen in the form

$$X(\phi,\ \phi^*) = E(\phi,\ \phi^*) - 2\alpha h^* c^* \partial l\, l,$$

where α is a free parameter and $E(\phi,\ \phi^*)$ is the generating functional of identical anticanonical transformations. The action (3.10.91) satisfies the classical master equation with both the same boundary condition and the set of gauge generators $\{R^i_\alpha\}$ but it corresponds to another set of gauge structure functions $F^\gamma_{\alpha\beta}$

$$\begin{aligned} F^1_{11} &= \delta(x-y_1)\partial_x \delta(x-y_2) - \delta(x-y_2)\partial_x \delta(x-y_1), \\ F^2_{21} &= \delta(x-y_1)\partial_x \delta(x-y_2) - 2\delta(x-y_2)\partial_x \delta(x-y_1), \\ F^1_{22} &= (1-\alpha)(\partial\phi)^2 \Big(\delta(x-y_1)\partial_x \delta(x-y_2) - \delta(x-y_2)\partial_x \delta(x-y_1) \Big) \end{aligned}$$

and non-vanishing matrices $M^{ij}_{\alpha\beta}$

$$
\begin{aligned}
M^{\phi h}_{22} &= -2(1+\alpha)(\partial\phi)\delta(x-y)\Big(\delta(y-y_2)\partial_y\delta(y-y_1)-\delta(y-y_1)\partial_y\delta(x-y_2)\Big),\\
M^{hh}_{22} &= -\alpha[\bar{\partial}_x-\bar{\partial}_y+(\partial_y-\partial_x)h(x)]\delta(x-y)\Big(\delta(x-y_2)\partial_x\delta(x-y_1)-\\
&\qquad -\delta(x-y_1)\partial_x\delta(x-y_2)\Big),\\
M^{hB}_{22} &= 2\alpha[3\partial B-2B\partial_y]\delta(x-y)\Big(\delta(x-y_2)\partial_x\delta(x-y_1)-\delta(x-y_1)\partial_x\delta(x-y_2)\Big).
\end{aligned}
$$

depending on the fields $\phi,\ h,\ B$.

Moreover it should be definitely stressed that arbitrariness described in (3.10.91) and preserved the closed form of solutions in ghost fields $c,\ l$ is not unique.

Making use anticanonical transformations with the generating functional X

$$X(\phi,\ \phi^*)=E(\phi,\ \phi^*)+6\beta_2\phi^*h^*B^*(\partial\phi)^{-2}\partial l\ l,$$

we obtain the action written in closed form

$$
\begin{aligned}
S=S_0+S_1 \quad &+ \int d^2x[c^*(\partial c\ c+\partial l\ l\ (\partial\phi)^2)+l^*(\partial l\ c+2\partial c\ l)\\
&-6\beta_2\ h^*B^*(\partial\phi)^{-2}(-\bar{\partial}\partial\phi+\partial h\ \partial\phi+\partial^2\phi\ h+(\partial\phi)^2\ \partial B+2\partial\phi\ \partial^2\phi\ B)\partial l\ l\\
&-3\beta_2\phi^*B^*\partial l\ l+2(1+\beta_2)\phi^*h^*\partial l\ l\ \partial\phi+\\
&+12\beta_2\phi^*h^*B^*(\partial\phi)^{-2}\partial l\ l\ \partial c],
\end{aligned}
\tag{3.10.92}
$$

and being solution to the classical master equation corresponding to realization of gauge algebra on the second level with non-analitical gauge structure functions described in (3.1.23), (3.1.24), (3.1.24), (3.1.25). It follows from (3.10.92) that in the case of non-analitical realization of gauge algebra on the second order it needs to complicate structure of gauge algebra by adding gauge structure functions of the third level.

Solution to CME for Freedman-Townsend model

For first-stage reducible gauge theories the action up to second order in ghost fields reads

$$
\begin{aligned}
S(\phi,\phi^*) &= S_0(A)+A^*_iR^i_\alpha C^\alpha+C^*_\alpha\left(Z^\alpha_{\alpha_1}C^{\alpha_1}-\frac{1}{2}F^\alpha_{\gamma\beta}C^\beta C^\gamma(-1)^{\varepsilon_\gamma}\right)\\
&+A^*_iA^*_j\left(\frac{1}{2}K^{ij}_{\alpha_1}C^{\alpha_1}(-1)^{\varepsilon_i}+\frac{1}{4}M^{ij}_{\alpha\beta}C^\beta C^\alpha(-1)^{\varepsilon_\alpha+\varepsilon_j}\right)+\\
&+C^*_{\alpha_1}P^{\alpha_1}_{\beta_1\alpha}C^{\beta_1}C^\alpha(-1)^{\varepsilon_\alpha}+C^*_\delta A^*_iQ^{\delta i}_{\alpha_1\alpha}C^{\alpha_1}C^\alpha(-1)^{\varepsilon_\delta+\varepsilon_\alpha}+\\
&+\bar{C}^*_\alpha B^\alpha+\bar{C}^*_{\alpha_1}B^{\alpha_1},
\end{aligned}
\tag{3.10.93}
$$

where structure functions $P^{\alpha_1}_{\beta_1\alpha},\ Q^{\delta i}_{\alpha_1\alpha}$ (3.1.14) define the gauge algebra on the second order.

For the Freedman-Townsend model (3.1.28) the action constructed by the rule (3.10.93)

$$
\begin{aligned}
S &= =\int d^4x\Big[-\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F^p_{\mu\nu}B^p_{\rho\sigma}+\frac{1}{2}A^p_\mu A^{p\mu}+\\
&+B^{*p\mu\nu}(D^{pq}_\mu C^q_\nu-D^{pq}_\nu C^q_\mu)+C^{*p\mu}D^{pq}_\mu C^q_1+\\
&+\frac{1}{2}B^{*p\mu\nu}B^{*q\alpha\beta}\varepsilon_{\mu\nu\alpha\beta}f^{pqr}C^r_1+\bar{C}^*_{p\mu}B^{p\mu}+\bar{C}^*_pB^p_1\Big]
\end{aligned}
\tag{3.10.94}
$$

gives the exact solution to the classical master equation.

3.11 BRST-invariant renormalization

Now let us consider the problem of renormalizability within of the BV formalism following [202], where this problem was first solved (for further discussions see [204, 8, 9, 109, 12]). It is well-known that quantum field theory contains divergences. Indeed, while the boson functional $S = S(\phi, \phi^*)$ (3.5.45) as well as $S_{ext} = S_{ext}(\phi, \phi^*)$ (3.7.48) satisfies the master equation (3.3.41) and does not contain divergences, the equation for the functional Γ

$$\exp\left\{\frac{i}{\hbar}\Gamma(\phi, \phi^*)\right\} == \int d\phi^{'} \exp\left\{\frac{i}{\hbar}[S_{ext}(\phi + \phi^{'}, \phi^*) - \frac{\delta\Gamma(\phi, \phi^*)}{\delta\phi^A}\phi^{A'}]\right\},$$

does contain divergences.

It will be proved that the BRST-symmetry is retained by renormalization. This means that the renormalized action S_R and the effective action Γ_R satisfy the same equations

$$\frac{1}{2}(S_R,\ S_R) = i\hbar\Delta S_R, \quad (\Gamma_R,\ \Gamma_R) = 0$$

as the corresponding nonrenormalized quantities S (here and elsewhere we drop the index ext) and Γ. Our proof is based on the standard assumption of the existence of a regularization respecting the Ward identities. Moreover, the proof is given within the framework of loop expansion.

Let us accordingly represent S and Γ in the form

$$S = \sum_{n=0}^{\infty} \hbar^n S_{(n)} = S_{(0)} + \hbar S_{(1)} + O(\hbar^2),$$

$$\Gamma = S + \hbar(\Gamma^{(1)}_{div} + \Gamma^{(1)}_{fin}) + O(\hbar^2) = S_{(0)} + \hbar(\Gamma^{(1)}_{div} + \bar{\Gamma}^{(1)}_{fin}) + O(\hbar^2),$$

where $\bar{\Gamma}^{(1)}_{fin} = \Gamma^{(1)}_{fin} + S_{(1)}$. The functional $S_{(0)}$ satisfies the equation

$$(S_{(0)},\ S_{(0)}) = 0,$$

while the $S_{(1)}$ satisfies the following linear equation:

$$(S_{(0)},\ S_{(1)}) = i\Delta S_{(0)}.$$

Besides, $\Gamma^{(1)}_{div}$ and $\Gamma^{(1)}_{fin}$ denote the divergent and finite parts of the one-loop approximation for Γ.

The functional $\Gamma^{(1)}_{div}$ determines the counterterms of the one-loop renormalized action S_{1R}:

$$S_{1R} = S - \hbar\Gamma^{(1)}_{div}$$

and satisfies the equation (because of (3.7.55))

$$(S_{(0)},\ \Gamma^{(1)}_{div}) = 0.$$

Let us consider

$$\begin{aligned}&\frac{1}{2}(S_{1R},\ S_{1R}) - i\hbar\Delta S_{1R} =\\&= \frac{1}{2}(S,\ S) - i\hbar\Delta S - \hbar(S,\ \Gamma^{(1)}_{div}) + \frac{1}{2}\hbar^2(\Gamma^{(1)}_{div},\ \Gamma^{(1)}_{div}) + i\hbar^2\Delta\Gamma^{(1)}_{div} =\\&= \hbar^2\left(\frac{1}{2}(\Gamma^{(1)}_{div},\ \Gamma^{(1)}_{div}) + i\Delta\Gamma^{(1)}_{div} - (S_{(1)},\ \Gamma^{(1)}_{div})\right) + O(\hbar^3).\end{aligned}$$

We find that S_{1R} satisfies the master equation

$$\frac{1}{2}(S_{1R},\ S_{1R}) - i\hbar\Delta S_{1R} = \hbar^2 E_2 + O(\hbar^3)$$

up to certain terms E_2

$$E_2 = \frac{1}{2}(\Gamma^{(1)}_{div},\ \Gamma^{(1)}_{div}) + i\Delta\Gamma^{(1)}_{div} - (S_{(1)},\ \Gamma^{(1)}_{div})$$

of the second order in $\hbar$.

Let us construct the effective action Γ_{1R} with the help of the action S_{1R}. This functional is finite in the one-loop approximation and satisfies the equation

$$\frac{1}{2}(\Gamma_{1R},\ \Gamma_{1R}) = \hbar^2 E_2 + O(\hbar^3).$$

Represent Γ_{1R} in the form

$$\begin{aligned}\Gamma_{1R} &= S + \hbar\Gamma^{(1)}_{fin} + +\hbar^2(\Gamma^{(2)}_{1,div} + \Gamma^{(2)}_{1,fin}) + O(\hbar^3) =\\&= S_{(0)} + \hbar\bar{\Gamma}^{(1)}_{fin} + +\hbar^2(\Gamma^{(2)}_{1,div} + \bar{\Gamma}^{(2)}_{1,fin}) + O(\hbar^3).\end{aligned}$$

The divergent part $\Gamma^{(2)}_{1,div}$ of the two - loop approximation for Γ_{1R} determines the two - loop renormalization for S_{2R}

$$S_{2R} = S_{1R} - \hbar^2\Gamma^{(2)}_{1,div}$$

and satisfies the equation

$$(S_{(0)},\ \Gamma^{(2)}_{1,div}) = E_2.$$

Let us now consider

$$\begin{aligned}&\frac{1}{2}(S_{2R},\ S_{2R}) - i\hbar\Delta S_{2R} =\\&= \frac{1}{2}(S_{1R},\ S_{1R}) - i\hbar\Delta S_{1R} - \hbar^2(S_{1R},\ \Gamma^{(2)}_{1,div}) + i\hbar^3\Delta\Gamma^{(2)}_{1,div} =\\&= \hbar^3\left((\Gamma^{(1)}_{div},\ \Gamma^{(2)}_{1,div}) + i\Delta\Gamma^{(2)}_{1,div} - (S_{(2)}, \Gamma^{(1)}_{div}) - (S_{(1)}, \Gamma^{(2)}_{1,div})\right) + O(\hbar^3) =\\&= \hbar^3 E_3 + O(\hbar^4).\end{aligned}$$

We find that S_{2R} satisfies the master equation up to terms E_3

$$E_3 = (\Gamma^{(1)}_{div},\ \Gamma^{(2)}_{1,div}) + i\Delta\Gamma^{(2)}_{1,div} - (S_{(2)}, \Gamma^{(1)}_{div}) - (S_{(1)}, \Gamma^{(2)}_{1,div})$$

of the third order in $\hbar$. Then the corresponding effective action Γ_{2R} generated by S_{2R} is finite in the two - loop approximation

$$\begin{aligned}\Gamma_{2R} &= S+\hbar\Gamma^{(1)}_{fin}+\hbar^2\Gamma^{(2)}_{1,fin}+\hbar^3(\Gamma^{(3)}_{2,div}+\Gamma^{(3)}_{2,fin})+O(\hbar^4)= \\ &= S_{(0)}+\hbar\bar{\Gamma}^{(1)}_{fin}+\hbar^2\bar{\Gamma}^{(2)}_{1,fin}+\hbar^3(\Gamma^{(3)}_{2,div}+\bar{\Gamma}^{(3)}_{2,fin})+O(\hbar^4)\end{aligned}$$

and satisfies the equation

$$\frac{1}{2}(\Gamma_{2R},\ \Gamma_{2R})=\hbar^3E_3+O(\hbar^4)$$

up to certain terms E_3 of the third order in $\hbar$.

Applying the induction method we establish that the totally renormalized action S_R

$$S_R=S-\sum_{n=1}^{\infty}\hbar^n\Gamma^{(n)}_{n-1,div} \tag{3.11.95}$$

satisfies the QME (3.3.41) exactly:

$$\frac{1}{2}(S_R,\ S_R)=i\hbar\Delta S_R, \tag{3.11.96}$$

while the renormalized effective action Γ_R is finite in each order of $\hbar$ powers:

$$\Gamma_R=S+\sum_{n=1}^{\infty}\hbar^n\Gamma^{(n)}_{n-1,fin}=S_{(0)}+\sum_{n=1}^{\infty}\hbar^n\bar{\Gamma}^{(n)}_{n-1,fin}, \tag{3.11.97}$$

and satisfies the identity

$$(\Gamma_R,\ \Gamma_R)=0. \tag{3.11.98}$$

Here, we have denoted by $\Gamma^{(n)}_{n-1,div}$ and $\Gamma^{(n)}_{n-1,fin}$ the divergent and finite parts, respectively, of the n - loop approximation for the effective action which is finite in (n-1)th approximation and is constructed from the action $S_{(n-1)R}$.

Thus, we have established the fact that the renormalized action S_R and the effective action Γ_R satisfy the master equation and the Ward identity, respectively.

We have thus presented the general features of the BV-quantization, constructed, in fact, as an explicit realization of the BRST-invariance principle and have showed how this principle can be effectively used in solving different problems within the BV-formalism.

Of course, we did not discuss all questions related to this method. Among them we would like to note the problem of unitarity of the S-matrix [140, 158, 183, 184, 154, 185, 105, 15], the problem of anomalies [30, 72, 29, 17, 60, 107, 7], the quantization problem of reducible theories [41], the cohomological aspects [122, 85, 86, 13, 16, 11, 124, 189, 190], the locality problem [125, 106, 13, 168], gauge and global symmetries [178, 4, 5, 61, 7], the geometry of the method [206, 175, 176, 177, 136, 137], the formulation and generalizations of the method in genegal coordinates [36, 37, 38, 20, 186], the equivalence of the Lagrangian (BV) and Hamiltonian (Batalin-Fradkin-Vilkovisky [88, 39, 87, 23, 24]) quantizations [78, 115, 116, 171, 164], the construction of quantum antibrackets [31, 32, 33], the properties of general gauge theories with external and composite fields [148, 19, 149, 81, 82], and so on.

Chapter 4

Sp(2)-Covariant Quantization

We have already seen that there is an example of gauge theory for which the quantum action is invariant not only under BRST-transformations but also under the antiBRST-transformations [66, 166]. A natural desire arises to find a quantization method based on the principle of BRST and antiBRST symmetry for general gauge theories. For a long time the opinion has existed that this is possible only for gauge theories with closed algebra and with structure coefficients independent of the fields (for example see [131, 187]).

Recently the quantization method based on the principle of BRST-antiBRST- symmetry has been suggested for general gauge theories [25, 26, 27] (for alternative approaches see [127, 68, 69, 51, 190]).

4.1 Configuration space

To construct the $Sp(2)$-quantization for general gauge theory described by the initial classical action $S_0(A)$ of fields A^i, it is necessary to introduce the total configuration space ϕ^A, which coinsides, in fact, with the total configuration space in the BV formalism (3.2.32), but there is difference in arrangement of the ghost and antighost fields:

$$\phi^A = (A^i,\ B^{\alpha|a_1\cdots a_s},\ C^{\alpha|a_0\cdots a_s},\ s=0,...,L;\ a_i=1,2),\quad \varepsilon(\phi^A)=\varepsilon_A. \tag{4.1.1}$$

Auxiliary fields $B^{\alpha|a_1\cdots a_s}$ and ghost fields $C^{\alpha|a_0\cdots a_s}$ are symmetric Sp(2) tensors of corresponding ranks. The following values of the Grassmann parity are ascribed to these fields:

$$\begin{aligned} \varepsilon(B^{\alpha|a_1\cdots a_s}) &= \varepsilon_{\alpha_s}+s\ (\text{mod}2),\\ \varepsilon(C^{\alpha|a_0\cdots a_s}) &= \varepsilon_{\alpha_s}+s+1\ (\text{mod}2),\quad s=0,...,L \end{aligned}$$

together with the following values of the ghost number:

$$gh(B^{\alpha_0})=0,\quad gh(B^{\alpha|a_1\cdots a_s})=\sum_{s'=1}^{s}(3-2a_{s'}),$$

$$gh(C^{\alpha|a_0\cdots a_s})=\sum_{s'=0}^{s}(3-2a_{s'}).$$

To each field ϕ^A of the total configuration space one introduces three sets of antifields $\phi^*_{Aa},\ \varepsilon(\phi^*_{Aa})=\varepsilon_A+1$ and $\bar\phi_A, \varepsilon(\bar\phi_A)=\varepsilon_A$. We know the meaning of antifields in the

BV-approach. They are sources of BRST transformations. In the extended BRST algebra, there are three kinds of transformations; namely, BRST-transformations, antiBRST-transformations and mixed transformations. The antifields ϕ^*_{Aa} form $Sp(2)$ doublets with respect to the index a and can be treated as sources of BRST- and antiBRST-transformations, while $\bar{\phi}_A$ are sources of combined transformation.

4.2 Extended antibrackets

On the space of fields ϕ^A and antifields ϕ^*_{Aa} one defines odd symplectic structures $(\, , \,)^a$, called the extended antibrackets

$$(F, G)^a \equiv \frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \phi^*_{Aa}} - (F \leftrightarrow G) \, (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}. \tag{4.2.2}$$

As usually the derivatives with respect to fields are understood as acting from the right and those with respect to antifields, as acting from the left.

The extended antibrackets (4.2.2) have the following properties:

$$\begin{aligned}
&\varepsilon((F, G)^a) = \varepsilon(F) + \varepsilon(G) + 1, \\
&(F, G)^a = -(G, F)^a (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}, \\
&(F, GH)^a = (F, G)^a H + (F, H)^a G (-1)^{\varepsilon(G)\varepsilon(H)}, \\
&((F, G)^{\{a}, H)^{b\}} (-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \text{cycl.perm.}(F, G, H) \equiv 0,
\end{aligned} \tag{4.2.3}$$

where curly brackets denote symmetrization with respect to the indices a, b of the $Sp(2)$ group:

$$A^{\{a} B^{b\}} \equiv A^a B^b + B^b A^a.$$

The last relations in (4.2.3) are the graded Jacobi identities for the extended antibrackets. In particular, for any bosonic functional S, $\varepsilon(S) = 0$, one can establish that

$$((S, S)^{\{a}, S)^{b\}} \equiv 0.$$

4.3 Operators V^a, Δ^a

In addition the operators V^a, Δ^a are introduced

$$V^a = \varepsilon^{ab} \, \phi^*_{Ab} \frac{\delta}{\delta \bar{\phi}_A}, \tag{4.3.4}$$

$$\Delta^a = (-1)^{\varepsilon_A} \frac{\delta_l}{\delta \phi^A} \frac{\delta}{\delta \phi^*_{Aa}}, \tag{4.3.5}$$

where ε^{ab} is the antisymmetric tensor for raising and lowering $Sp(2)$-indeces

$$\varepsilon^{ab} = -\varepsilon^{ba}, \quad \varepsilon^{12} = 1 \quad \varepsilon_{ab} = -\varepsilon^{ab}.$$

It can be readily established that the algebra of the operators (4.3.4), (4.3.5) has the form

$$\begin{aligned}
\Delta^{\{a} \Delta^{b\}} &= 0, \\
\Delta^{\{a} V^{b\}} + V^{\{a} \Delta^{b\}} &= 0. \\
V^{\{a} V^{b\}} &= 0.
\end{aligned} \tag{4.3.6}$$

The action of the operators Δ^a (4.3.5) on a product of functionals F and G gives

$$\Delta^a(F \cdot G) = (\Delta^a F)\cdot G + F\cdot(\Delta^a G)(-1)^{\varepsilon(F)} + (F,\ G)^a(-1)^{\varepsilon(F)} \tag{4.3.7}$$

while the action of the operators V^a (4.3.4) upon the extended antibrackets is given by the relations

$$\begin{aligned} V^a(F,G)^b &= (V^aF,\ G)^b - (-1)^{\varepsilon(F)}(F, V^aG)^b - \\ &\quad \varepsilon^{ab}\left(\frac{\delta F}{\delta\phi^A}\frac{\delta G}{\delta\bar{\phi}_A} - \frac{\delta G}{\delta\phi^A}\frac{\delta F}{\delta\bar{\phi}_A}(-1)^{\varepsilon(F)(\varepsilon(G)+1)}\right). \end{aligned}$$

Therefore only the symmetrized form of V^a acting on the extended antibracket observes the Leibniz rule

$$V^{\{a}(F,G)^{b\}} = (V^{\{a}F,G)^{b\}} - (-1)^{\varepsilon(F)}(F,V^{\{a}G)^{b\}}. \tag{4.3.8}$$

For any bosonic functional S we have

$$\frac{1}{2}V^{\{a}(S,S)^{b\}} = (V^{\{a}S,S)^{b\}}.$$

It is advantageous to introduce an operator $\bar{\Delta}^a$

$$\bar{\Delta}^a = \Delta^a + \frac{i}{\hbar}V^a$$

with the properties

$$\bar{\Delta}^{\{a}\bar{\Delta}^{b\}} = 0. \tag{4.3.9}$$

4.4 Extended quantum master equations

For a boson functional $S = S(\phi, \phi^*, \bar{\phi})$, we introduce extended quantum master equations

$$\frac{1}{2}(S,S)^a + V^aS = i\hbar\Delta^aS \tag{4.4.10}$$

with the boundary condition

$$S\Big|_{\phi^*=\bar{\phi}=\hbar=0} = S_0(A), \tag{4.4.11}$$

where $S_0(A)$ is the initial classical action. An equation similar to the extended action arises in the Yang - Mills theory invariant under the BRST - anti BRST symmetry. Indeed, let s, $\bar{\mathsf{s}}$ be the generators of BRST - antiBRST transformations in the Yang - Mills theory. The algebra of the operators has the form

$$\mathsf{s}^2 = \bar{\mathsf{s}}^2 = \mathsf{s}\bar{\mathsf{s}} + \bar{\mathsf{s}}\mathsf{s} = 0. \tag{4.4.12}$$

Let $S(\phi)$ be an action invariant under the BRST and antiBRST transformations

$$\mathsf{s}S(\phi) = 0, \quad \bar{\mathsf{s}}S(\phi) = 0$$

Consider the extended action $S_{ext} = S_{ext}(\phi, \phi^*, \bar{\phi})$

$$S_{ext} = S(\phi) + \phi^*_{A_1}\mathsf{s}\phi^A + \phi^*_{A_2}\bar{\mathsf{s}}\phi^A + \bar{\phi}_A\mathsf{s}\bar{\mathsf{s}}\phi^A.$$

In terms of S_{ext}, the property of invariance of $S(\phi)$ has the form

$$\mathsf{s}S(\phi) = \mathsf{s}S_{ext} - \phi^*_{A_2}\bar{\mathsf{s}}\mathsf{s}\phi^A = 0, \quad \bar{\mathsf{s}}S(\phi) = \bar{\mathsf{s}}S_{ext} - \phi^*_{A_1}\mathsf{s}\bar{\mathsf{s}}\phi^A = 0$$

or, equivalently,

$$\begin{aligned} &\frac{\delta S_{ext}}{\delta\phi^A}\mathsf{s}\phi^A + \phi^*_{A_2}\frac{\delta S_{ext}}{\delta\bar{\phi}_A} = 0, \quad \frac{\delta S_{ext}}{\delta\phi^A}\frac{\delta S_{ext}}{\delta\phi^*_{A_1}} + \phi^*_{A_2}\frac{\delta S_{ext}}{\delta\bar{\phi}_A} = 0,\\ &\frac{\delta S_{ext}}{\delta\phi^A}\bar{\mathsf{s}}\phi^A - \phi^*_{A_1}\frac{\delta S_{ext}}{\delta\bar{\phi}_A} = 0, \quad \frac{\delta S_{ext}}{\delta\phi^A}\frac{\delta S_{ext}}{\delta\phi^*_{A_2}} - \phi^*_{A_1}\frac{\delta S_{ext}}{\delta\bar{\phi}_A} = 0. \end{aligned} \tag{4.4.13}$$

We have exactly the l.h.s. of the extended quantum master equations.

The generating equation for the bosonic functional S is a set of two equations. It should be verified that these equations are compatible. The simplest way to establish this fact is to rewrite the extended master equations in an equivalent form of linear differential equations

$$\bar{\Delta}^a \exp\left\{\frac{i}{\hbar}S\right\} = 0. \tag{4.4.14}$$

Due to the properties of the operators $\bar{\Delta}^a$ (4.3.9), we immediately establish the compatibility of the equations.

4.5 Gauge fixing

The action S is gauge-degenerate. To lift the degeneracy, we should introduce a gauge. We denote the action modified by gauge as $S_{ext} = S_{ext}(\phi, \phi^*, \bar{\phi})$. The gauge should be introduced so as, first, to lift the generacy in ϕ and, second, to retain the extended master equation, which provides the invariance properties of the theory for S_{ext}. To meet these conditions, the gauge is introduced as

$$\exp\left\{\frac{i}{\hbar}S_{ext}\right\} = \exp\left\{-i\hbar\hat{T}(F)\right\}\exp\left\{\frac{i}{\hbar}S\right\} \tag{4.5.15}$$

where $F = F(\phi)$ is a bosonic functional fixing a gauge in the theory. The explicit form of the operator $\hat{T}(F)$ is

$$\hat{T}(F) = \frac{\delta F}{\delta\phi^A}\frac{\delta}{\delta\bar{\phi}_A} + \frac{i\hbar}{2}\varepsilon_{ab}\frac{\delta}{\delta\phi^*_{Aa}}\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\frac{\delta}{\delta\phi^*_{Bb}}. \tag{4.5.16}$$

Due to the properties of the operators $\bar{\Delta}^a$, it is not difficult to check the equality

$$\bar{\Delta}^a \exp\left\{-i\hbar\hat{T}(F)\right\} = \exp\left\{-i\hbar\hat{T}(F)\right\}\bar{\Delta}^a. \tag{4.5.17}$$

Therefore, the action S_{ext} satisfies the extended master equations

$$\bar{\Delta}^a \exp\left\{\frac{i}{\hbar}S_{ext}\right\} = 0. \tag{4.5.18}$$

4.6 Generating functional of Green's functions

We next define the generating functional $Z(J)$ of Green's functions by the rule

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{\mathit{eff}}(\phi) + J_A\phi^A]\right\}, \tag{4.6.19}$$

where

$$S_{\mathit{eff}} = S_{ext}(\phi, \phi^*, \bar{\phi})|_{\phi^*=\bar{\phi}=0}. \tag{4.6.20}$$

It can be represented in the form

$$\begin{aligned} Z(J) \;\; = \;\; & \int d\phi\, d\phi^*\, d\bar{\phi}\, d\lambda\, d\pi^a\; \exp\left\{\frac{i}{\hbar}\Big(S(\phi, \phi^*, \bar{\phi}) + \phi^*_{Aa}\pi^{Aa} + \right. \\ & \left. + \Big(\bar{\phi}_A - \frac{\delta F}{\delta\phi^A}\Big)\lambda^A - \frac{1}{2}\varepsilon_{ab}\pi^{Aa}\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\pi^{Bb} + J_A\phi^A\Big)\right\}, \end{aligned} \tag{4.6.21}$$

where we have introduced a set of auxiliary fields π^{Aa}, λ^A

$$\varepsilon(\pi^{Aa}) = \varepsilon_A + 1, \quad \varepsilon(\lambda^A) = \varepsilon_A.$$

4.7 Extended BRST symmetry

An important property of the integrand for $J_A = 0$ is its invariance under the following global transformations (which, for its part , is a consequence of the extended master equation for S_{ext})

$$\begin{aligned} &\delta\phi^A = \pi^{Aa}\mu_a, \quad \delta\phi^*_{Aa} = \mu_a\frac{\delta S}{\delta\phi^A}, \quad \delta\bar{\phi}_A = \varepsilon^{ab}\mu_a\phi^*_{Ab}, \\ &\delta\pi^{Aa} = -\varepsilon^{ab}\lambda^A\mu_b, \quad \delta\lambda^A = 0, \end{aligned} \tag{4.7.22}$$

where μ_a is an Sp(2) doublet of constant anticommuting Grassmann parameters. These transformations realize the extended BRST transformations in the space of the variables ϕ, ϕ^*, $\bar{\phi}$, π and λ.

4.8 Gauge independence of vacuum functional

The existence of these transformations enables one to establish the independence of the vacuum functional from the choice of gauge. Indeed, suppose $Z_F \equiv Z(0)$. We shall change the gauge $F \to F + \Delta F$. In the functional integral for $Z_{F+\Delta F}$ we make the above-mentioned change of variables with the parameters chosen as

$$\mu_a = \frac{i}{2\hbar}\varepsilon_{ab}\frac{\delta\Delta F}{\delta\phi^A}\pi^{Ab}. \tag{4.8.23}$$

Then we find

$$Z_F = Z_{F+\Delta F} \tag{4.8.24}$$

and therefore the S-matrix is gauge-independent.

4.9 Ward identities

Let us now derive the Ward identities, which follow from the fact that the boson functional $S(\phi, \phi^*, \bar{\phi})$ satisfies the extended master equations. To do this, we introduce the extended generating functional of Green's functions

$$\mathcal{Z}(J, \phi^*, \bar{\phi}) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_{ext}(\phi, \phi^*_a, \bar{\phi}) + J_A\phi^A]\right\}. \tag{4.9.25}$$

From this definition it follows that

$$\mathcal{Z}(J, \phi^*, \bar{\phi})|_{\phi^*=\bar{\phi}=0} = Z(J) \tag{4.9.26}$$

where $Z(J)$ has been introduced above (294), (296).

We have,

$$\int d\phi\ \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\bar{\Delta}^a \exp\left\{\frac{i}{\hbar}S_{ext}(\phi, \phi^*, \bar{\phi})\right\} = 0.$$

Integrating by parts, under the assumption that the integrated expression vanishes, we can write this equality as

$$\widehat{\omega}^a \mathcal{Z}(J, \phi^*, \bar{\phi}) = 0, \tag{4.9.27}$$

where

$$\widehat{\omega}^a = \left(J_A \frac{\delta}{\delta\phi^*_{Aa}} - \varepsilon^{ab}\phi^*_{Ab}\frac{\delta}{\delta\bar{\phi}_A}\right), \quad \widehat{\omega}^{\{a}\widehat{\omega}^{b\}} = 0. \tag{4.9.28}$$

Eqs. (4.9.27) are the Ward identities for the generating functional of Green's functions. For the generating functional $\mathcal{W}(J, \phi^*, \bar{\phi})$ of connected Green's functions we have

$$\widehat{\omega}^a \mathcal{W}(J, \phi^*, \bar{\phi}) = 0, \tag{4.9.29}$$

Finally, for the generating functional of vertex functions

$$\Gamma(\phi, \phi^*, \bar{\phi}) = \mathcal{W}(J, \phi^*, \bar{\phi}) - J_A\phi^A, \ \ \phi^A = \frac{\delta\mathcal{W}}{\delta J_A}$$

we obtain the Ward idntities

$$\frac{1}{2}(\Gamma, \Gamma)^a + V^a\Gamma = 0 \tag{4.9.30}$$

in the form of the classical part of the extended quantum master equations.

4.10 Extended BRST invariant renormalizability

In the same manner as in the case of gauge theories considered in the BV-method, here also one can prove the preservation of the extended BRST-symmetry under renormalization within the usual assumptions on perturbation theory as well as on a regularization respecting the Ward identities [27]. If

$$\frac{1}{2}(S, S)^a + V^a S = i\hbar\Delta^a S, \quad \frac{1}{2}(\Gamma, \Gamma)^a + V^a\Gamma = 0 \tag{4.10.31}$$

then

$$\frac{1}{2}(S_R, S_R)^a + V^a S_R = i\hbar\Delta^a S_R, \quad \frac{1}{2}(\Gamma_R, \Gamma_R)^a + V^a\Gamma_R = 0. \tag{4.10.32}$$

We shall consider two questions concerning the method of extended BRST quantization; namely, the physical unitarity [146], the gauge dependence of the generating functional of Green's functions [143, 149] and the explicit solutions for irreducible closed gauge theories [25].

4.11 Physical unitarity

One of the most important issues of the Lagrangian quantization of gauge theories is the unitarity problem. This long-standing problem was first explicitly formulated by Feynman [83]. For the Yang–Mills type theories, it was efficiently analyzed in Ref. [140] by Kugo and Ojima in the framework of a formalism discovered by them and based on the study of the physical subspace $\mathcal{V}_{\rm phys}$ of the total state vector space $\mathcal{V}$ with indefinite inner product $< \mid >$ (note that vector spaces having indefinite inner product are also commonly referred to as vector spaces with indefinite metric).

The subspace $\mathcal{V}_{\rm phys} \equiv \{|{\rm phys}>\}$ is specified by the operator $\hat{Q}_{\rm BRST}$ ($\hat{Q}^\dagger_{\rm BRST} = \hat{Q}_{\rm BRST}$)

$$\hat{Q}_{\rm BRST}|{\rm phys}>= 0 \tag{4.11.33}$$

being the generator of the BRST symmetry transformations and possessing an important nilpotency property

$$\hat{Q}^2_{\rm BRST} = 0. \tag{4.11.34}$$

In the Yang–Mills type theories, the nilpotency of the operator $\hat{Q}_{\rm BRST}$ follows immediately from the nilpotency of the BRST transformations.

Even though in arbitrary gauge theories the algebra of the BRST transformations is generally open (off-shell), one can still prove (on the assumption of the absence of anomalies) that within such theories, for the corresponding operator $\hat{Q}_{\rm BRST}$ the nilpotency property holds [154]. Thus, one can assume that the Noether charge operator $\hat{Q}_{\rm BRST}$ in the BV quantization scheme satisfies Eq. (4.11.34) and that the Kugo–Ojima formalism, discovered for the Yang–Mills type theories, applies to the analysis of the unitarity problem for general gauge theories (see also Ref. [105]).

In discussing the property (4.11.34), it is important to bear in mind that the widespread opinion that the nilpotency of the operator $\hat{Q}_{\rm BRST}$ guarantees the unitarity of a theory (see, for example, Ref. [63]) proves to be incorrect [154], and that a more accurate examination of physicality conditions fulfillment ensuring the unitarity of a theory is then required. To this end, we shall now recall the main results of analysis of the unitarity problem within the framework of the formalism proposed by Kugo and Ojima.

In Ref. [140] it was shown that if a theory satisfies the following conditions (physicality criteria) for the Hamiltonian $\hat{H}$ and the physical subspace $\mathcal{V}_{\rm phys}$ in the total state vector space $\mathcal{V}$ with indefinite inner product $< \mid >$

(i) hermiticity of the Hamiltonian $\hat{H} = \hat{H}^\dagger$ (or (pseudo-)unitarity of the total S-matrix $S^\dagger S = SS^\dagger = 1$),
(ii) invariance of $\mathcal{V}_{\rm phys}$ under the time development (or $S\mathcal{V}_{\rm phys} = S^{-1}\mathcal{V}_{\rm phys} = \mathcal{V}_{\rm phys}$) (PhC)

(iii) positive semi-definiteness of inner product $< \mid >$ in $\mathcal{V}_{\rm phys}$ ($\mathcal{V}_{\rm phys} \ni |\psi>$: $<\psi|\psi> \geq 0$),

then the physical S-matrix $S_{\rm phys}$ is consistently defined in a Hilbert space $H_{\rm phys}$ equipped with positive definite inner product (the probabilistic interpretation of the quantum theory thus secured). Namely, $H_{\rm phys}$ can be identified with a (completed) quotient space

$$\mathcal{V}_{\rm phys}/\mathcal{V}_0 \ni |\widetilde{\Phi}>, \;\; |\widetilde{\Phi}> = |\Phi> + \mathcal{V}_0, \;\; |\Phi> \in \mathcal{V}_{\rm phys}$$

of $\mathcal{V}_{\rm phys}$ with respect to the zero-norm subspace $\mathcal{V}_0$

$$\mathcal{V}_0 = \{|\chi> \in \mathcal{V}_{\rm phys} : <\chi|\chi> = 0\}, \;\; \mathcal{V}_{\rm phys} \perp \mathcal{V}_0,$$

where positive definite inner product in $\mathcal{V}_{\rm phys}/\mathcal{V}_0$ is defined by $<\widetilde{\Phi}|\widetilde{\Psi}> = <\Phi|\Psi>$. Given this, for the physical S-matrix in $H_{\rm phys}$

$$H_{\rm phys} = \overline{\mathcal{V}_{\rm phys}/\mathcal{V}_0}, \;\; S_{\rm phys}|\widetilde{\Phi}> = \widetilde{S|\Phi>}$$

the unitarity property holds

$$S^\dagger_{\rm phys} S_{\rm phys} = S_{\rm phys} S^\dagger_{\rm phys} = 1.$$

In this connection, note first of all that the subsidiary condition (4.11.33) ensures, on the assumption of hermiticity of the Hamiltonian, the fulfillment of the condition (PhC), (ii) of invariance of $\mathcal{V}_{\rm phys}$ under the time development ($\mathcal{V}^{\rm in}_{\rm phys} = \mathcal{V}^{\rm out}_{\rm phys}$). In [140], the analysis of the condition (PhC), (iii) for an arbitrary theory (4.11.34) was based on the study of representation of the algebra of the operator $\hat{Q}_{\rm BRST}$ and the ghost charge operator $i\hat{Q}_C$ ($[\hat{Q}_C, \hat{H}] = 0$)

$$[i\hat{Q}_C, \hat{Q}_{\rm BRST}] = \hat{Q}_{\rm BRST}$$

(the other commutators trivially vanish) in the one-particle subspace of the total Fock space $\mathcal{V}$.

The one-particle subspace of the theory generally consists of the so-called BRST-singlets and quartets [140]. By definition, the BRST-singlets are introduced as state vectors $|k, N>$ ($i\hat{Q}_C|k, N> = N|k, N>$) from the physical subspace $\mathcal{V}_{\rm phys}$ which cannot be represented in the form $|k, N> = \hat{Q}_{\rm BRST}|*>$ for any state $|*>$. Here, k stands for all the quantum numbers (except the ghost one) which specify the state. At that, the BRST-singlets that belong to the subspace of positive-definite norm (which implies $N = 0$) are called genuine ones and identified with physical states. (In this connection, note that the condition $N = 0$ alone does not provide in a general case the poisitive-defineteness of the subspace of BRST-singlet states [186, 80].) Meanwhile, all BRST-singlets with $N \neq 0$ possess zero norm and form pairs ($|k, -N>$, $|k, N>$) with non-vanishing inner product

$$<k, -N|k, N> = 1.$$

It should be pointed out that the presence of singlet pairs neccesarily leads to negative norm states in the physical subspace [140]. Finally, the states ($|k, N>$, $|k, -N>$, $|k, N+1>$, $|k, -(N+1)>$) such that

$$|k, N+1> = \hat{Q}_{\rm BRST}|k, N>, \;\; |k, -N> = \hat{Q}_{\rm BRST}|k, -(N+1)>,$$

$$<k, -(N+1)|k, N+1> = <k, -N|k, N> = 1$$

form a quartet. The states complexes just described (i.e. the BRST-singlets and quartets) obviously form representations of the algebra of operators $\hat{Q}_{\mathrm{BRST}}$, $i\hat{Q}_C$, while the one-particle subspace is representable as a direct sum of these subspaces [165] (different complexes being orthogonal to one another).

The study of Ref. [140] discovered a general mechanism, called the quartet one, by virtue of which (provided that BRST-singlets in the theory are all genuine ones) any state that belongs to the physical subspace $\mathcal{V}_{\mathrm{phys}}$ of the total Fock space and contains quartet particles has vanishing norm.

Thus, the requirement [140] that all BRST-singlets of the theory possess positive-definite norm (and, consequently, that singlet pairs be absent), providing the positive semi-definiteness (1.3), (iii) of inner product $<\ |\ >$ in $\mathcal{V}_{\mathrm{phys}}$, is a condition of the physical S-matrix unitarity in the Hilbert space $H_{\mathrm{phys}} = \overline{\mathcal{V}_{\mathrm{phys}}/\mathcal{V}_0}$.

Algebra of quantum extended BRST transformations

Here we shall discuss the algebraic properties of extended BRST symmetry transformations and prove the existence of operators required for the unitarity conditions analysis. To this end we now bring to mind the key points of this method.

Note first of all that the quantization involves introducing a complete set of fields ϕ^A and the set of the corresponding antifields ϕ^*_{Aa} (a=1, 2), $\bar{\phi}_A$ (the doublets of antifields ϕ^*_{Aa} play the role of sources of the BRST and antiBRST transformations while the antifields $\bar{\phi}_A$ are the sources of the mixed BRST and antiBRST transformations). The specific structure of configuration space of the fields ϕ^A (including the initial classical fields, the ghosts, the antighosts and the Lagrangian multipliers) is determined by the properties of original classical theory, i.e. by the linear dependence (reducible theories) or independence (irreducible theories) of generators of gauge transformations. Namely, the studies of Refs. [25, 26] have shown that the fields ϕ^A form components of irreducible completely symmetric $Sp(2)$-tensors.

The scheme developed in [25, 26, 27] explicitly possesses the extended BRST symmetry which, in terms of the generating functional of vertex functions $\Gamma = \Gamma(\phi, \phi^*_a, \bar{\phi})$ (extended effective action), implies the following Ward identities

$$\frac{1}{2}(\Gamma, \Gamma)^a + V^a\Gamma = 0. \tag{4.11.35}$$

The study of [27] proved the fact that the renormalized extended effective action satisfies the identities of the same form. In particular, Eq. (4.11.35), considered at $\phi^*_{Aa} = \bar{\phi}_A = 0$, results in the invariance of the effective action $\tilde{\Gamma} = \tilde{\Gamma}(\phi)$

$$\tilde{\Gamma} = \Gamma|_{\phi^*_a = \bar{\phi} = 0} \tag{4.11.36}$$

of the fields ϕ^A under the following transformations

$$\delta\phi^A = \left.\frac{\delta\Gamma}{\delta\phi^*_{Aa}}\right|_{\phi^*_a = \bar{\phi} = 0} \mu_a, \tag{4.11.37}$$

where μ_a is an $Sp(2)$-doublet of constant anticommuting infinitesimal parameters (we shall refer to Eq. (4.11.37) as quantum extended BRST symmetry transformations). Namely,

$$\delta\tilde{\Gamma} = \left.\frac{\delta\Gamma}{\delta\phi^A}\frac{\delta\Gamma}{\delta\phi^*_{Aa}}\right|_{\phi^*_a = \bar{\phi} = 0} \mu_a = -\varepsilon^{ab}\phi^*_{Ab}\left.\frac{\delta\Gamma}{\delta\bar{\phi}_A}\right|_{\phi^*_a = \bar{\phi} = 0} \mu_a = 0. \tag{4.11.38}$$

By virtue of Eq. (4.11.35), one readily finds that the algebra of the symmetry transformations (4.11.37), (4.11.38) is open off-shell

$$\begin{aligned} \delta_{(1)}\delta_{(2)}\phi^A &- \delta_{(2)}\delta_{(1)}\phi^A = \\ &= (-1)^{\varepsilon_A} \frac{\delta\tilde{\Gamma}}{\delta\phi^B} \frac{\delta^2\Gamma}{\delta\phi^*_{Bb}\delta\phi^*_{Aa}}\bigg|_{\phi^*_a=\bar{\phi}=0} \mu_{(1)\{a}\mu_{(2)b\}} \end{aligned} \tag{4.11.39}$$

(here, the symbol { } denotes the symmetrization with respect to the $Sp(2)$ indices: $A^{\{ab\}} = A^{ab} + A^{ba}$).

In this connection, note that the study of Ref. [154] investigated the properties of the symmetry transformations δ_α which form an open algebra

$$\delta_\alpha(\delta_\beta q^i) - \delta_\beta(\delta_\alpha q^i) = f^\gamma_{\alpha\beta}\delta_\gamma q^i + \Delta^i_{\alpha\beta} \tag{4.11.40}$$

within the Lagrangian formulation of an arbitrary non-degenerate theory. Here, q^i are configuration space variables, $f^\gamma_{\alpha\beta}$ are some structure coefficients (depending generally on q^i) and $\Delta^i_{\alpha\beta}$ are some functions vanishing on-shell. In Ref. [154] it was shown, on the assumption of the absence of anomalies, that within the quantum theory constructed in accordance with the Dirac procedure, the following relations hold

$$[\hat{Q}_\alpha,\ \hat{H}] = 0, \quad [\hat{Q}_\alpha,\ \hat{Q}_\beta] = f^\gamma_{\alpha\beta}\hat{Q}_\gamma, \tag{4.11.41}$$

where $\hat{H}$ is the Hamiltonian operator and $\hat{Q}_\alpha$ are the Noether charge operators generating, on the quantum level, the symmetry transformations δ_α.

The comparison of Eq. (4.11.39) with Eqs. (4.11.40), (4.11.41) yields the algebra of the operators of Hamiltonian $\hat{H}$ and Noether charges $\hat{Q}_{(1)} \equiv \hat{Q}^a\mu_{(1)a}$, $\hat{Q}_{(2)} \equiv \hat{Q}^a\mu_{(2)a}$ corresponding to the transformations $\delta_{(1)}$, $\delta_{(2)}$ (4.11.41), (4.11.39)

$$[\hat{Q}_{(1,2)},\ \hat{H}] = 0, \quad [\hat{Q}_{(1)},\ \hat{Q}_{(2)}] = 0. \tag{4.11.42}$$

By virtue of the arbitrariness of parameters $\mu_{(1)a}$, $\mu_{(2)a}$, Eq. (4.11.42) implies the relations

$$[\hat{Q}^a,\ \hat{H}] = 0, \quad [\hat{Q}^a,\ \hat{Q}^b]_+ = 0.$$

Hence it follows that within a general gauge theory (the anomalies out of account) there exists a doublet of nilpotent anticommuting operators $\hat{Q}^a$ generating the quantum transformations of the extended BRST symmetry.

Representation of the algebra of Q^a, Q_C

Let us consider the representation of algebra

$$[\hat{Q}^a, \hat{Q}^b]_+ = 0, \quad [i\hat{Q}_C, \hat{Q}^a] \ = \ -(-1)^a\hat{Q}^a \tag{4.11.43}$$

of the operators $\hat{L} = (\hat{Q}^a,\ \hat{Q}_C)$ in the one-particle subspace $\mathcal{V}^{(1)}$ of the total Fock space $\mathcal{V}$ with indefinite inner product $<\ |\ >$

$$\begin{gathered} \hat{L}\mathcal{V}^{(1)} \subset \mathcal{V}^{(1)}, \ < \Psi|\hat{L}\Phi > = < \hat{L}^\dagger\Psi|\Phi >, \ |\Psi >, \ |\Phi > \in \mathcal{V}^{(1)}, \\ (\hat{Q}^a)^\dagger = -(-1)^a\hat{Q}^a, \ (\hat{Q}_C)^\dagger = \hat{Q}_C. \end{gathered} \tag{4.11.44}$$

We shall demonstrate it here that the space $\mathcal{V}^{(1)}$ of representation of the algebra (4.11.43) is generally a direct sum

$$\mathcal{V}^{(1)} = \bigoplus_n \mathcal{V}^{(1)}_n, \;\; \hat{L}\mathcal{V}^{(1)}_n \subset \mathcal{V}^{(1)}_n, \quad \mathcal{V}^{(1)}_n \bigcap \mathcal{V}^{(1)}_{n'} = \emptyset, \;\; n \neq n', \tag{4.11.45}$$

where subspaces $\mathcal{V}^{(1)}_n$ include the following one-particle state complexes

(i) genuine BRST–antiBRST-singlets (physical particles),
(ii) pairs of BRST–antiBRST-singlets,
(iii) BRST-quartets,
(iv) antiBRST-quartets, (OPSC)
(v) BRST–antiBRST-quartets,
(vi) BRST–antiBRST-sextets,
(vii) BRST–antiBRST-octets.

Here, each of the state complexes (OPSC), (i)-(v), (vii) is itself a representation of the algebra (4.11.43). Note in this connection that even though the variety of all the state complexes (OPSC), (vi) is by construction invariant under the action of the operators $\hat{L}$, an arbitrary state complex (OPSC), (vi) is not necessarily a representation of the algebra (4.11.43).

BRST–antiBRST-quartets

In order to construct the basis of representation explicitly, note that for an arbitrary state $|\Phi>$, one of the following conditions holds

$$\frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\Phi> \;\; \neq \;\; 0, \tag{4.11.46}$$

$$\frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\Phi> \;\; = \;\; 0. \tag{4.11.47}$$

If a state $|\phi_{(k,N)}> \in \mathcal{V}^{(1)}_n$ $(i\hat{Q}_C|\phi_{(k,N)}> = N|\phi_{(k,N)}>)$ satisfies the condition (4.11.46), then, by virtue of Eq. (4.11.43), there exists a set of linearly independent states

$$|\phi_{(k,N)}>, \;\; \hat{Q}^a|\phi_{(k,N)}>, \;\; \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi_{(k,N)}>, \tag{4.11.48}$$

which form the basis of a four-dimensional representation of the algebra (4.11.43). Given this, owing to the properties (4.11.43), (4.11.45), the states

$$\hat{Q}^a|\phi_{(k,N)}>, \;\; \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi_{(k,N)}>$$

have vanishing norm, in particular, $|k,N> \equiv \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi_{(k,N)}>$

$$<k,N|k,N> = 0. \tag{4.11.49}$$

In accordance with Ref. [140], for an arbitrary one-particle zero-norm (4.11.49) state $|k,N>$, there exists some (generally not unique) one-particle state $|k,-N>$ such that

$$<k,-N|k,N> = 1 \tag{4.11.50}$$

(by virtue of Eq. (4.11.44), any states $|k,N>$, $|k',N'>$ can only have a non-vanishing inner product $<k',N'|k,N>$ when $N=-N'$). Given this, it is clear that the states $|k,N>$, $|k,-N>$ (4.11.49), (4.11.50) are linearly independent, and hence can be treated as basis state vectors in the subspace $\mathcal{V}^{(1)}$.

One readily establishes the fact that for any $|k,N>$ (4.11.49), the arbitrariness in a choice of the corresponding state $|k,-N>$ (4.11.50), in any subspace containing these vectors, can always be lifted by an appropriate choice of the basis. In fact, in the subspace of linearly independent states ($|k,-N>$, $\{|l,-N>\}$) with the properties $<k,-N|k,N>=<l,-N|k,N>=1$, it is always possible to choose a basis ($|k,-N>$, $\{|\overline{l,-N}>\equiv|l,-N>-|k,-N>\}$) such that $<\overline{l,-N}|k,N>=0$.

Note, owing to Eqs. (4.11.49), (4.11.50), that the basis in the subspace of states $|\Psi>=\{|l,N>, l\neq k\}$, $<k,N|\Psi>=0$ can always be chosen so as $<k,-N|l,N>=0$. Indeed, in order to go over from the basis states $|k,N>$, $\{|l,N>\}$

$$<k,N|k,N>=0,\ \ <k,-N|k,N>=1,$$

$$<k,N|l,N>=0,\ \ <k,-N|l,N>=1,\ \ \forall l$$

to an equivalent linearly independent set $|\overline{k,N}>$, $\{|\overline{l,N}>\}$

$$<\overline{k,N}|\overline{k,N}>=0,\ \ <k,-N|\overline{k,N}>=1,$$

$$<\overline{k,N}|\overline{l,N}>=0,\ \ <k,-N|\overline{l,N}>=0,\ \ \forall l$$

it is sufficient, for example, to identify

$$|\overline{k,N}>=|k,N>,\ \ |\overline{l,N}>=|l,N>-|k,N>,\ \ \forall l.$$

Thus, by means of an appropriate choice of the basis in an arbitrary subspace containing a pair $|k,N>$, $|k,-N>$ (4.11.49), (4.11.50), these states can always be made orthogonal to the remaining basis state vectors.

From Eqs. (4.11.49), (4.11.50) and the hermiticity assignment (4.11.44) it follows that there exists a set of four states

$$|\bar{\phi}_{(k,-N)}>,\ \hat{Q}^a|\bar{\phi}_{(k,-N)}>,\ \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\bar{\phi}_{(k,-N)}>, \tag{4.11.51}$$

which are also linearly independent and form the basis of a representation of the algebra (4.11.43). Here, $|\bar{\phi}_{(k,-N)}>$ is a state (4.11.46) chosen from the condition

$$\frac{1}{2}\varepsilon_{ab}<\bar{\phi}_{(k,-N)}|\hat{Q}^a\hat{Q}^b|\phi_{(k,N)}>=1. \tag{4.11.52}$$

By virtue of Eq. (4.11.52), the state vectors $|\bar{\phi}_a>\equiv(|\bar{\phi}_1>, |\bar{\phi}_2>)$ satisfying the normalization $<\bar{\phi}_1|\hat{Q}^1\phi>=<\bar{\phi}_2|\hat{Q}^2\phi>=1$ that corresponds to the zero-norm states $\hat{Q}^a|\phi>$ can be chosen in the form $|\bar{\phi}_a>=\varepsilon_{ba}(\hat{Q}^b)^\dagger|\bar{\phi}>$.

For a more detailed analysis of the states (4.11.46), (4.11.51), (4.11.52), we first suppose that some state $|\phi_{(k,N)}>$ (4.11.43) satisfies the condition

$$\frac{1}{2}\varepsilon_{ab}<\phi_{(k,N)}|\hat{Q}^a\hat{Q}^b|\phi_{(k,N)}>\neq 0. \tag{4.11.53}$$

Then, owing to Eq. (4.11.53), there exists such $\alpha\neq 0$ that the corresponding state $|\bar{\phi}_{(k,-N)}>$ (4.11.52) can be identified as

$$|\bar{\phi}_{(k,-N)}>=\alpha|\phi_{(k,N)}>. \tag{4.11.54}$$

Hence it is clear that $N = 0$, and that the representation subspaces corresponding to the vector sets (4.11.46), (4.11.52) coincide. For a set of basis vectors we choose, say, (4.11.46), i.e. $(|\phi_{(k,N=0)}> \equiv |k,0>)$

$$|k,0>, \ \hat{Q}^a|k,0>, \ \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|k,0> . \tag{4.11.55}$$

Given this, owing to Eq. (4.11.54), the relation holds

$$\frac{\alpha^*}{2}\varepsilon_{ab} < k,0|\hat{Q}^a\hat{Q}^b|k,0>= 1. \tag{4.11.56}$$

By virtue of Eq. (4.11.56), the set of states (4.11.55) can be represented in the form of both a BRST-quartet $((\hat{Q}^1)^\dagger = \hat{Q}^1)$

$$|k,0>, \ |\overline{k,0}>, \ |k,1>, \ |k,-1>,$$

$$|k,1>= \hat{Q}^1|k,0>, \ |\overline{k,0}>= \hat{Q}^1|k,-1>, \tag{4.11.57}$$

$$< \overline{k,0}|k,0>=< k,-1|k,1>= 1$$

(choosing for $|k,-1> \equiv -\alpha\hat{Q}^2|k,0>$), and an antiBRST-quartet $((i\hat{Q}^2)^\dagger = i\hat{Q}^2)$

$$|k,0>, \ |\overline{k,0}>, \ |k,-1>, \ |k,1>,$$

$$|k,-1>= i\hat{Q}^2|k,0>, \ |\overline{k,0}>= i\hat{Q}^2|k,1>, \tag{4.11.58}$$

$$< \overline{k,0}|k,0>=< k,-1|k,1>= 1$$

(choosing for $|k,1> \equiv -i\alpha^*\hat{Q}^1|k,0>$).

By construction, the variety of linear combinations of the vectors (4.11.55), (4.11.56) constitute a subspace (of states $|\Psi>$), which has non-degenerate inner product ($\forall|\Psi> \neq 0$, $\exists|\Psi'>$: $<\Psi|\Psi'> \neq 0$) and is invariant under the action of the operators $\hat{L}$.

In what follows, we shall consider the states (4.11.55), (4.11.56) (provided they do exist in a specific theory) as part of the basis state vectors in $\mathcal{V}^{(1)}$.

Eqs. (4.11.57), (4.11.58) imply, with allowance for Eqs. (4.11.43), (4.11.44), that the whole set ($|k,0>$, $|\overline{k,0}>$, $|k,-1>$, $|k,1>$) of states (4.11.55), (4.11.56) form two mutually orthogonal pair of state vectors (4.11.49), (4.11.50)

$$(|k,0>, |\overline{k,0}>), \ < \overline{k,0}|k,0>= 1 \ \ < \overline{k,0}|\overline{k,0}>= 0,$$

$$(|k,-1>, |k,1>), \ < k,-1|k,1>= 1, \ < k,-1|k,-1>= 0, \ < k,1|k,1>= 0,$$
$$< k,0|k,-1>=< k,0|k,1>= 0, \ < \overline{k,0}|k,-1>=< \overline{k,0}|k,1>= 0.$$

Owing to the above considered properties of states (4.11.49), (4.11.50), there can always be chosen such a basis in $\mathcal{V}^{(1)}$ that either pair ($|k,0>$, $|\overline{k,0}>$), ($|k,-1>$, $|k,1>$) of basis vectors (4.11.55), (4.11.56) in the subspace of states $|\Psi>$ is orthogonal to the remaining basis state vectors. Then, from the condition $<\Psi|\Phi>= 0$ ($|\Phi>$ is an arbitrary state not representable as a linear combination of the state vectors (4.11.55), (4.11.56)) it follows that the states $|\Phi>$

$$< \Psi|\hat{L}\Phi>=< \hat{L}^\dagger\Psi|\Phi>= 0$$

also form a subspace of the representation of algebra of the operators $\hat{L}$.

Repeating the above treatment with respect to the pointed out states $|\Phi>$, one can subsequently single out all the basis state complexes (4.11.55), (4.11.56), which we shall further call BRST–antiBRST-quartets (OPSC), (v).

Clearly, the BRST–antiBRST-quartet complexes (4.11.55), (4.11.56) exhaust all the states (4.11.53) (i.e. the condition (4.11.53) cannot be met by any linear combination of the remaining basis vectors). By construction, any two BRST–antiBRST-quartets have no elements in common. At the same time, the BRST–antiBRST-quartets are all chosen to be orthogonal both to one another and to the remaining basis states, which, as shown above, thus form a subspase of the representation of algebra of the operators $\hat{L}$.

BRST–antiBRST-octets

Turning ourselves to the analysis of state vectors $|\Phi>$ not representable as linear combinations of the above considered BRST–antiBRST-quartet states (4.11.53), we shall first of all proceed with the treatment of states (4.11.46) (or (4.11.52)), which are, of course, generally not restricted to the states (4.11.53) only.

To this end, we observe that, by construction, any pair $(|\phi_{(k,N)}>, |\bar{\phi}_{(k,-N)}>)$ of state vectors (4.11.52) which belong to the subspace of states $|\Phi>$ under consideration satisfies (as all the states $|\Phi>$ do) the conditions

$$< \phi_{(k,N)}|\hat{Q}^a\hat{Q}^b|\phi_{(k,N)} >= 0, \;\; < \bar{\phi}_{(k,-N)}|\hat{Q}^a\hat{Q}^b|\bar{\phi}_{(k,-N)} >= 0. \tag{4.11.59}$$

We shall now demonstrate, with Eq. (4.11.59) taken into account, that the states of the whole set (4.11.48), (4.11.51), (4.11.52) turn out to be linearly independent. Let us assume the reverse. Indeed, if among the numbers $(\beta,\ \beta_a,\ \tilde{\beta},\ \gamma,\ \gamma_a,\ \tilde{\gamma})$ there is a non-zero one, and if $(|\phi_{(k,N)} >\equiv |\phi>, |\bar{\phi}_{(k,-N)} >\equiv |\bar{\phi}>)$

$$\beta|\phi> +\beta_a\hat{Q}^a|\phi> +\frac{\tilde{\beta}}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi> +\gamma|\bar{\phi}> +\gamma_a\hat{Q}^a|\bar{\phi}> +\frac{\tilde{\gamma}}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\bar{\phi}>= 0,$$

then, owing to Eq. (4.11.43), hence follows the condition (for some $\alpha \neq 0$)

$$\frac{\alpha}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi_{(k,N)}>= \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\bar{\phi}_{(k,-N)}>,$$

obviously contradicting Eq. (4.11.59). In order to prove the above relation, suffice it to note that if $\beta=\gamma=\beta_a=\gamma_a=0$, then from the condition $\tilde{\beta}\neq 0$ it follows that $\tilde{\gamma}\neq 0$ (reversely, $\tilde{\gamma}\neq 0 \Rightarrow \tilde{\beta}\neq 0$) with $\alpha=\tilde{\beta}\tilde{\gamma}^{-1}$; in the case $\beta=\gamma=0$ the condition $\exists a : \beta_a \neq 0$ implies $\gamma_a \neq 0$ (similarly, $\gamma_a\neq 0 \Rightarrow \beta_a\neq 0$), here $\alpha=\beta_a\gamma_a^{-1}$ (no summation); finally, if $\beta\neq 0$ (or, equivalently, $\gamma\neq 0$), we have $\alpha=\beta\gamma^{-1}$.

By construction, the set of linearly independent states (4.11.48), (4.11.51), (4.11.52) form the basis of a representation subspace with non-degenerate inner product and can be considered as both a pair of BRST-quartets

$$\begin{gathered}(|\phi_{(k,N)}>,\ -\hat{Q}^1\hat{Q}^2|\bar{\phi}_{(k,-N)}>,\ \hat{Q}^1|\phi_{(k,N)}>,\ -\hat{Q}^2|\bar{\phi}_{(k,-N)}>),\\ (|\bar{\phi}_{(k,-N)}>,\ -\hat{Q}^1\hat{Q}^2|\phi_{(k,N)}>,\ \hat{Q}^1|\bar{\phi}_{(k,-N)}>,\ -\hat{Q}^2|\phi_{(k,N)}>)\end{gathered} \tag{4.11.60}$$

and a pair of antiBRST-quartets

$$\begin{gathered}(|\phi_{(k,N)}>,\ \hat{Q}^2\hat{Q}^1|\bar{\phi}_{(k,-N)}>,\ i\hat{Q}^2|\phi_{(k,N)}>,\ -i\hat{Q}^1|\bar{\phi}_{(k,-N)}>),\\ (|\bar{\phi}_{(k,-N)}>,\ \hat{Q}^2\hat{Q}^1|\phi_{(k,N)}>,\ i\hat{Q}^2|\bar{\phi}_{(k,-N)}>,\ -i\hat{Q}^1|\phi_{(k,N)}>).\end{gathered} \tag{4.11.61}$$

Note that, without the loss of generality, one can assume

$$< \bar{\phi}_{(k,-N)}|\phi_{(k,N)} >= 0,$$

since if there does exist such $\alpha \neq 0$ that

$$< \bar{\phi}_{(k,-N)}|\phi_{(k,N)} >= \alpha,$$

then one can choose a basis in the subspace (4.11.48) so as

$$|\phi'_{(k,N)} >,\ \hat{Q}^a|\phi'_{(k,N)} >,\ \frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi'_{(k,N)} >,$$

$$\frac{1}{2}\varepsilon_{ab} < \bar{\phi}_{(k,-N)}|\hat{Q}^a\hat{Q}^b|\phi'_{(k,N)} >= 1,\ \ < \bar{\phi}_{(k,-N)}|\phi'_{(k,N)} >= 0,$$

where $|\phi'_{(k,N)} >\equiv \alpha^{-1}|\phi_{(k,N)} > -\frac{1}{2}\varepsilon_{ab}\hat{Q}^a\hat{Q}^b|\phi_{(k,N)} >$. Hence, with Eqs. (4.11.43), (4.11.44), (4.11.52), (4.11.59) taken into account, it follows that the states (4.11.60) (or (4.11.61)) are representable as four mutually orthogonal pairs of states (4.11.49), (4.11.50)

$$(|\phi_{(k,N)} >,\ -\hat{Q}^1\hat{Q}^2|\bar{\phi}_{(k,-N)} >),\ (\hat{Q}^1|\phi_{(k,N)} >,\ -\hat{Q}^2|\bar{\phi}_{(k,-N)} >),$$

$$(|\bar{\phi}_{(k,-N)} >,\ -\hat{Q}^1\hat{Q}^2|\phi_{(k,N)} >),\ (\hat{Q}^1|\bar{\phi}_{(k,-N)} >,\ -\hat{Q}^2|\phi_{(k,N)} >),$$

and can therefore, when identified with elements of the basis in $\mathcal{V}^{(1)}$, be made orthogonal to the remaining basis state vectors (in the subspace of states $|\Phi >$ under consideration) by means of an appropriate choice of the latter.

Using the reasoning similar to the given above, one can subsequently single out all the state complexes (4.11.48), (4.11.51), (4.11.52), (4.11.59), which we shall refer to as BRST–antiBRST-octets (OPSC), (vii), in such a way that different BRST–antiBRST-octets be chosen mutually orthogonal and having no elements in common.

Thus, with allowance for Eqs. (4.11.48)–(4.11.61), we have described the structure of representations containing the state vectors (4.11.46).

BRST–antiBRST-sextets

Consider now the states $|\Phi >\in \mathcal{V}^{(1)}$ not representable as linear combinations of states (4.11.48), (4.11.51), (4.11.55) (i.e. those which do not belong to BRST–antiBRST-quartets or octets) and make use of a choice of the basis in $\mathcal{V}^{(1)}$ for which every state $|\Phi >$ is orthogonal to all the state vectors (4.11.48), (4.11.51), (4.11.55), and which thus ensures the invariance of subspace of the pointed out states $|\Phi >$ under the action of the operators $\hat{L}$.

From the previous treatment it follows immediately that the states $|\Phi >$ under consideration satisfy the condition (4.11.47) (all the states (4.11.46) are by construction exhausted by BRST–antiBRST-quartet and octet state vectors).

Given this, the following conditions generally hold $|\Phi >\equiv \{|\phi_{(k,N)} >\}$

$$\exists a:\ \ \hat{Q}^a|\phi_{(k,N)} >\neq 0, \tag{4.11.62}$$

$$\forall a:\ \ \hat{Q}^a|\phi_{(k,N)} >= 0. \tag{4.11.63}$$

Let us first turn ourselves to the states of the form (4.11.62). For such states the condition is valid ($|* >$ implies arbitrary one-particle states)

$$|\phi_{(k,N)} >\neq \hat{Q}^a|* >, \tag{4.11.64}$$

since otherwise the states $|\phi_{(k,N)}>$ under consideration would be some linear combinations of the states (4.11.48). An arbitrary state $|\phi_{(k,N)}>$ (4.11.47), (4.11.62), (4.11.64), in its turn, satisfies one of the three conditions

$$\begin{aligned}&\text{(i)}\ \ \hat{Q}^1|\phi_{(k,N)}>\neq 0,\ \ \hat{Q}^2|\phi_{(k,N)}>\neq 0,\\ &\text{(ii)}\ \ \hat{Q}^1|\phi_{(k,N)}>\neq 0,\ \ \hat{Q}^2|\phi_{(k,N)}>=0,\\ &\text{(iii)}\ \ \hat{Q}^1|\phi_{(k,N)}>=0,\ \ \hat{Q}^2|\phi_{(k,N)}>\neq 0.\end{aligned} \tag{TC}$$

If a state $|\phi_{(k,N)}>$ satisfies the condition (TC), (i), then, by virtue of Eq. (4.11.43), there exist linearly independent states

$$|\phi_{(k,N)}>,\ \ \hat{Q}^a|\phi_{(k,N)}>, \tag{4.11.65}$$

which form the basis of a three-dimensional representation of the algebra (4.11.43). At the same time, the states $\hat{Q}^a|\phi>$ (we omit, for the sake of brevity, the notations of quantum numbers) have vanishing norm

$$<\hat{Q}^1\phi|\hat{Q}^1\phi>=<\hat{Q}^2\phi|\hat{Q}^2\phi>=0.$$

From the above relations it follows, with allowance made for Eqs. (4.11.44), (4.11.49), (4.11.50), that there exist three linearly independent states

$$|\phi_a>,\ \ \frac{1}{2}(\hat{Q}^a)^{\dagger}|\phi_a>, \tag{4.11.66}$$

where the states $|\phi_a>\neq\hat{Q}^a|*>$, chosen without the loss of generality as eigenvectors for the ghost charge operator $i\hat{Q}_C$, satisfy the normalization conditions

$$<\phi_b|\hat{Q}^a\phi>=\delta^a_b \tag{4.11.67}$$

(the relations $i\hat{Q}_C|\phi_a>=-(N-(-1)^a)|\phi_a>$ immediately ensure the validity of the conditions $<\phi_2|\hat{Q}^1\phi>=<\phi_1|\hat{Q}^2\phi>=0$); at the same time, by virtue of Eqs. (4.11.44), (4.11.47), (4.11.67), we have

$$\begin{aligned}&\frac{1}{2}<(\hat{Q}^a)^{\dagger}\phi_a|\phi>=1,\\ &<(\hat{Q}^b)^{\dagger}\phi_b|\hat{Q}^a\phi_a>=0,\ \ <\phi_a|\phi>=0\end{aligned} \tag{4.11.68}$$

(the inequality $<\phi_a|\phi>\neq 0$ leads one to the condition $\exists a: N=N-(-1)^a$ and, therefore, does not hold for any N).

Let us show, with Eqs. (4.11.43), (4.11.44) taken into account, that the states of the whole set (4.11.65), (4.11.66)

$$|\phi>,\ \ \hat{Q}^a|\phi>,\ \ |\phi_a>,\ \ \frac{1}{2}(\hat{Q}^a)^{\dagger}|\phi_a> \tag{4.11.69}$$

are linearly independent. Indeed, assuming the reverse, i.e.

$$\beta|\phi>+\beta_a\hat{Q}^a|\phi>+\gamma^a|\phi_a>+\frac{\gamma}{2}(\hat{Q}^a)^{\dagger}|\phi_a>=0$$

(the numbers (β, β_a, γ^a, γ) not all vanishing), one arrives, by virtue of Eq. (4.11.47) and the normalization conditions (4.11.67), at the relation

$$\exists a:<(\hat{Q}^a)^{\dagger}\phi|\phi>\equiv\alpha^a\neq 0$$

representable as

$$\beta \neq 0 \Leftrightarrow \exists a: \ \gamma^a \neq 0, \ \alpha^a = (-1)^a \gamma^a / \beta,$$
$$\beta = \gamma = 0, \ \ \gamma \neq 0 \Leftrightarrow \exists a: \ \beta_a \neq 0, \ \alpha^a = -\gamma / \beta_a.$$

If we now suppose, for example, that $a = 1$, then, owing to Eq. (4.11.67) ($< \hat{Q}^1 \phi_1 | \phi >= 1$), the eigenvalues of the ghost charge operator $i\hat{Q}_C$ that correspond to the states $\hat{Q}^1 | \phi >$ and $\hat{Q}^1 | \phi_1 >$

$$i\hat{Q}_C | \hat{Q}^1 \phi >= (N+1) | \hat{Q}^1 \phi >,$$
$$i\hat{Q}_C | \hat{Q}^1 \phi_1 >= -N | \hat{Q}^1 \phi_1 >$$

must coincide, i.e. $N + 1 = -N$. In the case $a = 2$ we similarly have $N - 1 = -N$ and find that neither condition can be satisfied for an integer N. In what follows, we shall refer to the states (4.11.69), (4.11.67) as a BRST–antiBRST-sextet (OPSC), (vi).

Owing to Eqs. (4.11.67), (4.11.68), the bases (4.11.65), (4.11.66) ($|\phi >, \ \hat{Q}^a | \phi >) \equiv |e_i >$, ($|\phi_a >, \ \frac{1}{2} (\hat{Q}^a)^\dagger |\phi_a >) \equiv |f_i >$ are dual with respect to each other $< f_i | e_j >= \delta_{ij}$. Hence follows the non-degeneracy of bilinear form $< \ \ | \ \ >$ defined on the pair $X \equiv \{|e_i >\}$, $Y \equiv \{|f_i >\}$ of state spaces corresponding to the vector sets (4.11.65), (4.11.66). This fact implies that in the space Y exists the (unique) representation $\hat{\tilde{L}}^\dagger |f_i >= (\hat{L}^\dagger)_{ij} |f_j >$, $\hat{\tilde{L}}^\dagger = ((\hat{\tilde{Q}}^a)^\dagger, \ \hat{\tilde{Q}}^\dagger_C)$ of the algebra (4.11.43) conjugate to the representation $\hat{L} |e_i >= (\hat{L})_{ij} |e_j >$ defined in X, i.e. $(\hat{\tilde{L}}^\dagger)_{ij} = \overset{*}{(\hat{L})}_{ji}$. Namely,

$$(\hat{\tilde{Q}}^a)^\dagger |\phi_b >= \frac{1}{2} \delta^a_b (\hat{Q}^c)^\dagger |\phi_c >, \tag{4.11.70}$$
$$(\hat{\tilde{Q}}^a)^\dagger (\hat{Q}^b)^\dagger |\phi_b >= 0$$

(for the ghost charge operator $i\hat{Q}_C$, the basis states of the subspace Y are by construction eigenvectors, i.e. $\hat{\tilde{Q}}^\dagger_C |f_i >= \hat{Q}^\dagger_C |f_i >$).

Note that an arbitrary BRST–antiBRST-sextet (4.11.69), (4.11.67) is generally not invariant under the action of the operators $\hat{L}$. Indeed, if $\forall a$: $\hat{Q}^a |\phi_1 > \neq 0$, ($< \phi_1 \hat{Q}^1 | \phi >= 1$, $\forall a$: $\hat{Q}^a | \phi > \neq 0$, $|\phi > \equiv |k, N >$), then the state $|\phi_1 > \equiv |k, -(N+1) >$ gives rise to some BRST–antiBRST-sextet, which does not coincide with the given set (4.11.69), (4.11.67) (for example, it is clear that the state $\hat{Q}^2 |k, -(N+1) > \equiv |k, -(N+2) > \neq 0$ does not belong to the initial state vectors (4.11.69), (4.11.67)); and if $\forall a$: $\hat{Q}^a |\phi_2 > \neq 0$, ($< \phi_2 | \hat{Q}^2 \phi >= 1$, $|\phi_2 > \equiv |k, -(N-1) >$), we similarly have $\hat{Q}^1 |k, -(N-1) > \equiv |k, -(N-2) > \neq 0$ and find that there is another BRST–antiBRST-sextet associated with the state $|\phi_2 >$, which also differs from (4.11.69), (4.11.67).

Repeating the above considerations, we come to the variety of states that belong to all the BRST–antiBRST-sextets thus associated with the given set (4.11.69), (4.11.67). By construction, the linear combinations of these states form a subspace invariant under the action of the operators $\hat{L}$ and having non-degenerate inner product. The basis states of subspace concerned can in a general case be chosen as BRST- (or antiBRST-) quartets and singlets and thus made orthogonal to the remaining basis state vectors (in the subspace of states $|\Phi >$ under consideration) by means of their appropriate choice.

Note that if the set of state vectors (4.11.66) is invariant under the action of the operators $\hat{Q}^a$ (i.e. $(\hat{\tilde{Q}}^a)^\dagger |f_i >= (\hat{Q}^a)^\dagger |f_i >$), the states (4.11.69), (4.11.67), (4.11.70) can be represented in the form of a BRST-quartet

$$|\phi >, \ \ \hat{Q}^1 |\phi_1 >, \ \ \hat{Q}^1 |\phi >, \ \ |\phi_1 >,$$

$$< \hat{Q}^1\phi_1|\phi >=< \phi_1|\hat{Q}^1\phi >= 1$$

and a pair of BRST-singlets $(\hat{Q}^2|\phi >,\ |\phi_2 >)$

$$< \phi_2|\hat{Q}^2\phi >= 1,$$

$$\hat{Q}^1|\hat{Q}^2\phi >= \hat{Q}^1|\phi_2 >= 0,\ \ |\phi_2 >\neq \hat{Q}^1|* >,\ \ |\hat{Q}^2\phi >\neq \hat{Q}^1|* >, \tag{4.11.71}$$

as well as in the form of an antiBRST-quartet $((\hat{Q}^1)^\dagger|\phi_1 >= (\hat{Q}^2)^\dagger|\phi_2 >)$

$$|\phi >,\ \ -\hat{Q}^2|\phi_2 >,\ \ i\hat{Q}^2|\phi >,\ \ i|\phi_2 >,$$

$$< i\phi_2|i\hat{Q}^2\phi >= - < \hat{Q}^2\phi_2|\phi >= 1$$

and a pair of antiBRST-singlets $(\hat{Q}^1|\phi >,\ |\phi_1 >)$

$$< \phi_1|\hat{Q}^1\phi >= 1,$$

$$\hat{Q}^2|\hat{Q}^1\phi >= \hat{Q}^2|\phi_1 >= 0,\ \ |\phi_1 >\neq \hat{Q}^2|* >,\ \ |\hat{Q}^1\phi >\neq \hat{Q}^2|* > . \tag{4.11.72}$$

Let us consider the two-dimentional subspace of states $|\Psi >= \{|k,N >,\ |\overline{k,-N} >\}$ being linear combinations of vectors of the BRST-singlet pair (4.11.71) $(\hat{Q}^2|\phi >\equiv |k,N >,$ $|\phi_2 >\equiv |\overline{k,-N} >)$. There are two alternatives to be studied separately. First supposing that $N \neq 0$, we, as is well-known [140], have unphysical particles leading to negative norm states. If we now turn to the case $N = 0$, then from the non-degeneracy (4.11.71) of inner product in the space under consideration, it follows, by virtue of $|k,0 >\neq 0$, $< k,0|k,0 >= 0$, that there exists [140] a state $|\psi >= \beta|\overline{k,0} > +\gamma|k,0 >$, $\beta \neq 0$ (clearly, $i\hat{Q}_C|\psi >= 0$) having negative norm $< \psi|\psi >< 0$. Moreover, this implies that in the subspace of states $|\Psi >$ there can always be chosen such a basis $(|k,0 >,\ |\widetilde{k,0} >= |\overline{k,0} > + \alpha|k,0 >)$ that

$$< \widetilde{k,0}|k,0 >= 1,\ \ < \widetilde{k,0}|\widetilde{k,0} >< 0.$$

Quite similar considerations show that the antiBRST-singlet pair (4.11.72), too, always implies negative norm states and cannot evidently be treated as physical states (not for $N = 0$).

Let us show that any representation subspace (of states $|\Phi >$) having non-degenerate inner product and including a BRST–antiBRST-sextet complex always contains a BRST- or an antiBRST-singlet pair. Assuming the reverse, we single out, in the subspace under consideration, some states of the form

$$|k,N >,\ \ \forall a:\ \hat{Q}^a|k,N >\neq 0,$$

$$|k,-N+1 >,\ \ < k,-N+1|\hat{Q}^2|k,N >= 1$$

(such states $|k,N >$, $|k,-N+1 >$ must exist, since we consider a subspace containing some BRST–antiBRST-sextet). The above state $|k,-N+1 >$ satisfies the condition

$$\hat{Q}^1|k,-N+1 >\neq 0,$$

since we would otherwise deal with a BRST-singlet pair $(|k,-N+1 >,\ \hat{Q}^2|k,N >)$

$$(|k,-N+1 >,\ \hat{Q}^2|k,N >) \neq \hat{Q}^1|* >,\ \ \hat{Q}^1|k,-N+1 >= 0,\ \ \hat{Q}^1\hat{Q}^2|k,N >= 0,$$

$$< k, -N+1|\hat{Q}^2|k, N >= 1,$$

and hence there exists a state $|k, N-2>$ such that

$$< k, N-2|\hat{Q}^1|k, -N+1 >= 1.$$

Then, making allowance for the fact that the condition $\hat{Q}^2|k, N-2>= 0$ leads to an antiBRST-singlet pair

$$(|k, N-2>, \hat{Q}^1|k, -N+1>) \neq \hat{Q}^2|*>,$$

we have

$$\forall a: \ \hat{Q}^a|k, N-2>\neq 0.$$

By repetition of the above treatment, we find that for any integer $n \geq 0$, there exist some states of the form

$$|k, N-2n>, \ \ \forall a: \ \hat{Q}^a|k, N-2n>\neq 0,$$
$$|k, -N+2n+1>, \ \ < k, -N+2n+1|\hat{Q}^2|k, N-2n>= 1,$$
$$\hat{Q}^1|k, -N+2n+1>\neq 0,$$

and then for any integer $L \geq 0$, there exists such a state $|k, N>\neq 0$, $N \geq L+1$ that

$$\hat{Q}^1|k, N>\equiv |k, N+1>\neq 0.$$

Since the one-particle subspace $\mathcal{V}^{(1)}$ of an arbitrary L-stage reducible gauge theory is restricted to the states $|k, N>$, $|N| \leq L+1$, the above inequality $|k, N+1>\neq 0$, $N \geq L+1$ does not hold, and therefore the assumption of the absence of a BRST- (antiBRST-) singlet pair proves to be incorrect.

The above considerations imply that the variety of states that belong to the BRST–antiBRST-sextet complexes contain all the states (TC), (i); at the same time, the sextet representations (4.11.69), (4.11.70) generally include part of the states (TC), (ii), (iii), that is to say

$$(|\phi_1>, \ \ |\phi_2>) \neq \hat{Q}^a|*>, \tag{4.11.73}$$
$$\hat{Q}^1|\phi_1>= -\hat{Q}^2|\phi_2>\neq 0, \ \ \hat{Q}^2|\phi_1>= \hat{Q}^1|\phi_2>= 0.$$

Reversely, any states (4.11.73) belong to a BRST–antiBRST-sextet

$$|\phi>, \ \hat{Q}^1|\phi>, \ \hat{Q}^2|\phi>, \ |\phi_1>, \ |\phi_2>, \ \hat{Q}^1|\phi_1>= -\hat{Q}^2|\phi_2>, \tag{4.11.74}$$

where $|\phi>$ is chosen from the relations

$$< \hat{Q}^1\phi_1|\phi>= -< \hat{Q}^2\phi_2|\phi>= 1.$$

BRST- and antiBRST-quartets

The above treatment implies that for the further analysis of representations of the algebra (4.11.43) which contain the state vectors specified by the conditions (4.11.47), (4.11.62), (4.11.64), (TC), (ii), (iii) it is sufficient to confine ourselves to the states not repreresentable as linear combinations of BRST–antiBRST-sextet vectors. These states are without the loss of generality all orthogonal to the representation subspace containing the variety of BRST–antiBRST-sextet complexes and, therefore, belong to a subspace that is invariant under the action of the operators $\hat{L}$ and does not contain any BRST–antiBRST-sextet elements. For the states

$$|\phi>\neq \hat{Q}^a|*>,\ \hat{Q}^1|\phi>\neq 0,\ \hat{Q}^2|\phi>=0, \tag{4.11.75}$$

$$|\bar{\phi}>\neq \hat{Q}^a|*>,\ \hat{Q}^2|\bar{\phi}>\neq 0,\ \hat{Q}^1|\bar{\phi}>=0 \tag{4.11.76}$$

under consideration, the following supplementary conditions hold

$$\hat{Q}^1|\phi>\neq \hat{Q}^2|*>, \tag{4.11.77}$$

$$\hat{Q}^2|\bar{\phi}>\neq \hat{Q}^1|*>\ . \tag{4.11.78}$$

Let us show that the violation of Eq. (4.11.77), for instance, leads one to a contradiction. In fact, $|*>$ is, by definition, not representable as a linear combination of BRST–antiBRST-sextet states, and, consequently, the relation $\hat{Q}^1|\phi>=\hat{Q}^2|*>$ is only possible when $|*>$ belongs to the states (4.11.76), i.e. without the loss of generality, one has

$$\hat{Q}^1|\phi>=-\hat{Q}^2|\bar{\phi}>\ .$$

From the above relation it follows, by virtue of Eqs. (4.11.73)–(4.11.76), that the states $(|\phi>,\ |\bar{\phi}>,\ \hat{Q}^1|\phi>=-\hat{Q}^2|\bar{\phi}>)$ belong to some BRST–antiBRST-sextet (4.11.74). The inequality (4.11.78) is proved in a similar way. Eqs. (4.11.77), (4.11.78) imply, in particular, that the representation subspaces (4.11.75) and (4.11.76) respectively cannot be transformed into each another by the action of the operators $\hat{L}$.

By repetition of the given above reasoning with respect to Eqs. (4.11.75)–(4.11.78), we find that the state complexes of the form (4.11.75), (4.11.77) constitute some BRST-quartets (OPSC), (iii)

$$|\phi>,\ \ |\phi'>,\ \ \hat{Q}^1|\phi>,\ \ \hat{Q}^1|\phi'>,$$

$$<\phi'|\hat{Q}^1\phi>=<\hat{Q}^1\phi'|\phi>=1, \tag{4.11.79}$$

$$|\Phi>\equiv(|\phi>,\ |\phi'>),$$

$$\hat{Q}^2|\Phi>=0,\ |\Phi>\neq \hat{Q}^a|*>,\ \hat{Q}^1|\Phi>\neq \hat{Q}^2|*>$$

($|\phi>$ (4.11.79) is orthogonal to all the BRST–antiBRST-sextet states and, in particular, to any state $|\psi>$: $\forall a,\ \hat{Q}^a|\psi>\neq 0$; hence, $|\phi'>$ also satisfies Eqs. (4.11.75), (4.11.77)), representable as well in the form of two mutually orthogonal pair of antiBRST-singlets

$$(|\phi>,\ \hat{Q}^1|\phi'>),\ \ (|\phi'>,\ \hat{Q}^1|\phi>), \tag{4.11.80}$$

which we shall, as usual, identify with some basis state vectors, making them orthogonal to the remaining elements of the basis. Similarly, the states (4.11.76), (4.11.78), also considered as basis state vectors, constitute some antiBRST-quartets (3.4), (iv)

$$|\bar{\phi}>,\ \ |\bar{\phi}'>,\ \ i\hat{Q}^2|\bar{\phi}>,\ \ i\hat{Q}^2|\bar{\phi}'>,$$

$$<\bar{\phi}'|i\hat{Q}^2\bar{\phi}>=<i\hat{Q}^2\bar{\phi}'|\bar{\phi}>=1, \tag{4.11.81}$$

$$|\overline{\Phi}>\equiv(|\bar{\phi}>,\ |\bar{\phi}'>),$$
$$\hat{Q}^1|\overline{\Phi}>=0,\ |\overline{\Phi}>\neq\hat{Q}^a|*>,\ \hat{Q}^2|\overline{\Phi}>\neq\hat{Q}^1|*>$$

and BRST-singlet pairs

$$(|\bar{\phi}>,\ i\hat{Q}^2|\bar{\phi}'>),\ \ (|\bar{\phi}'>,\ i\hat{Q}^2|\bar{\phi}>),\qquad(4.11.82)$$

orthogonal both to one another and to the rest of the basis states. By construction, the state complexes (4.11.79) and (4.11.81) form bases of some representation subspaces with non-degenerate inner product.

Thus, with allowance for Eqs. (4.11.65)–(4.11.72), (4.11.75)–(4.11.82), we have described the structure of representations containing states of the form (4.11.47), (4.11.62), (4.11.64).

BRST–antiBRST-singlets

Finally, we turn to the states $|\Phi>$ that do not belong to the representation subspace constituted by the above considered BRST–antiBRST-quartets, sextets, octets and state complexes (4.11.75)–(4.11.78) (the subspace of states $|\Phi>$ is without the loss of generality orthogonal to all the state complexes just mentioned and is therefore invariant under the action of the operators $\hat{L}$). One readily finds that these restrictions can only be met by the states

$$|\Phi>\equiv\{|\phi_{(k,N)}>\}\neq\hat{Q}^a|*>$$

of the form (4.11.72) $\forall a:\ \hat{Q}^a|\Phi>=0$. Proceeding along the lines similar to the above presented ones, we firstly single out the states $|\Phi>\equiv(|\phi_{(k,-N)}>,\ |\phi_{(k,N)}>,\ N\neq 0)$

$$<\phi_{(k,-N)}|\phi_{(k,N)}>=1,\ \ <\phi_{(k,-N)}|\phi_{(k,-N)}>=<\phi_{(k,N)}|\phi_{(k,N)}>=0\qquad(4.11.83)$$

and the states $|\Phi>\equiv(|\phi_k>,\ |\bar{\phi}_k>,\ N=0)$

$$<\bar{\phi}_k|\phi_k>=1,\ \ <\phi_k|\phi_k>=0,\ \ ,<\bar{\phi}_k|\bar{\phi}_k><\ 0\qquad(4.11.84)$$

which we shall call BRST–antiBRST-singlet pairs (OPSC), (ii).

Taking into account that the state vectors (4.11.84) contain all the zero-norm states $|\Phi>,\ i\hat{Q}_C|\Phi>=0$ under consideration, one readily establishes the fact that the subspace of the remaining states (if any) must possess definite inner product. Indeed, assuming the reverse, i.e. that there exist at least two states $|k>,\ |\bar{k}>$ with the properties $<k|k>=1$, $<\bar{k}|\bar{k}>=-1$, we can easily find such $\alpha\neq 0$ that $|l>=\alpha|k>+|\bar{k}>,\ <l|l>=0$.

In this connection, to have a physically meaningful theory, we *require* that the subspace of the remaining states be positive definite and identify these states with physical particles $|\phi_k>$ (genuine BRST–antiBRST-singlets (OPSC), (i))

$$<\phi_k|\phi_k>=1,\ \ \hat{Q}^a|\phi_k>=0,\ \ |\phi_k>\neq\hat{Q}^a|*>.\qquad(4.11.85)$$

At the same time, it is clear that the state complexes (4.11.83), (4.11.84) and (4.11.85) are orthogonal to each other.

Thus, taking Eqs. (4.11.48)–(4.11.85) into account, we have in a general case described the structure (OPSC) of the one-particle state subspace $\mathcal{V}^{(1)}\supset\mathcal{V}^{(1)}_n$, $(\mathcal{V}^{(1)}_n\perp\mathcal{V}^{(1)}_{n'},\ \ n\neq n')$ as a space of the representation $\hat{L}\mathcal{V}\subset\mathcal{V},\ \ \hat{L}=(\hat{Q}^a,\ i\hat{Q}_C)$ of algebra (4.11.43) of the generators $\hat{Q}^a$ of extended BRST symmetry transformations and the ghost charge operator $i\hat{Q}_C$. By construction, indefinite inner product $<\ |\ >$ is non-degenerate in each subspace $\mathcal{V}^{(1)}_n$ (see the normalization conditions (4.11.52), (4.11.67), (4.11.79), (4.11.81), (4.11.83), (4.11.84), (4.11.85)), while they themselves have no elements in common $(\mathcal{V}^{(1)}_n\bigcap\mathcal{V}^{(1)}_{n'}=\emptyset,\ n\neq n')$ and form a direct sum (4.11.45) of representation subspaces.

Physical unitarity conditions

We now consider, with allowance for Eqs. (4.11.45), (OPSC), (4.11.48)–(4.11.85), the conditions of the physical S-matrix unitarity in the Hilbert space $H_{\mathrm{phys}} = \overline{\mathcal{V}_{\mathrm{phys}}/\mathcal{V}_0}$, where the physical subspace $\mathcal{V}_{\mathrm{phys}} \ni |\mathrm{phys}>$ is specified by the $Sp(2)$-covariant subsidiary condition

$$\hat{Q}^a|\mathrm{phys}>=0 \tag{4.11.86}$$

(which, clearly, ensures the invariance of $\mathcal{V}_{\mathrm{phys}}$ under the time development). By virtue of Eq. (4.11.86), the structure of $\mathcal{V}_{\mathrm{phys}}$ has the form

$$\mathcal{V}_{\mathrm{phys}} = \mathcal{V}^1_{\mathrm{phys}} \bigcap \mathcal{V}^2_{\mathrm{phys}},$$

where

$$\mathcal{V} \supset \mathcal{V}^1_{\mathrm{phys}},\;\; \hat{Q}^1\mathcal{V}^1_{\mathrm{phys}} = 0,$$
$$\mathcal{V} \supset \mathcal{V}^2_{\mathrm{phys}},\;\; \hat{Q}^2\mathcal{V}^2_{\mathrm{phys}} = 0.$$

In particular, for the zero-norm subspace $\mathcal{V}_0 \subset \mathcal{V}$ we have

$$\mathcal{V}_0 = \mathcal{V}^1_0 \bigcap \mathcal{V}^2_0, \qquad \mathcal{V}^1_0 \subset \mathcal{V}^1_{\mathrm{phys}},\;\; \mathcal{V}^2_0 \subset \mathcal{V}^2_{\mathrm{phys}}. \tag{4.11.87}$$

The analysis of representations (4.11.45), (OPSC), (4.11.48)–(4.11.85) on the basis of the quartet mechanism [140] shows that if BRST- and antiBRST-singlet pairs are absent in the theory, then, firstly, the remaining BRST–antiBRST-singlets have positive-definite norm, and, secondly, the state vectors from $\mathcal{V}_{\mathrm{phys}}$ containing particles of BRST–antiBRST-quartets (OPSC), (v) and octets (OPSC), (vii) (i.e. state complexes simultaneously representable as BRST- (4.11.57), (4.11.60) and antiBRST- (4.11.58), (4.11.61) quartets) belong to the zero-norm subspace $\mathcal{V}_0$ (4.11.87). At the same time, the state complexes (OPSC), (ii), (iii), (iv), (vi) generally contain BRST- (antiBRST-) singlet pairs (4.11.71), (4.11.72), (4.11.80), (4.11.82), (4.11.83), (4.11.84). In this connection, the unitarity conditions (providing positive semi-defineteness of $<\,|\,>$ in $\mathcal{V}_{\mathrm{phys}}$) of physical S-matrix in H_{phys} is, within the suggested approach, the requirement of absence of the pointed out state complexes, i.e. BRST–antiBRST-singlet pairs (OPSC), (ii), BRST-quartets (OPSC), (iii) (antiBRST-singlet pairs (4.11.80)), antiBRST-quartets (OPSC), (iv) (BRST-singlet pairs (4.11.82)) and BRST–antiBRST-sextets (OPSC), (vi).

4.12 Gauge dependence of Green's functions

First, consider an infinitesimal variation of the gauge functional $F \to F + \delta F$. It leads to a small variation of the action S_{ext}. This variation can be represented in terms of the functional δF depending only on the fields ϕ^A, in the form

$$\delta\bigg(\exp\left\{\frac{i}{\hbar}S_{ext}\right\}\bigg) = -i\hbar\hat{T}(\delta F)\exp\left\{\frac{i}{\hbar}S_{ext}\right\}$$

Then we have

$$\delta \mathcal{Z}(J,\phi^*,\bar{\phi}) = \frac{i\hbar}{2}\varepsilon_{ab}\int d\phi \exp\left\{\frac{i}{\hbar}J_A\phi^A\right\}\bar{\Delta}^a\bar{\Delta}^b\delta F \exp\left\{\frac{i}{\hbar}S_{ext}\right\}.$$

Performing two subsequent integration by parts in the above integral, we obtain

$$\delta \mathcal{Z}(J,\phi^*,\bar{\phi}) = \frac{i}{2\hbar}\varepsilon_{ab}\widehat{\omega}^b\widehat{\omega}^a\widehat{\delta F}\mathcal{Z}(J,\phi^*,\bar{\phi}), \tag{4.12.88}$$

where

$$\widehat{\delta F} \equiv \delta F\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right). \tag{4.12.89}$$

Therefore, the variation of the generating functional $\mathcal{W}$ of connected Green's functions has the form

$$\delta\mathcal{W} = \frac{i}{2}\varepsilon_{ab}\widehat{\omega}^b\widehat{\omega}^a\langle\widehat{\delta F}\rangle, \tag{4.12.90}$$

where

$$\langle\widehat{\delta F}\rangle \equiv \delta F\left(\frac{\hbar}{i}\frac{\delta}{\delta J}+\frac{\delta\mathcal{W}}{\delta J}\right)$$

is the vacuum expectation value of the operator $\widehat{\delta F}$.

For the generating functional of vertex functions $\Gamma = \Gamma(\phi,\phi^*,\bar{\phi})$ this results in

$$\delta\Gamma = \frac{i}{2}\varepsilon_{ab}\,\hat{\mathcal{B}}^b\,\hat{\mathcal{B}}^a\,\langle\langle\widehat{\delta F}\rangle\rangle, \tag{4.12.91}$$

where

$$\hat{\mathcal{B}}^a(\Gamma) = (\Gamma,\cdot)^a + V^a \equiv \frac{\delta\Gamma}{\delta\phi^A}\frac{\delta}{\delta\phi^*_{Aa}} + (-1)^{\varepsilon_A}\frac{\delta\Gamma}{\delta\phi^*_{Aa}}\frac{\delta_l}{\delta\phi^A} + V^a, \quad \hat{\mathcal{B}}^{\{a}\hat{\mathcal{B}}^{b\}} = 0, \tag{4.12.92}$$

$$\langle\langle\widehat{\delta F}\rangle\rangle = \delta F(\widehat{\phi}), \quad \widehat{\phi}^A = \phi^A + i\hbar(\Gamma''^{-1})^{AB}\frac{\delta_l}{\delta\phi^B}, \tag{4.12.93}$$

$$(\Gamma''^{-1})^{AC}(\Gamma'')_{CB} = \delta^A_B, \quad (\Gamma'')_{AB} = \frac{\delta_l}{\delta\phi^A}\left(\frac{\delta\Gamma}{\delta\phi^B}\right). \tag{4.12.94}$$

The gauge dependence of Γ can also be presented in the form [143]

$$\delta\Gamma(\phi,\phi^*,\bar{\phi}) = \frac{\delta\Gamma}{\delta\phi^A}G^A(\phi,\phi^*,\bar{\phi}) + \phi^*_{Aa}D^{Aa}(\phi,\phi^*,\bar{\phi}) \tag{4.12.95}$$

with some functionals G^A and D^{Aa}.

We can see that the generating functional of vertices, calculated on its extremals $\delta\Gamma/\delta\phi^A = 0$ does not depend on the gauge on the surface $\phi^*_{Aa} = 0$.

4.13 Irreducible gauge theories with a closed algebra

To illustrate the formalism, we consider irreducible gauge theories of so-called rank 1 with a closed algebra. Such theories are characterized by the fact that in the algebra of generators we have $M^{ij}_{\alpha\beta} = 0$, and the solution of any equation of the form $R^i_\alpha X^\alpha = 0$ is $X^\alpha = 0$. The majority of theories discussed in the literature belong to the indicated class (Yang - Mills, gravity, supergravity with auxiliary fields, ets.). From the viewpoint of extended BRST quantization, for all these theories the solution of extended master equantion exists as a linear functional in the antifields ϕ^*_{Aa} and $\bar{\phi}_A$

$$S(\phi,\ \phi^*,\ \bar{\phi}) = S_0(A) + \phi^*_{Aa}X^{Aa} + \bar{\phi}_A Y^A \qquad (4.13.96)$$

where X^{Aa} ($\varepsilon(X^{Aa}) = \varepsilon(\phi^*_{Aa})$) and Y^A ($\varepsilon(Y^A) = \varepsilon(\bar{\phi}_A)$) are functionals of the fields ϕ^A and have the sense of extended BRST- and mixed BRST-transformations respectively.

Substituting the functional (4.13.96) into the extended master equations, we obtain a system of equations for finding X^{Aa} and Y^A:

$$\frac{\delta S_0(A)}{\delta\phi^A}X^{Aa} = 0,$$

$$\frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb} + \frac{\delta X^{Ab}}{\delta\phi^B}X^{Ba} = 0,$$

$$Y^A = \frac{1}{2}\varepsilon_{ab}\frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb}$$

$$\frac{\delta Y^A}{\delta\phi^B}X^{Ba} = 0.$$

Let

$$X^{Aa} = (X_1^{ia},\ X_2^{\alpha a},\ X_3^{\alpha ab}),\quad Y^A = (Y_1^i,\ Y_2^\alpha,\ Y_3^{\alpha a}).$$

Then we obtain for solutions to the above equations

$$\begin{aligned}
&X_1^{ia} = R^i_\alpha C^{\alpha a},\\
&X_2^{\alpha a} = -\frac{1}{2}F^\alpha_{\gamma\beta}B^\beta C^{\gamma a} - \frac{1}{12}(-1)^{\varepsilon_\beta}(2F^\alpha_{\gamma\beta,j}R^j_\delta + F^\alpha_{\gamma\sigma}F^\sigma_{\beta\delta})C^{\delta b}C^{\beta a}C^{\gamma c}\varepsilon_{cb},\\
&X_3^{\alpha ab} = -\varepsilon^{ab}B^\alpha - \frac{1}{2}(-1)^{\varepsilon_\beta}F^\alpha_{\beta\gamma}C^{\gamma b}C^{\beta a},\\
&Y_1^i = R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha}R^i_{\alpha,j}R^j_\beta C^{\beta b}C^{\alpha a}\varepsilon_{ab},\\
&Y_2^a = 0,\ Y_3^{\alpha a} = -2X_2^{\alpha a}.
\end{aligned}$$

and the closed form of action $S = S(\phi, \phi^*, \bar{\phi})$ for any irreducible gauge theory of rank 1 with a closed algebra and initial action $S_0(A)$

$$\begin{aligned}
S \quad = \quad & S_0(A) + A^*_{ia}R^i_\alpha C^{\alpha a} - \frac{1}{2}B^*_{\alpha a}F^\alpha_{\gamma\beta}B^\beta C^{\gamma a} - \\
& -\frac{1}{12}(-1)^{\varepsilon_\beta}B^*_{\alpha a}(2F^\alpha_{\gamma\beta,j}R^j_\delta + F^\alpha_{\gamma\sigma}F^\sigma_{\beta\delta})C^{\delta b}C^{\beta a}C^{\gamma c}\varepsilon_{cb} - \varepsilon^{ab}C^*_{\alpha ab}B^\alpha - \\
& -\frac{1}{2}(-1)^{\varepsilon_\beta}C^*_{\alpha ab}F^\alpha_{\beta\gamma}C^{\gamma b}C^{\beta a} + \bar{A}_i R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha}\bar{A}_i R^i_{\alpha,j}R^j_\beta C^{\beta b}C^{\alpha a}\varepsilon_{ab} + \\
& +\bar{C}_{\alpha a}F^\alpha_{\gamma\beta}B^\beta C^{\gamma a} + +\frac{1}{6}(-1)^{\varepsilon_\beta}\bar{C}_{\alpha a}(2F^\alpha_{\gamma\beta,j}R^j_\delta + F^\alpha_{\gamma\sigma}F^\sigma_{\beta\delta})C^{\delta b}C^{\beta a}C^{\gamma c}\varepsilon_{cb}.
\end{aligned} \qquad (4.13.97)$$

Note that the existence of solutions of the extended classical master equations

$$\frac{1}{2}(S,S)^a + V^a S = 0$$

(for both irreducible and reducible gauge theories!) in the form (4.13.96) can be expressed in terms of global symmetries of action S [99]. Indeed, let us introduce the set of operators $\mathsf{s}^a(S)$

$$\mathsf{s}^a(S) = (-1)^{\varepsilon_A} \frac{\delta S}{\delta \phi^*_{Aa}} \frac{\delta_l}{\delta \phi^A} + V^a. \tag{4.13.98}$$

It is not very difficult to find the algebra of these operators

$$\{\mathsf{s}^a(S), \mathsf{s}^b(S)\} = 0$$

and to see that the action S is invariant under the following global supertransformations (extended BRST transformations):

$$\delta\phi^A = -\frac{\delta S}{\delta \phi^*_{Aa}}\mu_a = -\mathsf{s}^a(S)\phi^A \mu_a, \quad \delta\phi^*_{Aa} = 0, \quad \delta\bar{\phi}_A = \mu_a \varepsilon^{ab}\phi^*_{Ab} = \mu_a \mathsf{s}^a(S)\bar{\phi}_A.$$

Therefore, the operators $\mathsf{s}^a(S)$ should be considered as the symmetry operators of S

$$\mathsf{s}^a(S)S = 0.$$

Now, we shall show that in the class of gauges $F(\phi)$ depending only on the initial fields A^i

$$F(\phi) = F(A)$$

the generating functional is reduced to the standard FP result. Indeed, the integration over variables $\bar{\phi}_A$, ϕ^*_{Aa}, λ^A and π^{Aa} is trivial and yields

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_0(A) - \frac{1}{2}\varepsilon_{ab}X_1^{ia}\frac{\delta^2 F}{\delta A^i \delta A^j}X_1^{jb} + \frac{\delta F}{\delta A^i}Y_1^i + J_A\phi^A\right]\right\}.$$

Taking into account the explicit expressions for X_1^{ia} and Y_1^i, we come to

$$\frac{\delta F}{\delta A^i}Y_1^i - \frac{1}{2}\varepsilon_{ab}X_1^{ia}\frac{\delta^2 F}{\delta A^i \delta A^j}X_1^{jb} =$$
$$\frac{\delta F}{\delta A^i}R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha}\left(\frac{\delta F}{\delta A^i}R^i_{\alpha,j}R^j_\beta + \frac{\delta^2 F}{\delta A^i \delta A^j}R^j_\alpha R^i_\beta(-1)^{\varepsilon_i(\varepsilon_j+\varepsilon_\alpha)}\right)C^{\beta b}C^{\alpha a}\varepsilon_{ab}.$$

If we introduce the functions

$$\chi_\alpha = \frac{\delta F}{\delta A^i}R^i_\alpha$$

and identify $C^{\alpha 1} \equiv C^\alpha$ and $C^{\alpha 2} \equiv \bar{C}^\alpha$, then the functional integral can finally be written as

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}[S_0(A) + \bar{C}^\alpha \chi_{\alpha,i}R^i_\beta C^\beta + \chi_\alpha B^\beta + J_A\phi^A]\right\}. \tag{4.13.99}$$

Eq. (4.13.99) is the standard FP result for gauge theories with closed algebra when the gauge is introduced by means of the functional χ_α.

In the end of this Chapter we would like to list some problems considered in connection with the $Sp(2)$-covariant quantization. A geometric interpretation of the scheme was given in [123, 110, 201]. Reformulations based on the Schwinger-Dyson extended symmetry were presented in [68, 69]. Geometry underlying the $Sp(2)$ method was studied in [160]. Quantum Sp(2) antibrackets were introduced and studied in [34].

Chapter 5

Triplectic Quantization

In the $Sp(2)$ covariant approach one introduces a configuration space of fields ϕ^A (4.1.1). In addition one needs to introduce to each ϕ^A three kinds of antifields: $Sp(2)$-doublet $\phi^*_{Aa}, a = 1,2$ and $Sp(2)$-singlet $\bar{\phi}_A$. These three kinds of antifields are involved in the $Sp(2)$-method in a nonsymmetrical way. While the antifields ϕ^*_{Aa} are anticanonically conjugate with respect to the antibrackets (4.2.2), the antifields $\bar{\phi}_A$ have no corresponding conjugated fields. We have seen that in order to present a gauge fixing procedure of the $Sp(2)$-formalism in the form of a functional integral (4.6.21) explicitly, one needs to make use of auxiliary fields π^{Aa} to parametrize the differential operator containing the gauge functional F.

The main idea of the triplectic quantization proposed by Batalin, Marnelius and Semikhatov [28, 35, 29] was to consider fields π^{Aa} as anticanonical partners to the antifields $\bar{\phi}_A$ in the usual sense.

5.1 Configuration space and extended antibrackets

The starting point of triplectic quantization is the configuration space of fields $\phi^A,\ \epsilon(\phi^A) \equiv \epsilon_A$ which coincides with configuration space of $Sp(2)$ - method. Then to each of ϕ^A one introduces a pair of antifields $\phi^*_{Aa},\ a = 1,2$ with opposite statistics $\epsilon(\phi^*_{Aa}) = \epsilon_A + 1$. Next one introduces a set of pairs of fields $\pi^{Aa},\ \epsilon(\pi^{Aa}) \equiv \epsilon_A + 1$. On the space of variables introduced above one defines an extended antibrackets by the rule

$$(F,G)^a \equiv \left(\frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \phi^*_{Aa}} + \epsilon^{ab} \frac{\delta F}{\delta \pi^{Ab}} \frac{\delta G}{\delta \bar{\phi}_A} \right) - (F \leftrightarrow G)\ (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}. \tag{5.1.1}$$

The extended antibrackets have the properties which formally coincide with properties of extended antibrackets within the $Sp(2)$ formalism (4.2.3).

5.2 Operators $V^a,\ \Delta^a$

The operators $V^a,\ \Delta^a$ are introduced

$$V^a = \frac{1}{2}\left(\varepsilon^{ab}\ \phi^*_{Ab} \frac{\delta}{\delta \bar{\phi}_A} - (-1)^{\epsilon_A} \pi^{Aa} \frac{\delta_l}{\delta \phi^A} \right), \tag{5.2.2}$$

$$\Delta^a = (-1)^{\varepsilon_A} \frac{\delta_l}{\delta\phi^A} \frac{\delta}{\delta\phi^*_{Aa}} + (-1)^{\varepsilon_A+1} \epsilon^{ab} \frac{\delta_l}{\delta\pi^{Aa}} \frac{\delta}{\delta\bar{\phi}_A}. \quad (5.2.3)$$

It can be readily established that the algebra of operators (5.2.2), (5.2.3) has the form

$$V^{\{a}V^{b\}} = 0, \ \Delta^{\{a}\Delta^{b\}} = 0, \quad (5.2.4)$$

$$\Delta^a V^b + V^b \Delta^a = 0. \quad (5.2.5)$$

The action of the operators Δ^a ((5.2.3)) on a product of functionals F and G gives

$$\Delta^a(F \cdot G) = (\Delta^a F) \cdot G + F \cdot (\Delta^a G)(-1)^{\varepsilon(F)} + (F,\ G)^a(-1)^{\varepsilon(F)}. \quad (5.2.6)$$

Eq.(5.2.6) may be considered as an alternative definition of the extended antibracket (5.1.1). The action of the operators V^a (5.2.2) upon the extended antibrackets is given by the relations

$$V^a(F,G)^b = (V^a F,\ G)^b - (-1)^{\varepsilon(F)}(F, V^a G)^b. \quad (5.2.7)$$

Note that definition of the operators V^a (5.2.2) differs from the $Sp(2)$ one (see Eq.(4.3.4)). As a consequence, formulas (5.2.5) and (5.2.7) are valid without symmetrizazion in the indices a and b in comparison with the $Sp(2)$ formalism (see, (4.3.6), (4.3.8)).

It is usefully to introduce an operator $\bar{\Delta}^a$

$$\bar{\Delta}^a = \Delta^a + \frac{i}{\hbar} V^a \quad (5.2.8)$$

with the properties

$$\bar{\Delta}^{\{a}\bar{\Delta}^{b\}} = 0. \quad (5.2.9)$$

5.3 Vacuum functional

The vacuum functional in this approach is defined by the rule

$$Z(0) = \int d\phi d\phi^* d\pi d\bar{\phi} d\lambda \exp\left\{\frac{i}{\hbar}(S + X)\right\} \quad (5.3.10)$$

where boson functional $S = S(\phi, \phi^*, \pi, \bar{\phi}; \hbar)$ satisfies the following master equations

$$\bar{\Delta}^a \exp\left\{\frac{i}{\hbar} S\right\} = 0. \quad (5.3.11)$$

or, equivalently,

$$\frac{1}{2}(S,\ S)^a + V^a S = i\hbar\Delta^a S, \quad (5.3.12)$$

and boson functional $X = X(\phi, \phi^*, \pi, \bar{\phi}, \lambda; \hbar)$ is a hypergauge fixing action depending on new variables λ^A, $\epsilon(\lambda^A) = \epsilon_A$ and satisfing the following quantum equations:

$$\frac{1}{2}(X,\ X)^a - V^a X = i\hbar\Delta^a X, \quad (5.3.13)$$

which differs from Eq.(5.3.12) by the opposite sign of the V-term. It is expected that the classical part of the gauge fixing functional has the form

$$X|_{\hbar=0} = G_A \lambda^A + \mathcal{K} Y \quad (5.3.14)$$

where G_A and Y are functions and $\mathcal{K}$ is the differential operator

$$\mathcal{K} = \epsilon_{ab} V^a V^b. \quad (5.3.15)$$

5.4 Extended BRST symmetry

The vacuum functional (5.3.10) possesses an important property of invariance under the following global transformations

$$\begin{aligned} \delta\phi^A &= \Big(-\frac{\delta S}{\delta\phi^*_{Aa}} + \frac{\delta X}{\delta\phi^*_{Aa}} + \pi^{Aa}\Big)\mu_a, \\ \delta\phi^*_{Aa} &= \mu_a\Big(\frac{\delta S}{\delta\phi^A} - \frac{\delta X}{\delta\phi^A}\Big), \\ \delta\bar{\phi}_A &= \mu_a\varepsilon^{ab}\Big(\frac{\delta S}{\delta\pi^{Ab}} - \frac{\delta X}{\delta\pi^{Ab}} + \phi^*_{Ab}\Big), \\ \delta\pi^{Ab} &= \epsilon^{ab}\Big(-\frac{\delta S}{\delta\bar{\phi}_A} + \frac{\delta X}{\delta\bar{\phi}_A}\Big)\mu_a, \\ \delta\lambda^A &= 0, \end{aligned} \qquad (5.4.16)$$

where μ_a is an Sp(2) doublet of constant anticommuting Grassmann parameters. These transformations realize in the triplectic quantization the extended BRST transformations in the space of the variables ϕ, ϕ^*, $\bar{\phi}$, π^a and λ.

The transformations (5.4.16) can be presented in condensed notation

$$\delta\mathcal{G} = (\mathcal{G},\ -S + X)^a\mu_a + 2\mu_a V^a\mathcal{G}, \qquad (5.4.17)$$

where $\mathcal{G}$ denotes the complete set of variables.

5.5 Gauge independence

If we consider the transformations (5.4.16) with μ_a dependind on $\mathcal{G}$ and λ it is not difficult to obtain the following representation for vacuum functional

$$Z(0) = \int d\phi d\phi^* d\pi d\bar{\phi} d\lambda \exp\Big\{\frac{i}{\hbar}\Big[S + X - i\hbar(\mu_a, S)^a + i\hbar(\mu_a, X)^a + 2i\hbar V^a\mu_a\Big]\Big\} \qquad (5.5.18)$$

Let us make an additional change of variables in the integral (5.5.18)

$$\delta\mathcal{G} = \frac{1}{2}(\mathcal{G},\ \delta F_a)^a. \qquad (5.5.19)$$

This change gives

$$\begin{aligned} Z(0) \;=\; & \int d\phi d\phi^* d\pi d\bar{\phi} d\lambda \exp\Big\{\frac{i}{\hbar}[S + X - i\hbar(\mu_a, S)^a + i\hbar(\mu_a, X)^a + \\ & +2i\hbar V^a\mu_a + \frac{1}{2}(S,\ \delta F_a)^a + \frac{1}{2}(X,\ \delta F_a)^a - i\hbar\Delta^a\delta F_a]\Big\} \end{aligned} \qquad (5.5.20)$$

If we identify

$$\delta F_a(\mathcal{G}) \equiv \frac{2\hbar}{i}\mu_a(\mathcal{G},\ \lambda) \qquad (5.5.21)$$

then we have

$$Z(0) = \int d\phi d\phi^* d\pi d\bar{\phi} d\lambda \exp\Big\{\frac{i}{\hbar}\Big[S + X + \delta X\Big]\Big\} \qquad (5.5.22)$$

where notation has been introduced

$$\delta X = (X,\ \delta F_a)^a - V^a \delta F_a - i\hbar\Delta^a \delta F_a. \tag{5.5.23}$$

One can now check (for detailes, see [28])

$$(X,\ \delta X)^a - V^a \delta X = i\hbar\Delta^a \delta X \tag{5.5.24}$$

provided δF_a is chosen to have the following form

$$\delta F_a = \epsilon_{ab}\Big\{(X,\ \delta Y)^b - V^b \delta Y - i\hbar\Delta^b \delta Y\Big\}. \tag{5.5.25}$$

On the other hand, any small admissible variation of hypergauge fixing action δX in Eq.(5.3.10) has to satisfy Eqs.(5.5.24). It means that one can compensate for a variation of hypergauge fixing action in vacuum functional by suitable choice of δF_a in (5.5.19) (or δY in (5.5.25)). Therefore vacuum functional (5.3.10) does not depend on the gauge.

5.6 Modified triplectic quantization

Notice that the essential original point of the triplectic quantization consists in dividing the entire task of constructing the quantum effective action into the following two steps: first, the construction of the quantum action S, and second the construction of the corresponding gauge-fixing functional. Either problem is solved by means of an appropriate master equation.

Despite considering these new ideas as very promising, as to their concrete realization there exists [96] a different, modified scheme of the triplectic quantuzation, which – especially from some geometrical viewpoint – changes the meaning of the latter. Namely, remaining in the same configuration space of fields, and accepting the idea of a separate treatment of the two above mentioned actions, one proposes to change both systems of master equations by using a new set of two $Sp(2)$-doublets of generating operators: V^a and U^a. Such a modification is inspired by the experience of the superfield formulation of the $Sp(2)$ method [144] (see Chapter 7), in which the above operators acquire the geometrical interpretation of the generators of (super)transformations in a superspace spanned by fields and anti–fields. In this approach, the first master equation, determining the quantum action S is defined by means of the operators V^a, whereas the other master equation determining the gauge fixing functional X, is defined by means of the operators U^a. As in the original triplectic quantization, we may expect that the generating functional of Green's functions does not depend on the choice of gauge. It is important to emphasize that within the modified triplectic quantization the entire information contained in the initial classical action of the theory is conveyed to the quantum effective action via the corresponding boundary conditions. At the same time, the classical action obeys the first modified master equation in complete analogy with all previously known schemes of Lagrangian quantization. The original triplectic quantization gives no explicit relation to the initial classical action. If one assumes that such a classical action occurs, as usual, in the boundary condition to the solution of the master equation (with vanishing auxiliary fields and quantum corrections), then this classical action does not obey the master equation.

Using the same definitions of the extended antibrackets (5.1.1) and the operators Δ^a (5.2.3), let us introduce the following set of operators V^a and U^a:

$$V^a = \varepsilon^{ab}\phi^*_{Ab}\frac{\delta}{\delta\bar{\phi}_A}\ , \tag{5.6.26}$$

$$U^a = (-1)^{\varepsilon_A+1}\pi^{Aa}\frac{\delta_l}{\delta\phi^A}\ . \tag{5.6.27}$$

Notice that the operators V^a in Eq.(5.6.26) differ from the corresponding operators of the triplectic quantization (5.2.2), they coincide, at the same time, with the operators applied in the framework of the $Sp(2)$ method (4.3.4). The use of the operators U^a in Eq.(5.6.27) exhibits an essentially new feature as compared to both the $Sp(2)$ method and the triplectic quantization in its original version [28].

One easily establishes the following algebra of the operators (5.2.3), (5.6.26), (5.6.27):

$$\begin{aligned}
&\Delta^{\{a}\Delta^{b\}} = 0,\\
&V^{\{a}V^{b\}} = 0,\\
&\Delta^{\{a}V^{b\}} + V^{\{a}\Delta^{b\}} = 0,\\
&U^{\{a}U^{b\}} = 0,\\
&\Delta^{\{a}U^{b\}} + U^{\{a}\Delta^{b\}} = 0,\\
&V^aU^b + U^bV^a = 0,\\
&\Delta^aV^b + V^b\Delta^a + \Delta^aU^b + U^b\Delta^a = 0.
\end{aligned} \tag{5.6.28}$$

The action of the operators Δ^a (5.2.3) on the product of any two functionals F, G is given by Eq.(5.2.6). The action of each of the operators Δ^a, V^a and U^a (5.2.3), (5.6.26) and (5.6.27) on the extended antibrackets is given by the rule $(D^a = (\Delta^a, V^a, U^a))$

$$D^{\{a}(F,G)^{b\}} = (D^{\{a}F,G)^{b\}} - (F,D^{\{a}G)^{b\}}(-1)^{\varepsilon(F)}. \tag{5.6.29}$$

It is also useful to introduce the operators

$$\bar{\Delta}^a \equiv \Delta^a + \frac{i}{\hbar}V^a, \tag{5.6.30}$$

$$\tilde{\Delta}^a \equiv \Delta^a - \frac{i}{\hbar}U^a. \tag{5.6.31}$$

From Eqs.(5.6.28) it follows that the algebra of these operators has the form

$$\begin{aligned}
&\bar{\Delta}^{\{a}\bar{\Delta}^{b\}} = 0,\\
&\tilde{\Delta}^{\{a}\tilde{\Delta}^{b\}} = 0,\\
&\bar{\Delta}^{\{a}\tilde{\Delta}^{b\}} + \tilde{\Delta}^{\{a}\bar{\Delta}^{b\}} = 0.
\end{aligned} \tag{5.6.32}$$

Let us denote by $S = S(\phi, \phi^*, \pi, \bar{\phi})$ the quantum action, corresponding to the initial classical theory with the action S_0, and defined as a solution of the following master equations:

$$\frac{1}{2}(S,S)^a + V^aS = i\hbar\Delta^aS, \tag{5.6.33}$$

with the standard boundary condition

$$S|_{\phi^*=\bar{\phi}=\hbar=0} = S_0. \tag{5.6.34}$$

Eq.(5.6.33) can be represented in the equivalent form

$$\bar{\Delta}^a \exp\left\{\frac{i}{\hbar}S\right\} = 0. \tag{5.6.35}$$

Let us further define the vacuum functional as the following functional integral:

$$Z_X = \int d\phi\, d\phi^* d\pi\, d\bar{\phi}\, d\lambda \exp\left\{\frac{i}{\hbar}\left(S + X + \phi^*_{Aa}\pi^{Aa}\right)\right\}, \tag{5.6.36}$$

where $X = X(\phi, \phi^*, \pi, \bar{\phi}, \lambda)$ is a bosonic functional depending also on the new variables λ^A, $\varepsilon(\lambda) = \varepsilon_A$, which serve as gauge-fixing parameters. We require that the functional X satisfies the following master equation:

$$\frac{1}{2}(X, X)^a - U^a X = i\hbar\Delta^a X, \tag{5.6.37}$$

or, equivalently,

$$\tilde{\Delta}^a \exp\left\{\frac{i}{\hbar}X\right\} = 0. \tag{5.6.38}$$

Notice that the generating equations determining the quantum action S in Eq.(5.6.33) (or (5.6.35)) and the gauge-fixing functional X in Eq.(5.6.37) (or (5.6.38)) differ—along with the vacuum functional Z in Eq.(5.6.36)—from the corresponding definitions (5.3.12), (5.3.13), (5.3.10).

One can easily obtain the simplest solution of Eq. (5.6.37) (or Eq. (5.6.38)) determining the gauge-fixing functional X

$$\begin{aligned} X &= \left(\bar{\phi}_A - \frac{\delta F}{\delta\phi^A}\right)\lambda^A - \frac{1}{2}\varepsilon_{ab}U^a U^b F = \\ &= \left(\bar{\phi}_A - \frac{\delta F}{\delta\phi^A}\right)\lambda^A - \frac{1}{2}\varepsilon_{ab}\pi^{Aa}\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\pi^{Bb}, \end{aligned} \tag{5.6.39}$$

where $F = F(\phi)$ is a bosonic functional depending only on the fields ϕ^A. As a straightforward exercise, one makes sure that the functional X in Eq.(5.6.39) does satisfy Eq.(5.6.37). If we further demand that the quantum action S does not depend on the fields π^A, then the functional (5.6.36) with the gauge functional X in (5.6.39) becomes exactly the vacuum functional of the $Sp(2)$ quantization scheme (see (4.6.21).

Let us consider a number of properties inherent in the present scheme of triplectic quantization, i.e. modified according to Eq.(5.6.33)–(5.6.38). In the first place, the vacuum functional (5.6.36) is invariant under the following transformations:

$$\delta\mathcal{G} = (\mathcal{G}, -S + X)^a\mu_a + \mu_a(V^a + U^a)\mathcal{G}, \tag{5.6.40}$$

where μ_a is an $Sp(2)$ doublet of constant anticommuting parameters, and $\mathcal{G}$ stands for any of the variables ϕ, ϕ^*, π, $\bar{\phi}$. Eq.(5.6.40) defines the transformations of extended BRST symmetry, realized on the space of the variables ϕ, ϕ^*, π, $\bar{\phi}$. In the particular case, corresponding to the gauge-fixing boson chosen as in Eq.(5.6.39), we have

$$\delta\phi^A = -\left(\frac{\delta S}{\delta\phi^*_{Aa}} - \pi^{Aa}\right)\mu_a, \tag{5.6.41}$$

$$\delta\phi^*_{Aa} = \mu_a\left(\frac{\delta S}{\delta\phi^A} + \frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\lambda^B + \frac{1}{2}(-1)^{\varepsilon_A}\varepsilon_{bc}\pi^{Bb}\frac{\delta^3 F}{\delta\phi^B\delta\phi^A\delta\phi^C}\pi^{Cc}\right), \tag{5.6.42}$$

$$\delta\pi^{Aa} = \varepsilon^{ab}\left(\frac{\delta S}{\delta\bar{\phi}_A} - \lambda^A\right)\mu_b, \tag{5.6.43}$$

$$\delta\bar{\phi}_A = \mu_a\varepsilon^{ab}\left(\frac{\delta S}{\delta\pi^{Ab}} + \phi^*_{Ab}\right) + \mu_a\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\pi^{Ba}. \tag{5.6.44}$$

Consider now the question of gauge dependence in the case of the vacuum functional Z, Eq.(5.6.36). Any admissible variation δX should satisfy the equations

$$(X, \delta X)^a - U^a\delta X = i\hbar\Delta^a\delta X. \tag{5.6.45}$$

It is convenient to consider an $Sp(2)$ doublet of operators $\hat{S}^a(X)$, defined by the rule

$$(X,F)^a \equiv \hat{S}^a(X)\cdot F, \tag{5.6.46}$$

and possessing the properties

$$\hat{S}^{\{a}(X)\hat{S}^{b\}}(X) = \hat{S}^{\{a}\left(\frac{1}{2}(X,X)^{b\}}\right), \tag{5.6.47}$$

which follow from the generalized Jacobi identities for extended antibrackets. Eq.(5.6.45) can be, consequently, represented in the form

$$\hat{Q}^a(X)\delta X = 0, \tag{5.6.48}$$

where we have introduced an $Sp(2)$ doublet of operators $\hat{Q}^a$, defined by the rule

$$\hat{Q}^a(X) = \hat{S}^a(X) - i\hbar\tilde{\Delta}^a. \tag{5.6.49}$$

With allowance for Eq.(5.6.37) the operators $\hat{Q}^a$ form a set of nilpotent anticommuting operators, i.e.

$$\hat{Q}^{\{a}(X)\hat{Q}^{b\}}(X) = 0. \tag{5.6.50}$$

By virtue of Eq.(5.6.50), any bosonic functional of the form

$$\delta X = \frac{1}{2}\varepsilon_{ab}\hat{Q}^a(X)\hat{Q}^b(X)\delta Y, \tag{5.6.51}$$

with an arbitrary bosonic functional δY, is a solution of Eq.(5.6.47). Moreover, by analogy with the theorems proved in [27], one establishes the fact that any solution of Eq.(5.6.47)—vanishing when all the variables in δX are equal to zero—has the form (5.6.51), with a certain bosonic functional δY. In the particular case of the gauge functional X (5.6.39), its variation δX can be easily represented in the form of Eq.(5.6.51), i.e.

$$\delta X = -\frac{\delta(\delta F)}{\delta\phi^A}\lambda^A - \frac{1}{2}\varepsilon_{ab}\pi^{Aa}\frac{\delta^2(\delta F)}{\delta\phi^A\delta\phi^B}\pi^{Bb} = -\frac{1}{2}\varepsilon_{ab}\hat{Q}^a(X)\hat{Q}^b(X)\delta F \tag{5.6.52}$$

with $\delta Y = -\delta F$.

Let us denote by $Z_X \equiv Z$ the value of the vacuum functional (5.6.36) corresponding to the gauge condition chosen as a functional X.

In the vacuum functional $Z_{X+\delta X}$ we first make the change of variables (5.6.40), with $\mu_a = \mu_a(\mathcal{G},\lambda)$, and then, accompanying it with a subsequent change of variables

$$\delta\mathcal{G} = (\mathcal{G},\delta Y_a)^a, \quad \varepsilon(\delta Y_a) = 1, \tag{5.6.53}$$

with $\delta Y_a = -i\hbar\mu_a(\mathcal{G},\lambda)$, we arrive at

$$Z_{X+\delta X} = \int d\phi\, d\phi^* d\pi\, d\bar{\phi}\, d\lambda \exp\left\{\frac{i}{\hbar}\left(S + X + \delta X + \delta X_1 + \phi^*_{Aa}\pi^{Aa}\right)\right\}, \tag{5.6.54}$$

where we have used the notation

$$\delta X_1 = 2\Big((X,\delta Y_a)^a - U^a\delta Y_a - i\hbar\Delta^a\delta Y_a\Big) = 2\hat{Q}^a(X)\delta Y_a. \tag{5.6.55}$$

Let us choose the functional δY_a in the form

$$\delta Y_a = \frac{1}{4}\varepsilon_{ab}\hat{Q}^b\overline{\delta Y}, \quad \varepsilon(\overline{\delta Y}) = 0. \tag{5.6.56}$$

Then, representing δX as in Eq.(5.6.51), and identifying $\delta Y = -\overline{\delta Y}$, we find that

$$Z_{X+\delta X} = Z_X, \tag{5.6.57}$$

i.e. the vacuum functional (and hence, by virtue of the equivalence theorem [135], also the S matrix) does not depend on the choice of gauge.

Finally notice that investigations of the structure and properties of triplectic quantization are at starting point only [113, 1, 112, 96].

Chapter 6

Superfield BRST Quantization

In Section 4, we have presented the BV quantization method [40, 41] which may be applied for construction of suitable quantum theory for general gauge theories. The antisympletic manifold of the BV method contains the fields ϕ^A (including the initial classical fields, the ghosts, the antighosts and the Lagrangian multipliers) with assigned to them antifields ϕ^*_A of the opposite Grassmann parity, the usual sources J_A to the fields ϕ^A and finally, the auxiliary fields λ^A, introducing the gauge to the theory.

In turn, the Yang–Mills theories permit to realize the BRST symmetry transformations in superspace [53, 54, 126, 44]. At the same time, the crucial point of the formulations [53, 54, 126, 44, 130] is the manifest structure of configuration space of the theories concerned. On the other hand, no consistent form of Lagrangian quantization rules for general gauge theories that would enable one to give the BRST transformations a completely geometrical description has yet been discovered.

The purpose of this Section is formulation of Lagrangian quantization rules [145] within functional integration technique on the basis of a superfield approach, revealing the geometrical contents of the BRST symmetry. The functional integration over supervariables is understood as integration over their components. We also use the usual assumptions of both gauge invariant regularization and absence of anomalies.

6.1 Superspace, antibracket and operators Δ, U, V

Let us consider superspace $D+1$, parametrizied by coordinates (x^μ, θ); x^μ are the space-time coordinates, $\mu = (0, 1, \ldots, D-1)$; θ is a scalar Grassmann coordinate. Let $\Phi^A(\theta)$ be a set of superfields and $\Phi^*_A(\theta)$ be a set of the corresponding super-antifields

$$\varepsilon(\Phi^A) \equiv \varepsilon_A, \quad \varepsilon(\Phi^*_A) = \varepsilon_A + 1.$$

In terms of superfields and super-antifields we define an antibracket by the rule

$$\begin{aligned}(F, G) \quad = \quad & \int d\theta \bigg\{ \frac{\delta F}{\delta \Phi^A(\theta)} \frac{\partial}{\partial \theta} \frac{\delta G}{\delta \Phi^*_A(\theta)} (-1)^{\varepsilon_A + 1} \\ & -(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} (F \leftrightarrow G) \bigg\} ,\end{aligned} \qquad (6.1.1)$$

where $F = F[\Phi, \Phi^*]$, $G = G[\Phi, \Phi^*]$ are arbitrary functionals depending on supervariables. From the definition (6.1.1) one can find that the antibracket (6.1.1) obeys the same properties that the antibracket in the BV-formalism (3.2.36).

Let us also introduce operators Δ, U, V of the form

$$\Delta = -\int d\theta (-1)^{\varepsilon_A} \frac{\delta_l}{\delta\Phi^A(\theta)} \frac{\partial}{\partial\theta} \frac{\delta}{\delta\Phi^*_A(\theta)}, \tag{6.1.2}$$

$$U = -\int d\theta \frac{\partial\Phi^A(\theta)}{\partial\theta} \frac{\delta_l}{\delta\Phi^A(\theta)}, \tag{6.1.3}$$

$$V = -\int d\theta \frac{\partial\Phi^*_A(\theta)}{\partial\theta} \frac{\delta}{\delta\Phi^*_A(\theta)}. \tag{6.1.4}$$

Here, one has to take into account the expressions of the derivatives

$$\frac{\delta_l\Phi^A(\theta)}{\delta\Phi^B(\theta')} = (-1)^{\epsilon_A}\delta(\theta^{'} - \theta)\delta^A_B = (-1)^{\epsilon_A}\frac{\delta\Phi^A(\theta)}{\delta\Phi^B(\theta')},$$

$$\frac{\delta\Phi^*_A(\theta)}{\delta\Phi^*_B(\theta')} = (-1)^{\epsilon_A+1}\delta(\theta^{'} - \theta)\delta^B_A,$$

following from the definition of integration over the Grassmann variable θ

$$\int d\theta\, \theta = 1, \quad \int d\theta = 0, \quad F(\theta) = \int d\theta^{'}\delta(\theta^{'} - \theta)F(\theta^{'}),$$

$$\delta(\theta^{'} - \theta) = \theta^{'} - \theta.$$

The algebra of the operators (6.1.2), (6.1.3), (6.1.4) has the form

$$\begin{aligned} &\Delta^2 = 0 \quad U^2 = 0 \quad V^2 = 0, \\ &\Delta U + U\Delta = 0, \quad \Delta V + V\Delta = 0, \quad UV + VU = 0. \end{aligned} \tag{6.1.5}$$

The action of the operators $D = (\Delta, U, V)$ upon the antibracket is given by the following relation

$$D(F, G) = (DF, G) - (-1)^{\varepsilon(F)}(F, DG). \tag{6.1.6}$$

Finally, let us introduce the operator $\overline{\Delta}$

$$\overline{\Delta} = \Delta + \frac{i}{\hbar}V \tag{6.1.7}$$

with the properties

$$\begin{aligned} &\overline{\Delta}^2 = 0, \quad \overline{\Delta}U + U\overline{\Delta} = 0, \\ &\overline{\Delta}(F, G) = (\overline{\Delta}F, G) - (-1)^{\varepsilon(F)}(F, \overline{\Delta}G), \end{aligned} \tag{6.1.8}$$

readily verified with allowance made for Eqs. (6.1.5), (6.1.6).

6.2 Generating functional of Green's functions

Now we define the generating functional of Green's functions $Z = Z[\Phi^*]$ as a functional depending on the super-antifields in the form

$$Z[\Phi^*] \;=\; \int d\Phi^{'} d\Phi^{*'}\rho[\Phi^{*'}]\exp\left\{\frac{i}{\hbar}\left(S[\Phi^{'},\Phi^{*'}] \right.\right.$$
$$\left.\left. -U^{'}\Psi[\Phi^{'}] + (\Phi^{*'} - \Phi^*)\Phi^{'}\right)\right\}. \tag{6.2.9}$$

In Eq.(6.2.9), $S = S[\Phi, \Phi^*]$ is a quantum action satisfying the generating equation

$$\overline{\Delta}\exp\left\{\frac{i}{\hbar}S\right\} = 0, \tag{6.2.10}$$

or, equivalently,

$$\frac{1}{2}(S,S) + VS = i\hbar\Delta S, \tag{6.2.11}$$

$\Psi = \Psi[\Phi]$ is a fermion functional introducing the gauge; $\hbar$ is the Planck constant. Besides, the following notations

$$\rho[\Phi^*] = \delta\left(\int d\theta\Phi^*(\theta)\right), \quad \Phi^*\Phi = \int d\theta\Phi^*_A(\theta)\Phi^A(\theta) \tag{6.2.12}$$

are used.

An important property of the integrand in Eq. (6.2.9) for $\Phi^* = 0$ is its invariance under the following global supersymmetry transformations with a Grassmann parameter μ:

$$\delta\Phi^A(\theta) = \mu\frac{\partial\Phi^A(\theta)}{\partial\theta}\ ,$$
$$\delta\Phi^*_A(\theta) = \mu\frac{\partial\Phi^*_A(\theta)}{\partial\theta} + \mu\frac{\partial}{\partial\theta}\frac{\delta S}{\delta\Phi^A(\theta)}. \tag{6.2.13}$$

In fact, owing to Eqs. (6.1.2), (6.1.3), (6.1.4), (6.2.10), (6.2.11), the transformations (6.2.13) yield

$$\delta S = \mu\left(\frac{1}{2}(S,S) + (U+V)S\right) = i\hbar\mu\Delta S + \mu US, \tag{6.2.14}$$
$$\delta(\Phi^*\Phi) = -\mu US, \quad \delta\rho[\Phi^*] = 0, \quad \delta(U\Psi) = \mu U^2\Psi = 0,$$

and the corresponding Berezinian Y is equal to

$$Y = \exp\{\mu\Delta S\}. \tag{6.2.15}$$

6.3 Gauge independence

The transformations (6.2.13) permit one to establish the fact that the vacuum functional $Z_\Psi \equiv Z[0]$ is independent of a choice of the gauge. Indeed, we shall change the gauge by the rule $\Psi \to \Psi + \delta\Psi$. In the functional integral for $Z_{\Psi+\delta\Psi}$ we make the change of variables (

6.2.13) with the parameter $\mu = \mu[\Phi]$. By virtue of Eq. (6.2.13), the Berezinian $Y^{'}$ of the change of variables in question reads

$$Y^{'} = \exp\{\mu\Delta S - U\mu\}, \tag{6.3.16}$$

hence the set of variables $\Phi^{'} = \Phi + \delta\Phi$, $\Phi^{*'} = \Phi^* + \delta\Phi^*$ is equivalent to the initial set Φ, Φ^*. Owing to Eqs. (6.2.14), (6.2.15) $Z_{\Psi+\delta\Psi}$ takes on the form

$$\begin{aligned} Z_{\Psi+\delta\Psi} &= \int d\Phi\, d\Phi^* \rho[\Phi^*] \exp\left\{ \frac{i}{\hbar}\Big(S[\Phi,\Phi^*] - U\Psi[\Phi] \right. \\ &\left. -U(\delta\Psi[\Phi] - i\hbar\mu[\Phi]) + \Phi^*\Phi \Big) \right\} . \end{aligned} \tag{6.3.17}$$

Then, choosing for the parameter μ the functional

$$\mu = -\frac{i}{\hbar}\delta\Psi \ , \tag{6.3.18}$$

we find that $Z_{\Psi+\delta\Psi} = Z_\Psi$ and conclude that the S-matrix is gauge independent. Note that by virtue of the definitions (6.1.1), (6.1.3), (6.1.4), the transformations (6.2.13) take on the form

$$\begin{aligned} &\delta\Phi^A(\theta) = \mu U \Phi^A(\theta) \ , \\ &\delta\Phi^*_A(\theta) = \mu V \Phi^*_A(\theta) + \mu\Big(S, \Phi^*_A(\theta) \Big) \ . \end{aligned} \tag{6.3.19}$$

Eq. (6.3.19) implies that from the geometrical viewpoint the operators U and V (6.1.3), (6.1.4) can be considered as generators of supertranslations realized on the supervariables $\Phi^A(\theta)$ and $\Phi^*_A(\theta)$ respectively.

6.4 Ward identity

Another consequence of the validity of the transformations (6.2.13) are the Ward identities for the generating functional of Green's functions. In fact, making in the functional integral (6.2.9) the change of variables (6.2.13) and taking Eqs. (6.2.14), (6.2.15) into account, we arrive at the relation

$$\begin{aligned} &\int d\Phi^{'} d\Phi^{*'} \rho[\Phi^*] \int d\theta \frac{\partial\Phi^*_A(\theta)}{\partial\theta} \Phi^{'A}(\theta) \exp\left\{ \frac{i}{\hbar}\Big(S[\Phi^{'},\Phi^{*'}] \right. \\ &\left. -U^{'}\Psi[\Phi^{'}] + (\Phi^{*'} - \Phi^*)\Phi^{'} \Big) \right\} = 0 \ , \end{aligned} \tag{6.4.20}$$

representable, with allowance made for Eqs. (6.1.4), (6.2.9), in the form

$$-\int d\theta \frac{\partial\Phi^*_A(\theta)}{\partial\theta} \frac{\delta}{\delta\Phi^*_A(\theta)} Z[\Phi^*] = VZ[\Phi^*] = 0 \ . \tag{6.4.21}$$

Geometrically, the Ward identities (6.4.21) imply the fact that the functional $Z[\Phi^*]$ is invariant under supertranslations of Φ^*_A with respect to the coordinate θ.

It appears very important to establish a relation between the superfield approach in question and the BV quantization rules. To this end, note that the components of superfields $\Phi^A(\theta)$ and super-antifields $\Phi^*_A(\theta)$ are defined by expansions in θ

$$\Phi^A(\theta) = \phi^A + \lambda^A\theta\ , \quad \Phi^*_A(\theta) = \phi^*_A - \theta J_A\ , \tag{6.4.22}$$
$$\varepsilon(\phi^A) = \varepsilon(J_A) = \varepsilon_A\ ,\ \ \varepsilon(\phi^*_A) = \varepsilon(\lambda^A) = \varepsilon_A + 1$$

and coincide with the set of variables in the BV quantization scheme (the choice of signs in Eq. (6.4.22) is due to considerations of convenience).

6.5 Component representation

Consider by virtue of Eq. (6.4.22) the component form of the basic definitions and relations given above.

First, the antibracket (6.1.1) and the operator Δ (6.1.2) can be represented in terms of the component fields as follows

$$(F, G) = \frac{\delta F}{\delta\phi^A}\frac{\delta G}{\delta\phi^*_A} - (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}(F \leftrightarrow G)\ , \tag{6.5.23}$$

$$\Delta \;\; = \;\; (-1)^{\varepsilon_A}\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\phi^*_A}\ . \tag{6.5.24}$$

Eqs. (6.5.23), (6.5.24) coincide with the usual definitions of the antibracket and the operator Δ in the framework of the BV quantization method (see Eqs. (3.2.35), (3.2.41)).

Second, the corresponding component expressions for the operators U, V (6.1.3), (6.1.4) read

$$U = -(-1)^{\varepsilon_A}\lambda^A\frac{\delta_l}{\delta\phi^A}\ , \tag{6.5.25}$$

$$V = -J_A\frac{\delta}{\delta\phi^*_A}\ . \tag{6.5.26}$$

In virtue of Eqs. (6.5.23), (6.5.25), (6.5.26) we find that the transformations (6.2.13) (or, equivalently, (6.3.19)) take on the form

$$\delta\phi^A = \lambda^A\mu, \quad \delta\lambda^A = 0,$$
$$\delta\phi^*_A = \mu\left(\frac{\delta S}{\delta\phi^A} - J_A\right), \quad \delta J_A = 0. \tag{6.5.27}$$

Note that for $J = 0$ the component form (6.5.27) of Eqs. (6.2.13), (6.3.19) coincides formally with the BRST transformations (3.5.46) in the BV quantization scheme. In this connection, Eqs. (6.2.13), (6.3.19) may be considered as a superfield form of the BRST symmetry transformations.

Next, making use of Eq. (6.5.26), one readily obtains the component representation of the Ward identities (6.4.21) for the functional $Z(J, \phi^*) \equiv Z[\Phi^*]$

$$J_A\frac{\delta}{\phi^*_A}Z(J, \phi^*) = 0. \tag{6.5.28}$$

It should also be pointed out that Eq. (6.5.28) realizes the usual form of the Ward identities for gauge theories.

All things considered, the connection between the superfield approach concerned and the BV quantization scheme is then established as follows. To begin with, note that owing to Eqs. (6.2.12), (6.4.22), the integration measure in Eq. (6.2.9) is given in terms of the component variables by

$$d\Phi\, d\Phi^*\, \rho(\Phi^*) = d\phi\, d\phi^*\, d\lambda\, dJ\, \delta(J), \tag{6.5.29}$$

and the component representation of the functional $\Phi^*\Phi$ (6.2.12) reads

$$\Phi^*_A\Phi^A = \phi^*_A\lambda^A - J_A\phi^A\,. \tag{6.5.30}$$

It turns out to be sufficient for our purpose to confine ourselves to a special choice of solution to the generating equation (6.2.11) in the form of a functional $\overline{S} = \overline{S}[\Phi, \Phi^*]$ independent on the variables λ^A

$$\frac{\delta\overline{S}}{\delta\lambda^A} = \int d\theta\, \theta\frac{\delta\overline{S}}{\delta\Phi^A(\theta)} = 0$$

and linear in J_A

$$\overline{S}[\Phi, \Phi^*] = S(\phi, \phi^*) + J_A\phi^A, \tag{6.5.31}$$

where $S = S(\phi, \phi^*)$ satisfies the usual QME (3.3.41)

$$\frac{1}{2}(S, S) = i\hbar\Delta S. \tag{6.5.32}$$

Let us now choose the boundary condition to Eq. (6.5.32) in the form

$$\overline{S}\big|_{\Phi^*=\hbar=0} = S_0, \tag{6.5.33}$$

where S_0 is a classical gauge invariant action (note that Eq. (6.5.33) is compatible with the generating equation (6.2.11)). Then, making use of Eqs. (6.5.25), (6.5.29), (6.5.30) and supposing $\Psi = \Psi(\phi)$, we arrive at the following representation of the generating functional of Green's functions $Z = Z(J)$ for the fields ϕ^A

$$\begin{aligned} Z(J) \;\;=\;\; & Z[\Phi^*]\big|_{\phi^*=0} = \int d\phi\, d\phi^*\, d\lambda \exp\left\{\frac{i}{\hbar}\left[S(\phi, \phi^*)\right.\right. \\ & \left.\left.+\left(\phi^*_A - \frac{\delta\Psi}{\delta\phi^A}\right)\lambda^A + J_A\phi^A\right]\right\}. \end{aligned} \tag{6.5.34}$$

The above relation defines, with allowance made for Eqs. (6.5.31)–(6.5.33), the generating functional of Green's functions in the framework of the BV quantization formalism.

6.6 Generalization of gauge fixing

Note that there exists [97] generalization of gauge fixing procedure within superfield BRST quantization which allows to present the BRST transformations in more symmetrical way and simultaneously to povide a natural generalization of the BV-formalism. To this end, we define the vacuum functional Z as the following functional integral:

$$Z = \int d\Phi \; d\Phi^* \rho[\Phi^*] \exp\left\{ \frac{i}{\hbar}\Big(S[\Phi, \Phi^*] + X[\Phi, \Phi^*] + \Phi^* \Phi \Big) \right\}. \tag{6.6.35}$$

Here, $S = S[\Phi, \Phi^*]$ obeys the generating equation (6.2.11) while the (bosonic) gauge-fixing functional $X = X[\Phi, \Phi^*]$ is required to satisfy the equation

$$\frac{1}{2}(X, X) - UX = i\hbar\Delta X. \tag{6.6.36}$$

We have used the same definitions of antibracket (,) (6.1.1), operators Δ (6.1.2), U (6.1.3), V (6.1.4) and the weight functional $\rho[\Phi^*]$ (6.2.12).

It is convenient to recast the equations (6.2.11), (6.6.36) into the equivalent form

$$\bar{\Delta} \exp\left\{ \frac{i}{\hbar} S \right\} = 0, \tag{6.6.37}$$

$$\tilde{\Delta} \exp\left\{ \frac{i}{\hbar} X \right\} = 0, \tag{6.6.38}$$

using the operators

$$\bar{\Delta} = \Delta + \frac{i}{\hbar} V, \quad \tilde{\Delta} = \Delta - \frac{i}{\hbar} U, \tag{6.6.39}$$

whose algebra reads as follows:

$$\bar{\Delta}^2 = 0, \quad \tilde{\Delta}^2 = 0, \quad \bar{\Delta}\tilde{\Delta} + \tilde{\Delta}\bar{\Delta} = 0. \tag{6.6.40}$$

Using the nilpotency of the operator U, we observe that any functional $X = U\Psi[\Phi]$, with $\Psi[\Phi]$ being an arbitrary fermionic functional, is obviously a solution of Eq. (6.6.36). The above expression gives the precise form of the gauge-fixing functional proposed in (6.2.9) when formulating the rules of superfield BRST quantization.

A remarkable property of the integrand in (6.6.35) is its invariance under the following transformations of global supersymmetry with an anticommuting parameter μ:

$$\begin{aligned} \delta\Phi^A(\theta) &= \mu U \Phi^A(\theta) + (\Phi^A(\theta), X - W)\mu, \\ \delta\Phi^*_A(\theta) &= \mu V \Phi^*_A(\theta) + (\Phi^*_A(\theta), X - W)\mu. \end{aligned} \tag{6.6.41}$$

Eqs. (6.6.41) being symmetrical ones are the transformations of BRST symmetry in the framework of superfield quantization based on the gauge-fixing functional X introduced as a solution of the corresponding generating equation (6.6.36).

It is not difficult to prove the gauge-dependence of the vacuum functional Z, Eq. (6.6.35). Note, in the first place, that any admissible variation δX of the gauge-fixing functional X should satisfy the equation

$$(X, \delta X) - U\delta X = i\hbar\Delta\delta X,$$

which can be represented in the form

$$\widehat{Q}(X)\delta X = 0. \tag{6.6.42}$$

Here, we have introduced the graded linear, nilpotent operator $\hat{Q}(X)$,

$$\widehat{Q}(X) = \widehat{\mathcal{B}}(X) - i\hbar\tilde{\Delta}, \quad \widehat{Q}^2(X) = 0, \tag{6.6.43}$$

where $\widehat{\mathcal{B}}(X)$ stands for an operator acting by the rule

$$(X, F) \equiv \widehat{\mathcal{B}}(X)F, \tag{6.6.44}$$

and possessing the property

$$\widehat{\mathcal{B}}^2(X) = \widehat{\mathcal{B}}\left(\frac{1}{2}(X, X)\right). \tag{6.6.45}$$

By the nilpotency of the operator $\widehat{Q}(X)$, any functional of the for

$$\delta X = \widehat{Q}(X)\delta\Psi, \tag{6.6.46}$$

with $\delta\Psi$ being an arbitrary fermionic functional, obeys the equation (6.6.42). Furthermore, as in the theorems proved by the study of [25, 27], one can establish the fact that any solution δX of Eq. (6.6.42), vanishing when all the variables entering δX are equal to zero, has the form (6.6.46), with a certain fermionic functional $\delta\Psi$.

Let $Z_X \equiv Z$ be the value of the vacuum functional (6.6.35) related to the gauge condition chosen as a functional X. In the vacuum functional $Z_{X+\delta X}$ we now make the change of variables (6.6.41) with a functional $\mu = \mu[\Phi, \Phi^*]$, accompanied by an additional change

$$\delta\Phi^A = (\Phi^A, \delta Y), \quad \delta\Phi^*_A = (\Phi^*_A, \delta Y), \quad \varepsilon(\delta Y) = 1, \tag{6.6.47}$$

where $\delta Y = -i\hbar\mu[\Phi, \Phi^*]$. We obtain

$$Z_{X+\delta X} = \int d\Phi\, d\Phi^* \rho[\Phi^*] \exp\left\{\frac{i}{\hbar}\Big(S + X + \delta X + \delta X_1 + \Phi^*\Phi\Big)\right\}. \tag{6.6.48}$$

In (6.6.48), we have denoted

$$\delta X_1 = 2\Big((X, \delta Y) - U\delta Y - i\hbar\Delta\delta Y\Big) = 2\widehat{Q}(X)\delta Y. \tag{6.6.49}$$

Let the functional δY be chosen in the form (recall that $\delta X = \widehat{Q}(X)\delta\Psi$)

$$\delta Y = -\frac{1}{2}\delta\Psi. \tag{6.6.50}$$

Thereby we find

$$Z_{X+\delta X} = Z_X, \tag{6.6.51}$$

which implies the fact that the vacuum functional (and, hence, the S-matrix, by the equivalence theorem [135]) does not depend on the gauge.

In component form, restricting ourselves to functionals S independent of λ^A, and taking into account (6.2.12), we arrive at the following representation of the vacuum functional in Eq. (6.6.35):

$$Z = \int d\phi\, d\phi^*\, d\lambda \exp\left\{\frac{i}{\hbar}\Big[S(\phi, \phi^*) + X(\phi, \phi^*, \lambda) + \phi^*_A\lambda^A\Big]\right\}, \tag{6.6.52}$$

where $S = S(\phi, \phi^*)$ obeys the QME (3.3.41).

The above result may be considered as an extention of the BV quantization procedure [39] to a more general case of gauge-fixing. In fact, as stated above, the functional $X = U\Psi[\Phi]$ is a solution of the generating equation (6.6.36). From the component representation of the operator U

$$U = -(-1)^{\varepsilon_A}\lambda^A \frac{\delta_l}{\delta\phi^A} \ ,$$

provided the functional Ψ is independent of the fields λ^A, $\Psi = \Psi(\phi)$, it follows that the gauge-fixing functional X

$$X(\phi, \lambda) = -\frac{\delta\Psi(\phi)}{\delta\phi^A}\lambda^A$$

becomes identical with the gauge applied by the BV quantization method, thus leading to the usual expression for the vacuum functional

$$Z = \int d\phi\; d\phi^*\; d\lambda \exp\left\{\frac{i}{\hbar}\left[S(\phi, \phi^*) + \left(\phi^*_A - \frac{\delta\Psi}{\delta\phi^A}\right)\lambda^A\right]\right\}. \tag{6.6.53}$$

It is well-known that the fermionic functional Ψ can always be chosen so as to ensure the non-degeneracy of (6.6.53), which implies the fact that there always exists at least one permissible choice of gauge (i.e. satisfying the generating equation (6.6.36)) which leads to the correct vacuum functional in (6.6.35).

Note that there has been a fairly large amount of papers [130, 58, 59, 145, 2, 133, 97] devoted to various superfield extensions of the BV-quantization method for gauge theories. Thus, in [130, 58, 133] a superspace formulation of the action and BRST transformations for Yang-Mills theories was found; in [2] a superfield representation of the generating operator Δ in the BV-method was suggested; in [59] a superspace formulation of the BV-formalism was given, in [145, 97] a closed superfield form of the BV quantization method [39] was obtained. In the study of [21, 22], a superfield quantization in canonical formalism has been proposed.

Recall once again that the Lagrangian quantization rules for general gauge theories on a basis of a superfield realization of the standard BRST symmetry [145] allow to consider both the BRST transformations and the Ward identities from geometrical point of view. The Ward identities (6.4.21) imply the invariance of the generating functional $Z[\Phi^*]$ under translations in superspace (x^μ, θ) with respect to the Grassmann coordinate θ. The BRST transformations of fields are realized in the form of translantions in superspace along the coordinate θ.

Chapter 7

Superfield extended BRST Quantization

In this Chapter we will demonstrate a possibility to construct for general gauge theories a superfield covariant quantization [144] based on the BRST-antiBRST invariance principle.

7.1 Superspace, extended antibrackets, operators Δ^a, V^a, U^a

As usually, the condensed notations by De Witt [75] are used. Scalar anticommuting coordinates θ^a form an $Sp(2)$–doublet. Lowering the $Sp(2)$ indices is given by the rule $\theta_a = \varepsilon_{ab}\theta^b$. Derivatives with respect to θ^a are understood as the left-hand ones. Integration over θ^a is given by

$$\int d^2\theta = 0, \quad \int d^2\theta\ \theta^a = 0, \quad \int d^2\theta\ \theta^a\theta^b = \varepsilon^{ab}.$$

For any function $f(\theta)$ the equalities hold

$$\int d^2\theta\ \frac{\partial f(\theta)}{\partial\theta^a} = 0.$$

Any function of θ can be represented in the form

$$f(\theta) = f_0 + f_a\theta^a + \frac{1}{2}f_3\theta_a\theta^a.$$

Now let us introduce a superspace with coordinates (x^μ, θ^a), where x^μ $(\mu = 0, 1, ..., d-1)$ are the space-time coordinates and θ^a $(a = 1, 2)$ are anticommuting scalar coordinates. Let $\Phi^A(\theta)$, $\varepsilon(\Phi^A(\theta)) \equiv \varepsilon_A$ be a set of superfields with the following restriction

$$\Phi^A(\theta)\big|_{\theta=0} = \phi^A,$$

where ϕ^A are the fields of configuration space in the $Sp(2)$–covariant Lagrangian quantization [25, 26]. With each superfield$\Phi^A(\theta)$ we associate one supersource $\bar{\Phi}_A(\theta)$ of *the same* Grassmann parity

$$\varepsilon(\bar{\Phi}_A(\theta)) = \varepsilon_A.$$

In terms of supervariables $\Phi^A(\theta), \bar{\Phi}_A(\theta)$ we define for any functionals $F = F(\Phi, \bar{\Phi})$, $G = G(\Phi, \bar{\Phi})$ the super-antibrackets

$$\begin{aligned}(F, G)^a \quad = \quad & \int d^2\theta \left\{ \frac{\delta F}{\delta \Phi^A(\theta)} (-1)^{\varepsilon_A + 1} \frac{\partial}{\partial \theta_a} \frac{\delta G}{\delta \bar{\Phi}_A(\theta)} \right. \\ & \left. -(F \leftrightarrow G)(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} \right\}\end{aligned} \tag{7.1.1}$$

which have the algebraic properties coinciding with properties of the extended antibrackets in the $Sp(2)$-formalism (4.2.2).

Let us introduce the operators Δ^a, V^a and U^a by the rule

$$\Delta^a = -\int d^2\theta \frac{\delta_l}{\delta \Phi^A(\theta)} \frac{\partial}{\partial \theta_a} \frac{\delta}{\delta \bar{\Phi}_A(\theta)}, \tag{7.1.2}$$

$$V^a = \int d^2\theta \frac{\partial \bar{\Phi}_A(\theta)}{\partial \theta_a} \frac{\delta}{\delta \bar{\Phi}_A(\theta)}, \tag{7.1.3}$$

$$U^a = \int d^2\theta \frac{\partial \Phi^A(\theta)}{\partial \theta_a} \frac{\delta_l}{\delta \Phi^A(\theta)}. \tag{7.1.4}$$

Operators V^a and U^a have simple geometrical interpretation in terms of representation of the translation operators along Grassmann variables θ^a realized on supervariables Φ^*_A and Φ^A respectively.

One can easily check that the algebra of the operators (7.1.2), (7.1.3), (7.1.4) coincides with the algebra of correspoinding operators used in the modified version of triplectic quantization (5.6.28) while the action of these operators $D^a = (\Delta^a, V^a, U^a)$ upon the super-antibrackets is given by the relations (5.6.29).

It is also convenient to introduce the extended operators $\bar{\Delta}^a$

$$\bar{\Delta}^a = \Delta^a + \frac{i}{\hbar} V^a. \tag{7.1.5}$$

These operators satisfy the relations

$$\bar{\Delta}^{\{a} \bar{\Delta}^{b\}} = 0. \tag{7.1.6}$$

7.2 Generating functional of Green's functions

The basic object of the superfield quantization in question is the quantum action $S = S(\Phi, \bar{\Phi})$. We require S to be a solution to the following generating equations

$$\bar{\Delta}^a \exp\left\{ \frac{i}{\hbar} S \right\} = 0$$

or, equivalently,

$$\frac{1}{2}(S, S)^a + V^a S = i\hbar \Delta^a S. \tag{7.2.7}$$

The generating functional of the Green functions $Z = Z(\bar{\Phi})$ for superfields $\Phi^A(\theta)$ we define as

$$\begin{aligned} Z(\bar{\Phi}) \quad = \quad & \int [d\Phi'][d\bar{\Phi}']\rho(\bar{\Phi}') \exp\left\{\frac{i}{\hbar}\left[S(\Phi', \bar{\Phi}') + \bar{\Phi}'\Phi' - \right.\right. \\ & \left.\left. - \frac{1}{2}\varepsilon_{ab} U'^a U'^b F(\Phi') - \bar{\Phi}\Phi'\right]\right\}, \end{aligned} \tag{7.2.8}$$

where $F(\Phi)$ is the boson gauge functional, $\rho(\bar{\Phi})$ is the weight functional having the form of functional δ–function

$$\rho(\bar{\Phi}) = \delta\left(\int d^2\theta \ \bar{\Phi}(\theta)\right), \tag{7.2.9}$$

and the notation

$$\bar{\Phi}\Phi \equiv \int d^2\theta \ \bar{\Phi}_A(\theta)\Phi^A(\theta) \tag{7.2.10}$$

is used.

7.3 Extended BRST symmetry and gauge independence

The introduced above generating functional (7.2.8) possesses two important properties. Firstly, the integrand in (7.2.8) for $\bar{\Phi} = 0$ is invariant under the transformations of global supersymmetry

$$\delta\Phi^A(\theta) = \mu_a \frac{\partial \Phi^A(\theta)}{\partial \theta_a}, \quad \varepsilon(\mu_a) = 1, \tag{7.3.11}$$

$$\delta\bar{\Phi}_A(\theta) = \mu_a \frac{\partial \bar{\Phi}_A(\theta)}{\partial \theta_a} + \mu_a \frac{\partial}{\partial \theta_a} \frac{\delta S}{\delta \Phi^A(\theta)} \tag{7.3.12}$$

on account of the generating equations (7.2.7) and invariance of the weight functional (7.2.9) under the transformations (7.3.12)

$$\delta\rho(\bar{\Phi}) = 0. \tag{7.3.13}$$

In (7.3.11), (7.3.12) μ_a is a $Sp(2)$–doublet of the constant anticommuting Grassmann parameters. Secondly, the vacuum functional $Z(0)$ does not depend on a choice of the gauge boson F within the superfield scheme proposed (7.2.7), (7.2.8), (7.2.9). Indeed, suppose $Z_F \equiv Z(0)$. We shall change the gauge $F(\Phi) \to F(\Phi) + \delta F(\Phi)$. In the functional integral for $Z_{F+\delta F}$ we make the change of variables (7.3.11), (7.3.12), choosing for the parameters μ_a

$$\mu_a = -\frac{i}{2\hbar}\varepsilon_{ab} U^b \delta F(\Phi).$$

Taking into account properties of U^a, (7.2.7) and (7.3.13), we find that

$$Z_{F+\delta F} = Z_F \tag{7.3.14}$$

and, hence, the S–matrix is gauge-invariant.

The transformations (7.3.11), (7.3.12) realize the BRST- and antiBRST- symmetry in the superfield approach to quantum gauge theory. Allowing for (7.1.1), (7.1.3), (7.1.4) one can rewrite these transformations in the form

$$\delta\Phi^A(\theta) = \mu_a U^a \Phi^A(\theta), \tag{7.3.15}$$
$$\delta\bar{\Phi}_A(\theta) = \mu_a V^a \bar{\Phi}_A(\theta) + \mu_a (W, \bar{\Phi}_A(\theta))^a. \tag{7.3.16}$$

From (7.3.15), (7.3.16) we conclude that the BRST-antiBRST transformations are realized as supertranslations in the θ^a–directions on supervariables $\Phi^A(\theta)$. This gives a geometric interpretation of the BRST- and antiBRST- symmetry for arbitrary gauge theory.

7.4 Ward identities

The invariance of the vacuum functional $Z(0)$ under the BRST- and antiBRST- transformations leads to the presence of gauge Ward identities. Let us consider the derivation of these identities. To do this we shall use the standard assumptions on functional integral properties, in particular,

$$\int [d\Phi][d\bar{\Phi}]\rho(\bar{\Phi})\,\frac{\delta F(\Phi,\bar{\Phi})}{\delta\Phi} = 0,\ \int [d\Phi][d\bar{\Phi}]\rho(\bar{\Phi})\,\frac{\delta F(\Phi,\bar{\Phi})}{\delta\bar{\Phi}} = 0. \tag{7.4.17}$$

Taking into account the explicit form of the operators $\bar{\Delta}^a$ and (7.4.17), we have the following equalities

$$\int [d\Phi'][d\bar{\Phi}']\rho(\bar{\Phi}')\bar{\Delta}'^a \exp\left\{\frac{i}{\hbar}[S(\Phi',\bar{\Phi}') + \bar{\Phi}'\Phi' - \right. \tag{7.4.18}$$
$$\left. - \frac{1}{2}\varepsilon_{ab}U'^a U'^b F(\Phi') - \bar{\Phi}\Phi']\right\} = 0.$$

Let us act on the exponential by the operators $\bar{\Delta}^a$ and take into account the algebra of the operators Δ^a, V^a, U^a (5.6.28) and (7.2.7). We obtain

$$V^a Z(\bar{\Phi}) = 0. \tag{7.4.19}$$

Equations (7.4.19) represent the superfield form of Ward identities for generating functional of Green functions.

From (7.4.19) one can establish a new (geometric) interpretation of the Ward identities in quantum gauge theory. Indeed, the Ward identities express the invariance of the generating functionals Z under supertranslations in the θ^a–directions.

7.5 Component representation

It is useful to compare the suggested superfield extended BRST quantization for general gauge theories with the $Sp(2)$–covariant Lagrangian quantization [25, 26, 27] considered in Section 4 and with the triplectic quantization presented in Section 5. To this end, we present, first of all, the above superfield quantization scheme in the component form.

For the supervariables $\Phi^A(\theta)$ and $\bar{\Phi}_A(\theta)$ we shall use the following notations in the θ^a–expansions

$$\Phi^A(\theta) = \phi^A + \pi^{Aa}\theta_a + \frac{1}{2}\lambda^A\theta_a\theta^a,$$
$$\varepsilon(\pi^{Aa}) = \varepsilon_A + 1,\ \varepsilon(\lambda^A) = \varepsilon_A,$$
$$\bar{\Phi}_A(\theta) = \bar{\phi}_A - \theta^a\phi^*_{Aa} - \frac{1}{2}\theta_a\theta^a J_A,$$
$$\varepsilon(\phi^*_{Aa}) = \varepsilon_A + 1,\ \varepsilon(J_A) = \varepsilon_A.$$

Then the operators Δ^a (7.1.2), V^a (7.1.3), U^a (7.1.4) and the super-antibrackets (7.1.1) have the form

$$(F,G)^a = \frac{\delta F}{\delta\phi^A}\frac{\delta G}{\delta\phi^*_{Aa}} + \varepsilon^{ab}\frac{\delta F}{\delta\pi^{Ab}}\frac{\delta G}{\delta\bar{\Phi}_A} - -(F\leftrightarrow G)\,(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}, \tag{7.5.20}$$

$$\Delta^a = (-1)^{\varepsilon_A}\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\phi^*_{Aa}} + (-1)^{\varepsilon_A+1}\varepsilon^{ab}\frac{\delta_l}{\delta\pi^{Ab}}\frac{\delta}{\delta\bar{\phi}_A}, \tag{7.5.21}$$

$$V^a = \varepsilon^{ab}\phi^*_{Ab}\frac{\delta}{\delta\bar{\phi}_A} - J_A\frac{\delta}{\delta\phi^*_{Aa}}, \tag{7.5.22}$$

$$U^a = (-1)^{\varepsilon_A}\varepsilon^{ab}\lambda^A\frac{\delta_l}{\delta\pi^{Ab}} - (-1)^{\varepsilon_A}\pi^{Aa}\frac{\delta_l}{\delta\phi^A}. \tag{7.5.23}$$

From (7.5.20), (7.5.21) there follow the definitions of super-antibrackets and Δ^a used in construction of the superfield extended BRST quantization lead to analogous objects of triplectic quantization (5.1.1), (5.2.3).

In the component form the gauge-fixing action reads

$$\frac{1}{2}\varepsilon_{ab}U^aU^bF(\Phi) = \frac{1}{2}\varepsilon_{ab}\pi^{Aa}\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\pi^{Bb} + \frac{\delta F}{\delta\phi^A}\lambda^A - -\frac{1}{2}\varepsilon^{ab}\lambda^A\frac{\delta^2 F}{\delta\pi^{Aa}\delta\pi^{Bb}}\lambda^B + \pi^{Aa}\frac{\delta^2 F}{\delta\phi^A\delta\pi^{Ba}}\lambda^B. \tag{7.5.24}$$

For the functional $\bar{\Phi}\Phi$ (7.2.10), we have

$$\bar{\Phi}\Phi = \bar{\phi}_A\lambda^A + \phi^*_{Aa}\pi^{Aa} - J_A\phi^A.$$

The integration measure

$$[d\Phi][d\bar{\Phi}]\rho(\bar{\Phi}) = [d\phi][d\phi^*][d\pi][d\bar{\phi}][d\lambda][dJ]\delta(J) \tag{7.5.25}$$

coincides, in fact, with the measures in functional integrals of the $Sp(2)$–quantization [25, 26, 27] and the triplectic quantization [28, 35, 96].

One can now readily establish a connection with both the $Sp(2)$-covariant quantization. To do this, we note that due to the special form of the integration measure (7.5.25) it is sufficient to consider a solution to the generating equations (7.2.7) when $J_A = 0$ and to require, in addition, that the functional S be independent on the variables π^{Aa}, λ^A, i.e. $S = S(\phi, \phi^*, \bar{\phi})$. From component representations (7.5.20), (7.5.21), (7.5.22) we can conclude that the functional $S = S(\phi, \phi^*, \bar{\phi})$ satisfies the generating equations of the $Sp(2)$–covariant quantization method. Choose the gauge fixing functional F as to depend on the variables ϕ^A only

$$F = F(\phi).$$

Next, suppose that the boundary condition for $S(\phi, \phi^*, \bar{\phi})$ has the form

$$S(\phi, \phi^*, \bar{\phi}) \mid_{\phi^*=\bar{\phi}=\hbar=0}= S_0, \tag{7.5.26}$$

where S_0 is a classical gauge invariant action. Then we find in (7.2.8) the exact form of the generating functional of Green functions in the $Sp(2)$–covariant quantization.

In turn, let us consider the case when S does not depend on λ^A and $J_A = 0$. Then the action $S = S(\phi, \phi^*, \pi, \bar{\phi})$ satisfies the generating equations of modified triplectic quantization (5.6.33). Note that the action $X = \frac{1}{2}\varepsilon_{ab}U^aU^bF(\Phi)$ (7.5.24) satisfies the generating equations for gauge fixing functional (5.6.37) within the modified triplectic method. Then the generating fuctional in (7.2.8) presents the functional (5.6.36) corresponding to special choice of the gauge in the modified triplectic formalism.

The form (7.4.19) of the Ward identities for $Z(\bar{\Phi})$, rewritten in terms of the components

$$J_A \frac{\delta Z}{\delta \phi^*_{Aa}} - \varepsilon^{ab}\phi^*_{Ab}\frac{\delta Z}{\delta \bar{\phi}_A} = 0, \tag{7.5.27}$$

coincides with the one derived in [25].

The BRST-antiBRST symmetry transformations (7.3.11), (7.3.12) (or, equivalently, (7.3.15), (7.3.16)) for arbitrary gauge theories acquire, within the superfield formalism, a clear geometric meaning, since they are realized as supertranslations in superspace (x^μ, θ^a) along the Grassmann coordinates θ^a.

The superfield description provides a new outlook of the Ward identities in the quantum theory of gauge fields, thus revealing their geometric contents. Indeed, the identities (7.4.19) for the generating functional of Green's functions $Z = Z(\bar{\Phi})$ are nothing but the fact that Z is invariant under supertranslations in superspace. Also revealed are the role and geometric origin of the operators V^a, U^a which realize the supertranslations in terms of the variables $\Phi^A(\theta)$, $\bar{\Phi}_A(\theta)$.

Chapter 8

osp(1,2)–Covariant Quantization

In Chapter 4 we have presented the general method for quantizing gauge theories in the Lagrangian formalism proposed in [25, 26, 27] which is based on extended BRST symmetry, i.e. simultaneous invariance under both BRST and antiBRST transformations.

Although this formalism is seemingly manifestly $Sp(2)$-covariant, among the solutions of the master equations, despite those allowed by the above requirements, there are both $Sp(2)$-symmetric and $Sp(2)$-nonsymmetric ones. The symmetric solutions may be singled out by the explicit requirement of invariance under $Sp(2)$ transformations by additional master equations whose generating differential operators $\bar{\Delta}_\alpha$ ($\alpha = 0, +, -$) are related to the generators of the symplectic group $Sp(2)$. The algebra of these operators may be chosen to obey the orthosymplectic superalgebra $osp(1, 2)$. Moreover, if also *massive* fields should be considered to circumvent possible infrared singularities occuring in the process of subtracting ultraviolet divergences, without breaking the extended BRST symmetry, then this algebra appears necessarily. Let us also mention that the $osp(1, 2)$ superalgebra is present in many problems in which $N = 1$ superconformal symmetry is involved; e.g. in the minimal $N = 1$ superconformal models this symmetry appears in the light-cone approach to two-dimensional supergravity [169].

The goal of the present Chapter will be to generalize the $Sp(2)$ - quantization procedure to another one being $osp(1, 2)$-covariant [98, 99] and to get an answer on the intrigue question: What happens if we extend in a non-trivial way the usual algebra of generating operators Δ^a in the $Sp(2)$ formalisn to that when the fundamental property of nilpotency for every operator $\bar{\Delta}^a, a = 1, 2$ will be lost?. We will show that an answer to this question consists in the concluzion of gauge independence violation of the S-matrix when the characteristic parameter destroying the nilpotency of $\bar{\Delta}^a, a = 1, 2$ is not equal to zero.

8.1 New algebraic structure

The total configuration space ϕ^A of $osp(1, 2)$ coincides with the configuration space of $Sp(2)$ method (see (4.1.1)). To realize the $osp(1, 2)$ symmetry, one needs, except antifields ϕ^*_{Aa} and $\bar{\phi}_A$, also to introduce additional variables η_A, $\epsilon(\eta_A) \equiv \epsilon_A$.

Now let us introduce the extended antibrackets $(F, G)^a$ in the same manner as in the

$Sp(2)$-formalism Eqs.(4.2.2) and a *new* even graded algebraic structure $\{F,G\}_\alpha$ by the rule

$$\{F,G\}_\alpha = (\sigma_\alpha)_B{}^A\left(\frac{\delta F}{\delta\phi^A}\frac{\delta G}{\delta\eta_B} + (-1)^{\epsilon(F)\epsilon(G)}\frac{\delta G}{\delta\phi^A}\frac{\delta F}{\delta\eta_B}\right), \tag{8.1.1}$$

where we used the following notations

$$(\sigma_\alpha)^B{}_A \equiv (\sigma_\alpha)^b{}_a (P_+)^{Ba}_{Ab}. \tag{8.1.2}$$

Here, we have introduced the matrix $(P_+)^{Ba}_{Ab}$

$$\begin{aligned}(P_+)^{Ba}_{Ab} \;\; \equiv \;\; & \{\delta^i_j\delta^a_b \; (if A=i, B=j); \quad \delta^{\beta_0}_{\alpha_0}\delta^a_b \; (if A=\alpha_0, B=\beta_0);\\ & \delta^{\beta_0}_{\alpha_0}(\delta^a_b\delta^{a_0}_{b_0} + \delta^a_{a_0}\delta^{b_0}_b) \; (if A=\alpha_0 a_0, B=\beta_0 b_0); \quad 0 \; (otherwise)\},\end{aligned} \tag{8.1.3}$$

and the matrices $\sigma_\alpha(\alpha = 0,+,-)$,which are defined by the rule

$$(\sigma_0)^b{}_a = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}, \quad (\sigma_+)^b{}_a = \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}, \quad (\sigma_-)^b{}_a = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}.$$

From the definitions (4.2.2) and (8.1.1) it follows

$$\begin{aligned}\epsilon(\{F,G\}_\alpha) = \epsilon(F)+\epsilon(G), \quad \{F,G\}_\alpha = \{G,F\}_\alpha(-1)^{\epsilon(F)\epsilon(G)},\\ \epsilon((F,G)^a) = \epsilon(F)+\epsilon(G)+1, \quad (F,G)^a = -(G,F)^a(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)},\end{aligned} \tag{8.1.4}$$

i.e. $\{F,G\}_\alpha$ $((F,G)^a)$ defines an even (odd) graded bracket. Furthermore, it holds

$$\begin{aligned}\{F,GH\}_\alpha = \{F,G\}_\alpha H + G\{F,H\}_\alpha(-1)^{\epsilon(F)\epsilon(G)},\\ (F,GH)^a = (F,G)^a H + G(F,H)^a(-1)^{(\epsilon(F)+1)\epsilon(G)},\end{aligned} \tag{8.1.5}$$

i.e. both brackets act on the algebra of functions under multiplications.

Next, one can arrive at the following Jacobi identities satisfied by two brackets:

$$\{\{F,G\}_{[\alpha},H\}_{\beta]}(-1)^{\epsilon(F)\epsilon(H)} + \text{cyclic}(F,G,H) \equiv 0, \tag{8.1.6}$$

$$((F,G)^{\{a},H)^{b\}}(-1)^{(\epsilon(F)+1)(\epsilon(H)+1)} + \text{cyclic}(F,G,H) \equiv 0, \tag{8.1.7}$$

$$\left(\{(F,G)^a,H\}_\alpha - (\{F,G\}_\alpha,H)^a(-1)^{\epsilon(G)}\right)(-1)^{\epsilon(F)(\epsilon(H)+1)} + \tag{8.1.8}$$

$$+\text{cyclic}(F,G,H) \equiv 0.$$

where the square (curly) bracket means antisymmetrization (symmetrization) in the indices α and β (a and b), respectively. Identities (8.1.8) are usual Jacobi ones for extended antibrackets while Eqs.(8.1.6), (8.1.9) present new type of Jacobi identities in this formalism.

Then the operators $\bar{\Delta}^a_m$, $\bar{\Delta}_\alpha$ are introduced

$$\bar{\Delta}^a_m = \Delta^a + \frac{i}{\hbar}V^a_m, \quad \bar{\Delta}_\alpha = \Delta_\alpha + \frac{i}{\hbar}V_\alpha, \tag{8.1.9}$$

where we used the notations

$$\Delta^a = (-1)^{\epsilon_A}\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\phi^*_{Aa}}, \quad \Delta_\alpha = (-1)^{\epsilon_A}(\sigma_\alpha)_B{}^A\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\eta_B}. \tag{8.1.10}$$

$$V^a_m = \epsilon^{ab}\phi^*_{Ab}\frac{\delta}{\delta\bar{\phi}_A} - \eta_A\frac{\delta}{\delta\phi^*_{Aa}} + m^2(P_+)^{Ba}_{Ab}\bar{\phi}_B\frac{\delta}{\delta\phi^*_{Ab}} - m^2\epsilon^{ab}(P_-)^{Bc}_{Ab}\phi^*_{Bc}\frac{\delta}{\delta\eta_A} \tag{8.1.11}$$

$$V_\alpha = \bar{\phi}_B(\sigma_\alpha)^B{}_A \frac{\delta}{\delta\bar{\phi}_A} + \left(\phi^*_{Ab}(\sigma_\alpha)^b{}_a + \phi^*_{Ba}(\sigma_\alpha)^B{}_A\right)\frac{\delta}{\delta\phi^*_{Aa}} + \eta_B(\sigma_\alpha)^B{}_A\frac{\delta}{\delta\eta_A} \qquad (8.1.12)$$

and the following abbrevations:

$$(P_-)^{Ba}_{Ab} \equiv (P_+)^{Ba}_{Ab} - (P_+)^B_A\delta^a_b + \delta^B_A\delta^a_b, \quad (P_+)^B_A \equiv \delta^b_a(P_+)^{Ba}_{Ab}.$$

It is known that the extended antibrackets, being odd graded, may be exstracted from the action of second-order operators Δ^a on a product of two functionals F and G

$$\Delta^a(FG) = (\Delta^a F)G + F(\Delta^a G)(-1)^{\epsilon(F)} + (F,G)^a(-1)^{\epsilon(F)}.$$

A similar statement is valid for the new even graded brackets (8.1.1)

$$\Delta_\alpha(FG) = (\Delta_\alpha F)G + F(\Delta_\alpha G) + \{F,G\}_\alpha$$

in contrast with the Poisson bracket which is the even graded bracket defined on a phase space but for which does not exist a creating operator in the sence discussed here. The reason is different symmetry properties of these two kinds of brackets: the Poisson bracket is antisymmetrical while new brackets are symmetrical.

From definitions (8.1.10), (8.1.11), (8.1.12) one can straightforwardly derive the following algebra of operators $\Delta^a, \Delta_\alpha, V^a_m, V_\alpha$

$$[\Delta_\alpha,\Delta_\beta] = 0, \qquad \{\Delta^a,\Delta^b\} = 0, \qquad [\Delta_\alpha,\Delta^a] = 0, \qquad (8.1.13)$$

$$[V_\alpha,V_\beta] = \epsilon_{\alpha\beta}{}^\gamma V_\gamma, \qquad \{V^a_m,V^b_m\} = -m^2(\sigma^\alpha)^{ab}V_\alpha, \qquad [V_\alpha,V^a_m] = V^b_m(\sigma_\alpha)_b{}^a. \quad (8.1.14)$$

$$[\Delta_\alpha,V_\beta] + [V_\alpha,\Delta_\beta] = \epsilon_{\alpha\beta}{}^\gamma\Delta_\gamma, \qquad (8.1.15)$$

$$\{\Delta^a,V^b_m\} + \{V^a_m,\Delta^b\} = -m^2(\sigma^\alpha)^{ab}\Delta_\alpha, \qquad (8.1.16)$$

$$[\Delta_\alpha,V^a_m] + [V_\alpha,\Delta^a] = \Delta^b(\sigma_\alpha)_b{}^a. \qquad (8.1.17)$$

Applying the identities (8.1.13) – (8.1.17) to a product of two functions F and G, one can ectablish the following relations which define the action of the operators Δ^a, Δ_α, V^a_m and V_α upon the brackets:

$$\begin{aligned}
\Delta_{[\alpha}\{F,G\}_{\beta]} &= \{\Delta_{[\alpha}F,G\}_{\beta]} + \{F,\Delta_{[\alpha}G\}_{\beta]},\\
\Delta^{\{a}(F,G)^{b\}} &= (\Delta^{\{a}F,G)^{b\}} + (F,\Delta^{\{a}G)^{b\}}(-1)^{\epsilon(F)+1},\\
\Delta_\alpha(F,G)^a - \Delta^a\{F,G\}_\alpha(-1)^{\epsilon(F)} &= (\Delta_\alpha F,G)^a + (F,\Delta_\alpha G)^a -\\
&\quad -\{\Delta^a F,G\}_\alpha(-1)^{\epsilon(F)} - \{F,\Delta^a G\}_\alpha,
\end{aligned}$$

$$\begin{aligned}
V_{[\alpha}\{F,G\}_{\beta]} &= \epsilon_{\alpha\beta}{}^\gamma\{F,G\}_\gamma + \{V_{[\alpha}F,G\}_{\beta]} + \{F,V_{[\alpha}G\}_{\beta]},\\
V^{\{a}_m(F,G)^{b\}} &= -m^2(\sigma^\alpha)^{ab}\{F,G\}_\alpha + (V^{\{a}_m F,G)^{b\}} + (F,V^{\{a}_m G)^{b\}}(-1)^{\epsilon(F)+1},\\
V_\alpha(F,G)^a - V^a_m\{F,G\}_\alpha(-1)^{\epsilon(F)} &= (F,G)^b(\sigma_\alpha)_b{}^a + (V_\alpha F,G)^a + (F,V_\alpha G)^a -\\
&\quad -\{V^a_m F,G\}_\alpha(-1)^{\epsilon(F)} - \{F,V^a_m G\}_\alpha.
\end{aligned}$$

For any bosonic functional S, $\varepsilon(S)=0$ the following relations hold:

$$\{\{S,S\}_{[\alpha},S\}_{\beta]}\equiv 0,\quad ((S,S)^{\{a},S)^{b\}}\equiv 0,\quad \{(S,S)^a,S\}_\alpha-(\{S,S\}_\alpha,S)^a\equiv 0,\quad (8.1.18)$$

$$\begin{aligned}\tfrac{1}{2}\Delta_{[\alpha}\{S,S\}_{\beta]} &= \{\Delta_{[\alpha}S,S\}_{\beta]},\\ \tfrac{1}{2}\Delta^{\{a}(S,S)^{b\}} &= (\Delta^{\{a}S,S)^{b\}},\\ \tfrac{1}{2}\big(\Delta_\alpha(S,S)^a-\Delta^a\{S,S\}_\alpha\big) &= (\Delta_\alpha S,S)^a-\{\Delta^a S,S\}_\alpha,\end{aligned}\qquad(8.1.19)$$

and

$$\begin{aligned}\tfrac{1}{2}V_{[\alpha}\{S,S\}_{\beta]} &= \{V_{[\alpha}S,S\}_{\beta]}+\tfrac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma}\{S,S\}_\gamma,\\ \tfrac{1}{2}V_m^{\{a}(S,S)^{b\}} &= (V_m^{\{a}S,S)^{b\}}-\tfrac{1}{2}m^2(\sigma^\alpha)^{ab}\{S,S\}_\alpha,\\ \tfrac{1}{2}\big(V_\alpha(S,S)^a-V_m^a\{S,S\}_\alpha\big) &= (V_\alpha S,S)^a-\{V_m^a S,S\}_\alpha+\tfrac{1}{2}(S,S)^b(\sigma_\alpha)_b{}^a.\end{aligned}\qquad(8.1.20)$$

As long as $m\neq 0$ (the new (mass) parameter of the approach), the operators $\bar{\Delta}_m^a$ are neither nilpotent nor do they anticommute among themselves; instead, together with the operators $\bar{\Delta}_\alpha$ they generate a superalgebra isomorphic to $osp(1,2)$ (see **Appendix A**):

$$[\bar{\Delta}_\alpha,\bar{\Delta}_\beta]=(i/\hbar)\epsilon_{\alpha\beta}{}^{\gamma}\bar{\Delta}_\gamma,$$

$$[\bar{\Delta}_\alpha,\bar{\Delta}_m^a]=(i/\hbar)\bar{\Delta}_m^b(\sigma_\alpha)_b{}^a,$$

$$\{\bar{\Delta}_m^a,\bar{\Delta}_m^b\}=-(i/\hbar)m^2(\sigma^\alpha)^{ab}\bar{\Delta}_\alpha,\qquad(8.1.21)$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor, $\epsilon_{0+-}=1$. From Eqs.(8.1.21) we see that when $m=0$ it follows the usual anticommutative relations (4.3.6) for operators $\bar{\Delta}^a$ of $Sp(2)$ - quantization.

8.2 Generating equations

Let us introduce a boson action $S_m=S_m(\phi,\phi^*,\bar{\phi},\eta)$ which is required to satisfy the generating equations of $osp(1,2)$ - quantization:

$$\bar{\Delta}_m^a\exp\{(i/\hbar)S_m\}=0,\qquad(8.2.22)$$

$$\bar{\Delta}_\alpha\exp\{(i/\hbar)S_m\}=0\qquad(8.2.23)$$

or equivalently

$$\frac{1}{2}(S_m,\ S_m)^a+V_m^aS_m=i\hbar\Delta^aS_m,$$

$$\frac{1}{2}\{S_m,\ S_m\}_\alpha+V_\alpha S_m=i\hbar\Delta_\alpha S_m$$

with the usual boundary condition

$$S_m|_{\phi^*=\bar{\phi}=\eta=\hbar=0}=S_0.$$

In order to lift the degeneracy of S_m we follow the general gauge-fixing procedure introducing the gauge fixed action

$$\exp\{(i/\hbar)S_{m,\text{ext}}\} = \hat{U}_m(F)\exp\{(i/\hbar)S_m\},$$

where the operator $\hat{U}_m(F) = \exp\{(\hbar/i)\hat{T}_m(F)\}$ is defined by the rule

$$\hat{T}_m(F) = \frac{1}{2}\epsilon_{ab}\{\bar{\Delta}^b_m, [\bar{\Delta}^a_m, F]\} + (i/\hbar)^2 m^2 F$$

If the gauge-fixing boson functional is assumed to depend only on the fields, $F = F(\phi^A)$, then one gets

$$\hat{T}_m(F) = \frac{\delta F}{\delta\phi^A}\left(\frac{\delta}{\delta\bar{\phi}_A} - \tfrac{1}{2}m^2(P_-)^A_B\frac{\delta}{\delta\eta_B}\right) - \frac{\hbar}{2i}\epsilon_{ab}\frac{\delta}{\delta\phi^*_{Aa}}\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\frac{\delta}{\delta\phi^*_{Bb}} + \frac{i}{\hbar}m^2 F. \quad (8.2.24)$$

When $m = 0$ operator $\hat{T}_m(F)$ (8.2.24) coincide with operator $\hat{T}(F)$ (4.5.16) in the $Sp(2)$-scheme.

Let us prove that $S_{m,\text{ext}}$ obeys the generating equations (8.2.22) and (8.2.23) as well. Clearly, since $\bar{\Delta}^a_m$, $\bar{\Delta}_\alpha$ and $\hat{U}_m(F)$ do not commute with each other this proof will be more involved than in the $Sp(2)$- approach. This is due to the fact that, neither

$$[\bar{\Delta}^a_m, \hat{T}_m(F)] = \tfrac{1}{2}(i/\hbar)m^2(\sigma_\alpha)^a{}_b[\bar{\Delta}^b_m, [\bar{\Delta}^\alpha, F]]$$

nor

$$[\bar{\Delta}_\alpha, \hat{T}_m(F)] = \tfrac{1}{2}\epsilon_{ab}\{\bar{\Delta}^b_m, [\bar{\Delta}^a_m, [\bar{\Delta}_\alpha, F]]\} + (i/\hbar)^2 m^2[\bar{\Delta}_\alpha, F]$$

does vanish, since due to the nonlinearity of $\bar{\Delta}_\alpha$ one cannot require the strong condition $[\bar{\Delta}_\alpha, F] = 0$. However, a direct verification shows that $\hat{T}_m(F)$ commutes with any term on the right-hand side of both previous relations, i.e. it holds

$$[\hat{T}_m(F), [\bar{\Delta}^a_m, \hat{T}_m(F)]] = 0, \qquad [\hat{T}_m(F), [\bar{\Delta}_\alpha, \hat{T}_m(F)]] = 0.$$

Then, by the help of (8.2.22) and (8.2.23) one obtains

$$[\bar{\Delta}^a_m, \hat{U}_m(F)] = (\hbar/i)\hat{U}_m(F)[\bar{\Delta}^a_m, \hat{T}_m(F)], \quad [\bar{\Delta}_\alpha, \hat{U}_m(F)] = (\hbar/i)\hat{U}_m(F)[\bar{\Delta}_\alpha, \hat{T}_m(F)].$$

Let us require

$$[\bar{\Delta}_\alpha, F]S_m \equiv (\sigma_\alpha)_B{}^A\frac{\delta F}{\delta\phi^A}\frac{\delta S_m}{\delta\eta_B} = 0,$$

then, taking into account that S_m solves the generating equations, it is easily seen that $[\bar{\Delta}^a_m, \hat{U}_m(F)]$ and $[\bar{\Delta}_\alpha, \hat{U}_m(F)]$ vanish after acting on $\exp\{(i/\hbar)S_m\}$,

$$[\bar{\Delta}^a_m, \hat{U}_m(F)]\exp\{(i/\hbar)S_m\} = 0, \quad [\bar{\Delta}_\alpha, \hat{U}_m(F)]\exp\{(i/\hbar)S_m\} = 0.$$

Summarizing, we have the results

$$\bar{\Delta}^a_m\exp\{(i/\hbar)S_{m,\text{ext}}\} = 0, \quad \bar{\Delta}_\alpha\exp\{(i/\hbar)S_{m,\text{ext}}\} = 0, \qquad (8.2.25)$$

i.e. the gauge-fixed action $S_{m,\text{ext}}$ satisfies the same generating equations (8.2.22) and (8.2.23) as S_m.

8.3 Vacuum functional

Now let us define the vacuum functional the theory in question

$$Z_m(0) = \int d\phi^A \exp\{(i/\hbar) S_{m,\mathrm{eff}}(\phi^A)\},$$

where

$$S_{m,\mathrm{eff}}(\phi^A) = S_{m,\mathrm{ext}}(\phi^A, \phi^*_{Aa}, \bar{\phi}_A, \eta_A)|_{\phi^*_a = \bar{\phi} = \eta = 0}.$$

It can be represented in the form

$$Z_m(0) = \int d\phi^A \, d\eta_A \, d\phi^*_{Aa} \, d\pi^{Aa} \, d\bar{\phi}_A \, d\lambda^A \, \delta(\eta_A) \exp\{(i/\hbar)(S_{m,\mathrm{ext}} + W_X)\} \qquad (8.3.26)$$

with

$$W_X = (\eta_A - \tfrac{1}{2} m^2 (P_+)^B_A \bar{\phi}_B)\phi^A + \phi^*_{Aa}\pi^{Aa} + \bar{\phi}_A(\lambda^A - \tfrac{1}{2} m^2 (P_-)^A_B \phi^B).$$

where we have extended the space of variables by introducing the auxiliary fields π^{Aa} and λ^A.

Then we express $\delta(\eta_A)$ by

$$\delta(\eta_A) = \int d\zeta^A \exp\{(i/\hbar)\eta_A \zeta^A\}$$

and change in (8.3.26) the integration variables ϕ^A and λ^A according to $\phi^A \to \phi^A + \zeta^A$ and $\lambda^A \to \lambda^A + \frac{1}{2} m^2 ((P_-)^A_B - (P_+)^A_B)\zeta^B$. Then, for $Z_m(0)$ this yields

$$Z_m(0) = \int d\phi^A \, d\eta_A \, d\zeta^A \, d\phi^*_{Aa} \, d\pi^{Aa} \, d\bar{\phi}_A \, d\lambda^A \exp\{(i/\hbar)(S^{\zeta}_{m,\mathrm{ext}} + W_X)\},$$

where $S^{\zeta}_{m,\mathrm{ext}}$ is obtained from $S_{m,\mathrm{ext}}$ by performing the replacement $\phi^A \to \phi^A + \zeta^A$.

The term W_X may be cast into the $osp(1,2)$-invariant form

$$W_X = (\tfrac{1}{2}\epsilon_{ab}(V^b_m - U^b_m)(V^a_m - U^a_m) + m^2)X, \quad X = \bar{\phi}_A \phi^A, \quad (V_\alpha + U_\alpha)X = 0,$$

with V^a_m and V_α defined in (8.1.11) and (8.1.12), satisfying the $osp(1,2)$-superalgebra

$$[V_\alpha, V_\beta] = \epsilon_{\alpha\beta}{}^{\gamma} V_\gamma, \qquad [V_\alpha, V^a_m] = V^b_m (\sigma_\alpha)_b{}^a, \qquad \{V^a_m, V^b_m\} = -m^2 (\sigma^\alpha)^{ab} V_\alpha$$

and the operators U^a_m and U_α are defined according to

$$\begin{aligned} U^a_m &= -(-1)^{\epsilon_A} \pi^{Aa} \frac{\delta_l}{\delta\phi^A} + (-1)^{\epsilon_A} \epsilon^{ab} \lambda^A \frac{\delta_l}{\delta\pi^{Ab}} + \\ & (-1)^{\epsilon_A} m^2 \epsilon^{ac} (P_+)^{Ab}_{Bc} \phi^B \frac{\delta_l}{\delta\pi^{Ab}} - (-1)^{\epsilon_A} m^2 (P_-)^{Aa}_{Bb} \pi^{Bb} \frac{\delta_l}{\delta\lambda^A} \end{aligned} \qquad (8.3.27)$$

$$\begin{aligned} U_\alpha &= \phi^B (\sigma_\alpha)_B{}^A \frac{\delta_l}{\delta\phi^A} + \lambda^B (\sigma_\alpha)_B{}^A \frac{\delta_l}{\delta\lambda^A} + \\ & \left(\pi^{Ab}(\sigma_\alpha)_b{}^a + \pi^{Ba}(\sigma_\alpha)_B{}^A\right) \frac{\delta_l}{\delta\pi^{Aa}} + \zeta^B (\sigma_\alpha)_B{}^A \frac{\delta_l}{\delta\zeta^A}. \end{aligned} \qquad (8.3.28)$$

The operators U_m^a and U_α obey the $osp(1,2)$-superalgebra as well

$$[U_\alpha, U_\beta] = -\epsilon_{\alpha\beta}{}^\gamma U_\gamma, \qquad [U_\alpha, U_m^a] = -U_m^b(\sigma_\alpha)_b{}^a, \qquad \{U_m^a, U_m^b\} = m^2(\sigma^\alpha)^{ab}U_\alpha$$

Inserting into (8.3.26) the relation (8.2.24) and integrating by parts this yields

$$Z_m(0) = \int d\phi^A\, d\eta_A\, d\zeta^A\, d\phi^*_{Aa}\, d\pi^{Aa}\, d\bar{\phi}_A\, d\lambda^A \exp\{(i/\hbar)(S_m^\zeta + W_F^\zeta + W_X)\} \qquad (8.3.29)$$

with the following expression for W_F:

$$W_F = -\frac{\delta F}{\delta\phi^A}(\lambda^A + \tfrac{1}{2}m^2(P_+)^A_B\phi^B) - \tfrac{1}{2}\epsilon_{ab}\pi^{Aa}\frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\pi^{Bb} + m^2 F$$

which may be recast into the $osp(1,2)$-invariant form

$$W_F = (\tfrac{1}{2}\epsilon_{ab}U_m^b U_m^a + m^2)F, \quad U_\alpha F = 0.$$

(Again, S_m^ζ and W_F^ζ are obtained from S_m and W_F, respectively, by carrying out the replacement $\phi^A \to \phi^A + \zeta^A$.)

8.4 Global supersymmetries

We assert now that (8.3.26) is invariant under the following (global) transformations:

$$\begin{aligned}
\delta_m\phi^A &= \mu_a U_m^a\phi^A, \quad \delta_m\zeta^A = 0, \quad \delta_m\bar{\phi}_A = \mu_a V_m^a\bar{\phi}_A, \\
\delta_m\pi^{Ab} &= \mu_a U_m^a\pi^{Ab}, \quad \delta_m\phi^*_{Ab} = \mu_a V_m^a\phi^*_{Ab} + \mu_a(S_m^\zeta, \phi^*_{Ab})^a, \\
\delta_m\lambda^A &= \mu_a U_m^a\lambda^A U_m^a, \quad \delta_m\eta_A = \mu_a V_m^a\eta_A,
\end{aligned} \qquad (8.4.30)$$

where μ_a, $\epsilon(\mu_a) = 1$, is a $Sp(2)$-doublet of constant anticommuting parameters. The transformations (8.4.30) realize the m-extended BRST symmetry in the space of variables ϕ^A, $\bar{\phi}_A$, ϕ^*_{Aa}, η_A, π^{Aa}, λ^A and ζ^A.

Moreover, it is straightforward to check that (8.3.26) is also invariant under the following transformations:

$$\begin{aligned}
\delta\phi^A &= \theta^\alpha U_\alpha\phi^A, \quad \delta\zeta^A = \theta^\alpha U_\alpha\zeta^A, \quad \delta\bar{\phi}_A = \theta^\alpha V_\alpha\bar{\phi}_A, \\
\delta\pi^{Ab} &= \theta^\alpha U_\alpha\pi^{Ab}, \quad \delta\phi^*_{Ab} = \theta^\alpha V_\alpha\phi^*_{Ab}, \quad \delta\lambda^A = \theta^\alpha U_\alpha\lambda^A, \\
\delta\eta_A &= \theta^\alpha V_\alpha\eta_A + \theta^\alpha\{S_m^\zeta, \eta_A\}_\alpha,
\end{aligned} \qquad (8.4.31)$$

where θ^α, $\epsilon(\theta^\alpha) = 0$, are constant commuting parameters. The transformations (8.4.31) realize the $Sp(2)$-symmetry in the space of variables ϕ^A, $\bar{\phi}_A$, ϕ^*_{Aa}, η_A, π^{Aa}, λ^A and ζ^A.

In principle, for a general gauge functional F, μ_a may be assumed to depend on all these variables ϕ^A, $\bar{\phi}_A$, ϕ^*_{Aa}, η_A, π^{Aa}, λ^A and ζ^A. As long as F depends only on the fields it is sufficient for μ_a to depend on ϕ^A and π^{Aa} only. Then the symmetry of the vacuum functional $Z_m(0)$ with respect to the transformations (8.4.30) and (8.4.31) permits to study the question whether the mass dependent terms of the action violate the independence of the S-matrix on the choice of the gauge.

Indeed, let us change the gauge-fixing functional $F(\phi) \to F(\phi) + \delta F(\phi)$. Then the gauge-fixing term W_F changes according to

$$W_F \to W_{F+\delta F} = W_F + W_{\delta F}, \quad W_{\delta F} = (\tfrac{1}{2}\epsilon_{ab}U_m^b U_m^a + m^2)\delta F(\phi). \qquad (8.4.32)$$

Now, performing in (8.3.29) the transformations (8.4.30), we choose

$$\mu_a = \mu_a(\phi, \pi) \equiv -\frac{i}{2\hbar}\epsilon_{ab} U^b_m \delta F(\phi).$$

This induces the factor $\exp(U^a_m \mu_a)$ in the integration measure. Combining its exponent with W_F leads to

$$W_F \to W_F + (\hbar/i)\mu_a U^a_m = W_F - \tfrac{1}{2}\epsilon_{ab} U^b_m U^a_m \delta F(\phi) = W_F - W_{\delta F} + m^2 \delta F(\phi).$$

By comparison with (8.4.32) we see that the mass term $m^2 F$ in W_F violates the independence of the vacuum functional $Z_m(0)$ on the choice of the gauge. This result, together with the equivalence theorem [135], is sufficient to prove that the same is true also for the S-matrix.

One may try to compensate this undesired term $m^2 \delta F(\phi)$ by means of an additional change of variables. But this change should not destroy the form of the action arrived at the previous stage. However, an additional change of variables leads to a Berezinian which is equal to one because σ_α are traceless. Therefore, the unwanted term could never be compensated and the S-matrix within this formalism becomes gauge dependent when $m \neq 0$. It means to obtain physical results in this formalism one needs after performing of all calculations to take the limit $m \to 0$ and to wait for the gauge independence of the S-matrix.

8.5 Ward identities

Finally, we shall derive the Ward identities for the extended BRST- and the $Sp(2)$-symmetries. To begin with, let us introduce the generating functional of the Green functions:

$$Z_m(J_A; \phi^*_{Aa}, \bar\phi_A, \eta_A) = \int d\phi^A \exp\{(i/\hbar)\big(S_{m,\mathrm{ext}}(\phi^A, \phi^*_{Aa}, \bar\phi_A, \eta_A) + J_A \phi^A\big)\}. \quad (8.5.33)$$

If we multiply Eqs. (8.2.25) from the left by $\exp\{(i/\hbar)J_A\phi^A\}$ and integrate over ϕ^A we get

$$\begin{aligned} \int d\phi^A \exp\{(i/\hbar)J_A\phi^A\}\bar\Delta^a_m \exp\{(i/\hbar)S_{m,\mathrm{ext}}(\phi^A, \phi^*_{Aa}, \bar\phi_A, \eta_A)\} &= 0, \\ \int d\phi^A \exp\{(i/\hbar)J_A\phi^A\}\bar\Delta_\alpha \exp\{(i/\hbar)S_{m,\mathrm{ext}}(\phi^A, \phi^*_{Aa}, \bar\phi_A, \eta_A)\} &= 0. \end{aligned} \quad (8.5.34)$$

Now, integrating by parts and assuming the integrated expressions to vanish, we can rewrite the resulting equalities by the help of the definition (8.5.33) as

$$\begin{aligned} (J_A \frac{\delta}{\delta \phi^*_{Aa}} - V^a_m) Z_m(J_A; \phi^*_{Aa}, \bar\phi_A, \eta_A) &= 0, \\ \big((\sigma_\alpha)_B{}^A J_A \frac{\delta}{\delta \eta_B} - V_\alpha\big) Z_m(J_A; \phi^*_{Aa}, \bar\phi_A, \eta_A) &= 0, \end{aligned} \quad (8.5.35)$$

which are the Ward identities for the generating functional of Green's functions.

Introducing as usual the generating functional of the vertex functions,

$$\begin{aligned} \Gamma_m(\phi^A; \phi^*_{Aa}, \bar\phi_A, \eta_A) &= (\hbar/i)\ln Z_m(J_A; \phi^*_{Aa}, \bar\phi_A, \eta_A) - J_A\phi^A, \\ \phi^A &= (\hbar/i)\frac{\delta \ln Z_m(J_A; \phi^*_{Aa}, \bar\phi_A, \eta_A)}{\delta J_A}, \end{aligned} \quad (8.5.36)$$

we obtain

$$\tfrac{1}{2}(\Gamma_m, \Gamma_m)^a + V_m^a \Gamma_m = 0, \qquad \tfrac{1}{2}\{\Gamma_m, \Gamma_m\}_\alpha + V_\alpha \Gamma_m = 0. \tag{8.5.37}$$

For Yang-Mills theories the first identities in (8.5.37) are the Slavnov-Taylor identities of the extended BRST symmetries. Furthermore, choosing for σ_α the representation mentioned above the second identities in (8.5.37) express for $\alpha = 0$ the ghost number conservation and, in Yang-Mills theories, for $\alpha = (+, -)$ the Delduc-Sorella identities of the $Sp(2)$-symmetry [102].

Appendix A

Lie Groups and Lie Algebras, Lie Superalgebras and Lie Groups with Grassmann Structure

A *Lie group* G is defined by the following properties:

1. G is an abstract group,
2. G is an analytic manifold of dimension $d_G = n$, i.e. their elements depend analytically on the local group parameters, $g(\underline{\xi}), \underline{\xi} = (\xi_1, \ldots, \xi_n)$,
3. the map $\big(g(\underline{\xi}), g(\underline{\xi}')\big) \mapsto g(\underline{\xi})g^{-1}(\underline{\xi}')$ is analytic.

Usually the parametrization will be chosen such that $g(\underline{0}) = e$ (the unit element).

A Lie group may be considered as the group of continuous transformations acting on some (vector) space V with elements $x \in V$ according to

$$g(\underline{\xi}) : \; x \mapsto x'(\underline{\xi}) = (gx)(\underline{\xi}) \quad \text{with} \quad x = (gx)(0). \tag{A.0.1}$$

Given a linear independent basis $\{e_i\}$ of the space V the infinitesimal transformations of the coordinates in this basis, $x = x^i e_i$, are given by

$$dx^i = u_a^i(x) d\xi^a \quad \text{with} \quad u_a^i(x) = \left. \frac{\partial (gx)^i}{\partial \xi^a} \right|_{\underline{\xi}=0}. \tag{A.0.2}$$

An infinitesimal change of a function $F(x)$ on V is given by

$$dF(x) = \frac{\partial F}{\partial x^i} dx^i = d\xi^a X_a F \quad \text{with} \quad X_a = u_a^i(x) \frac{\partial}{\partial x^i} \tag{A.0.3}$$

being the *infinitesimal generators* of the Lie group. The quantities $u_a^i(x)$ define a *velocity field* on the space V which determine the *orbit* of x under the group actions generated by X_a; the condition of integrability reads

$$u_a^j(x) \frac{\partial u_b^i(x)}{\partial x^j} - u_b^j(x) \frac{\partial u_a^i(x)}{\partial x^j} = f_{ab}{}^c u_c^i(x) \quad \text{with} \quad f_{ab}{}^c = -f_{ba}{}^c; \tag{A.0.4}$$

the quantities $f_{ab}{}^{c}$ are called *structure constants* of the Lie group. The equation (xxx) for the generators of the gauge transformations on the functionals of the (bosonic) fields is a generalization of (A.0.4).

The infinitesimal generators X_a of the Lie group form a linear independent basis of the *Lie algebra* Lie (G) of the group G. Because of eq. (A.0.4) the obey the following commutation relations

$$[X_a, X_b] = f_{ab}{}^{c} X_c \tag{A.0.5}$$

which uniquely determine the Lie algebra (having arbitrary elements $X = \xi^a X_a \in \text{Lie}\,(G)$) with the Lie product defined by

$$(\text{Lie}\,(G), \text{Lie}\,(G)) \ni (X, Y) \mapsto X \circ Y \equiv [X, Y] \in \text{Lie}\,(G) \quad \forall X, Y \in \text{Lie}\,(G). \tag{A.0.6}$$

Because of the *Jacobi identity* an analogous relation for the structure constants follow:

$$[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] \equiv 0 \tag{A.0.7}$$
$$f_{ab}{}^{d} f_{dc}{}^{e} + f_{bc}{}^{d} f_{da}{}^{e} + f_{ca}{}^{d} f_{db}{}^{e} \equiv 0. \tag{A.0.8}$$

A Lie group is called *abelian* if all its generators commute, i.e. if $f_{ab}{}^{c} \equiv 0$. A subset of the generators, $X_\rho, \rho = 1, \ldots, r < n$ generates a subgroup $H \subset G$ iff $f_{\rho\sigma}{}^{\tau} = 0$ for $\rho, \sigma \leq r, \tau > r$; this subgroup is called an *invariant subgroup* iff $f_{\rho\sigma}{}^{\tau} = 0$ for $\rho \leq r, \tau > r$.

By the help of the structure constants a symmetric second rank tensor g_{ab}, the so-called *Cartan metric*, can be introduced:

$$g_{ab} = f_{ad}{}^{c} f_{bc}{}^{d}, \tag{A.0.9}$$

which serves to specify the Lie groups. A Lie group is called *semi-simple* iff the Cartan metric is non-degenerate, i.e. $\det|g_{ab}| \neq 0$, and it is *compact* if the Cartan metric is positive (or negative) definite. Furthermore, by the help of the Cartan metric the group indices can be raised and lowered. Especially, it can be shown that

$$f_{abc} = f_{ab}{}^{d} g_{dc} \tag{A.0.10}$$

can be choosen totally antisymmetric; from this it follows that a semi-simple Lie group does not have any abelian invariant subgroup (besides the unit element). A Lie group is called *simple* if it has no invariant subgroup besides the unit element. In the case of semi-simple Lie groups the connection withGrassmann Variables, Berezinian and All That the correcponding Lie algebra is given by [1]

$$g(\underline{\xi}) = \exp\{\xi^a X_a\} \quad \text{with} \quad X_a = \left.\frac{\partial g(\underline{\xi})}{\partial \xi^a}\right|_{\underline{\xi}=0} \tag{A.0.11}$$

and the generators X_a are skew-hermitian $X_a^\dagger = -X_a$.

Let us furthermore note that by the help of the Cartan metric an infinitesimal line element on the group manifold is defined,

$$ds^2(\underline{\xi}) = g_{ab} d\xi^a d\xi^b, \tag{A.0.12}$$

[1] Contrary to the normal use in physical context where the generators of the group transformations are taken to be hermitian operators and being related to the observables of the theory here we have taken the mathematicians convention. therwise, we had to to change the generators according to $X_a \to -iX_a$.

which is left (and right) invariant under the action of the group. Therefore, the group manifold is a Riemannian (or pseudo Riemannian) space if the metric is definite (or indefinite). If the group is compact there exists a left (and right) invariant measure $\mu(\cdot)$, the *Haar measure*, such that for any function f there exists an integral over the group,

$$I_\mu(f) = \int_G f(g)d\mu(g) \quad \text{with} \quad d\mu(g_0 g) = d\mu(g g_0) = d\mu(g); \tag{A.0.13}$$

in terms of the group parameters it is given according to

$$d\mu(g) = \prod_a d\xi^a \rho(\underline{\xi}) = \prod_a d\xi'^a \rho(\underline{\xi}') \tag{A.0.14}$$

with the measure function $\rho(\underline{\xi})$ ensuring invariance under parameter changes. Of course, the group volume is given by $\mu(1) = \mathrm{vol}\,(G) < \infty$. That measure may be constructed by starting with the observation that

$$\omega(g^{-1}, dg) := g^{-1}(\underline{\xi})dg(\underline{\xi}) \equiv g^{-1}(\underline{\xi})\frac{\partial g(\underline{\xi})}{\partial \xi^a}d\xi^a = \omega^a(\underline{\xi})X_a \tag{A.0.15}$$

defines a left (and right) invariant differential form on the Lie algebra, such that the volume element may be given by

$$dg = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n = \rho(\underline{\xi})d\xi^1 \wedge d\xi^2 \wedge \ldots \wedge d\xi^n. \tag{A.0.16}$$

Remark: In the case of Euler's parametrization of the group $SO(3)$,

$$g(\phi, \theta, \psi) = R_z(\phi)R_y(\theta)R_z(\psi), \qquad 0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi, \quad 0 \le \psi \le 2\pi,$$

we obtain for the normalized measure

$$dg = \frac{1}{8\pi^2}\sin\theta d\phi d\theta d\psi, \qquad \mathrm{vol}\, G = 1.$$

Because of the Jacobi identity the structure constants determine a (matrix) representation of the Lie algebra, the so-called *adjoint* or regular representation,

$$\mathrm{Lie}\,(G) \ni X \mapsto \mathrm{ad}\, X : [X, Y] = (\mathrm{ad}\, X)Y \tag{A.0.17}$$

which is uniquely determined through the following equivalence for their basis elements: $(X_a)_b{}^c \equiv (\mathrm{ad}X_a)_b{}^c = f_{ab}{}^c$. It can be understood as the action of the (fixed) element $\xi^a X_a$ of the Lie algebra on an arbitrary basis element X_b to give another basis element X_c according to $\xi^a\,[X_a, X_b] = \xi^a(\mathrm{ad}X_a)_b{}^c X_c$, i.e. the Lie algebra itself – by their property to be a vector space – serves as representation space for arbitrary elements $X = \xi^a X_a$ of the Lie algebra. In terms of the adjoint representation it is possible to introduce a bilinear form, the *Killing form* which, taken for the basis elements, defines the Cartan metric:

$$K(X, Y) := \mathrm{tr}\,((\mathrm{ad}X) \cdot (\mathrm{ad}Y)) \quad \Longrightarrow \quad g_{ab} = K(X_a, X_b). \tag{A.0.18}$$

The adjoint representation of the Lie group G is defined on Lie(G) according to

$$\mathrm{Ad}\, g :\; X \mapsto g\, X\, g^{-1} \qquad \forall g \in G, \quad X \in \mathrm{Lie}(G), \tag{A.0.19}$$

or, equivalently, in the case of semi-simple Lie groups

$$\mathrm{Ad}\, g(\underline{\xi}) = \exp\left\{\xi^a(\mathrm{ad}X_a)\right\}. \tag{A.0.20}$$

The above introduced notions of Lie groups and Lie algebras have a natural extension to Z_2–graded Lie supergroups and Lie superalgebras. Let us first consider the latter ones because they are used in Chapter 8. A *Lie superalgebra* $\mathcal{G}$ (over the field R or C) is an associative, Z_2–graded algebra that is a direct sum of two vector spaces $\mathcal{G}_{(e)} \equiv \mathcal{G}_{(0)}$ and $\mathcal{G}_{(o)} \equiv \mathcal{G}_{(1)}$ of even and odd elements, respectively, in which a (Z_2–graded) product, the so-called *Lie superbracket* or *supercommutator*, $[\cdot,\cdot\}$, is defined with

$$\left[\mathcal{G}_{(i)}, \mathcal{G}_{(j)}\right\} \subset \mathcal{G}_{(k)} \quad \text{with} \quad k = i + j (\text{mod} 2), \qquad (Z_2 - \text{gradation}), \qquad (A.0.21)$$

$$[X_i, X_j\} = -(-1)^{\epsilon(X_i)\epsilon(X_j)} [X_j, X_i\}, \qquad (\text{graded antisymmetry}), \qquad (A.0.22)$$

$$(-1)^{\epsilon(X_i)\epsilon(X_k)} [X_i, [X_j, X_k\}\} + (-1)^{\epsilon(X_j)\epsilon(X_i)} [X_j, [X_k, X_i\}\}$$
$$+(-1)^{\epsilon(X_k)\epsilon(X_j)} [X_k, [X_i, X_i\}\} \equiv 0, \qquad (Z_2 - \text{graded Jacobi identity}) \qquad (A.0.23)$$

If $\epsilon(X_i)\epsilon(X_j) = 0$ the supercommutator coincides with the usual commutator, otherwise it is the anticommutator, i.e.

$$[X, Y\} = XY - (-1)^{\epsilon(X)\epsilon(Y)} YX, \qquad \forall\, X, Y \in \mathcal{G}. \qquad (A.0.24)$$

The even (or bosonic) part $\mathcal{G}_0$ of a Lie superalgebra $\mathcal{G}$ is an ordinary Lie algebra, while the odd (or fermionic) part $\mathcal{G}_1$ is not an algebra but, because of $\mathcal{G}_0\, \mathcal{G}_1 \subset \mathcal{G}_1$, it is a module where $\mathcal{G}_0$ is represented. To be more explicit let us write a Lie superalgebra as follows:

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^{\gamma} X_\gamma, \quad [X_\alpha, Y_a] = f_{\alpha a}{}^{b} Y_b, \quad \{Y_a, Y_b\} = f_{ab}{}^{\gamma} X_\gamma. \qquad (A.0.25)$$

Here we used the obvious notation: $X_\alpha \in \mathcal{G}_0, Y_a \in \mathcal{G}_1$.

The most simple example of a Lie superalgebra, generalizing the simplest nontrivial Lie algebra $su(2)$, is the *orthosymplectic* superalgebra $osp(1,2)$ which plays a crucial role in Chapter 8 of this book.

The supercommutation relations of the superalgebra $osp(1,2)$ in the Cartan-Weyl basis according to (A.0.25) read:

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_0, \qquad (A.0.26)$$

$$[L_0, R_\pm] = \pm\tfrac{1}{2} R_\pm, \qquad [L_\pm, R_\mp] = -R_\pm, \qquad [L_\pm, R_\pm] = 0, \qquad (A.0.27)$$

$$\{R_\pm, R_\pm\} = \pm\tfrac{1}{2} L_\pm, \qquad \{R_+, R_-\} = \tfrac{1}{2} L_0, \qquad (A.0.28)$$

and for the fundamental representation these generators are given by:

$$L_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; L_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; L_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$R_+ = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \; R_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

The superalgebra (A.0.26)–(A.0.28) in the main part is obtained by the following identifications:

$$(\hbar/i)\overline{\Delta}_0 = 2L_0, \qquad (\hbar/i)\overline{\Delta}_\pm = L_\pm, \qquad (\hbar/i)\overline{\Delta}^1_m = 2mR_+, \qquad (\hbar/i)\overline{\Delta}^2_m = 2mR_-;$$

so the relations (A.0.26)–(A.0.28) may be rewritten as follows ($\alpha = 0, +, -$):

$$[\overline{\Delta}_\alpha, \overline{\Delta}_\beta] = (i/\hbar)\epsilon_{\alpha\beta\gamma}\overline{\Delta}^\gamma, \qquad (A.0.29)$$

$$[\overline{\Delta}_\alpha, \overline{\Delta}^a_m] = (i/\hbar)\overline{\Delta}^b_m(\sigma_\alpha)_b{}^a, \qquad (A.0.30)$$

$$\{\overline{\Delta}^a_m, \overline{\Delta}^b_m\} = -(i/\hbar)m^2(\sigma_\alpha)^{ab}\overline{\Delta}^\alpha. \qquad (A.0.31)$$

Here, eq. (A.0.29) is a realization of the Lie algebra $sl(2)$ with the antisymmetric structure coefficients $\epsilon_{\alpha\beta\gamma}$ which are determined by $\epsilon_{0+-} = 1$; eq. (A.0.30) determines the fundamental representation $(\sigma_\alpha)_b{}^a$ of that Lie algebra with $\sigma_\alpha\sigma_\beta = g_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta\gamma}\sigma^\gamma$, on the spinorial doublet of the odd generators $\overline{\Delta}^a_m$; and eq. (A.0.31) constitutes the anticommutator of the odd generators with the structure coefficients given by $(\sigma_\alpha)^{ab}$. Raising and lowering of indicies is obtained by

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta; \quad \epsilon^{ab} = -\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{ac}\epsilon_{cb} = \delta^a_b.$$

From eqs. (A.0.27) and (A.0.28) the following realization of the structure coefficients in eqs. (A.0.30) and (A.0.31) may be read off:

$$(\sigma_+)_a{}^b = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad (\sigma_-)_a{}^b = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad (\sigma_0)_a{}^b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and by raising the first index according to $(\sigma_\alpha)^{ab} = \epsilon^{ac}(\sigma_\alpha)_c{}^b$ we get

$$(\sigma_+)^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (\sigma_-)^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (\sigma_0)^{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The quadratic Casimir operator of the supergroup $osp(1,2)$ is given by $C_2 = \frac{1}{2}\epsilon_{ab}\overline{\Delta}^b_m\overline{\Delta}^a_m + m^2\overline{\Delta}^\alpha\overline{\Delta}_\alpha$.

A *(linear) representation* π of a Lie superalgebra $\mathcal{G}$ is obtained as a homomorphism of $\mathcal{G}$ into the superalgebra of endomorphisms of a Z_2–graded vector space $\mathcal{V} = \mathcal{V}_{(0)} \oplus \mathcal{V}_{(1)}$ such that

$$\pi(cX) = c\pi(X), \quad \pi(X+Y) = \pi(X) + \pi(Y), \qquad (A.0.32)$$

$$\pi([X,Y\}) = [\pi(X), \pi(Y)\} \qquad (A.0.33)$$

$\forall X, Y \in \mathcal{G}, c \in C$. The *dimension* resp. *superdimension* of the representation is the dimension resp. graded dimension of the vector space $\mathcal{V}$, i.e.

$$\dim\pi = \dim\mathcal{V}_0 + \dim\mathcal{V}_1, \quad \mathrm{sdim}\,\pi = \dim\mathcal{V}_0 - \dim\mathcal{V}_1. \qquad (A.0.34)$$

The *adjoint representation* is obtain according to

$$(\mathrm{ad}\,X)\,Y := [X, Y\}, \quad \text{i.e.} \quad \mathrm{ad} : \mathcal{G} \mapsto \mathrm{End}\mathcal{G} \qquad (A.0.35)$$

and the *Killing form* is obtained by

$$K(X_i, X_j) = \mathrm{str}(\mathrm{ad}(X_i)\mathrm{ad}(X_j)) = (-1)^{\epsilon(X_j)}C_{im}{}^nC_{jn}{}^m = g_{ij} \qquad (A.0.36)$$

The Killing form is an *inner product*, this means that it is
consistent, i.e. $K(X,Y) = 0\,\forall X \in \mathcal{G}_0$, $\forall Y \in \mathcal{G}_1$,
supersymmetric, i.e. $K(X,Y) = (-1)^{\epsilon(X)\epsilon(Y)} K(Y,X)$, and
invariant, i.e. $K([X,Y\},Z) = K(X,[Y,Z\})$. There are analogous results as in the case of ordinary Lie algebras. Namely, it holds

(1) A Lie superalgebra $\mathcal{G}$ with a non-degenerate Killing form is a direct sum of simple Lie superalgebras each having a non-degenerate Killing form.

(2) A Lie superalgebra is called *simple* if it does not contain any non-trivial ideal.

(3) A necessary condition for a Lie superalgebra to be simple is that
(i) the representation of $\mathcal{G}_0$ on $\mathcal{G}_1$ is faithful and irreducible,
(ii) $\{\mathcal{G}_1, \mathcal{G}_1\} = \mathcal{G}_0$

A *Lie supergroup* or, more correctly, a *Lie group with Grassmann structure* is associated with a (simple) Lie superalgebra. Let $\mathsf{G}(n) = \mathsf{G}(n)_0 \oplus \mathsf{G}(n)_1$ be the complex Grassmann algebra of order n, and let $A(\mathsf{G}$ be the *Grassmann enveloppe* of a superalgebra $\mathcal{A}$) which consists of formal linear combinations $\sum_i \eta^i a_i$ of elements $\eta^i \in \mathsf{G}$ and $a_i \in \mathcal{A}$ both being either even or odd. Then, the commutator $[X,Y] := \sum_{ij} \eta^i \eta'^j [a_i, a_j\}$ confers $A(\mathsf{A})$ with a *Lie algebra* structure. Now, a supergroup A associated with the superalgebra $\mathcal{A}$ is – according to the definition of Berezin – the exponential map of the Grassmann envelope $A(\mathsf{A})$.

The Lie supergroups of linear transformations are obtained from the *even* (square) $(m+n) \times (m+n)$–supermatrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad A, D \quad \text{even}, \quad B, C \quad \text{odd}, \tag{A.0.37}$$

defining *supermatrix groups* for an arbitrary field K:
$GL(m,n|\mathsf{K}) \ni M$ being even and invertible
$SL(m,n|\mathsf{K}) \ni M : \mathrm{sdet} M = 1$
$U(m,n) : M \in GL(m,n|\mathsf{C}, M\, M^+ = 1$
$OSP(m, n = 2p|\mathsf{K}) : M^{\mathrm{st}} H M = H$ with

$$H = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2p} \end{pmatrix}, \qquad J_{2p} = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}; \tag{A.0.38}$$

a special one is the supergroup $OSP(1,2|\mathsf{C})$.

Appendix B

Path Integral Representation of Transition Amplitude

In Chapter 1 we presented the generating functional $Z(J)$ of Green's functions by the path integral (1.7.28) over trajectories in phase space. Its derivation rests on a corresponding representation of the matrix elements of the time ordered product $T\left(\hat{q}^{i_1}(t_1)...\hat{q}^{i_n}(t_n)\right)$ between eigenstates of the position operator. Here we like to present this derivation. Thereby we restrict ourselves to a quantum mechanical system with one degree of freedom only (for a more detailed exposition see, for example, [170]).

The *eigenstates* of the position operator are introduced as follows:

$$\begin{aligned} \hat{q}(t)|q,t\rangle &= q|q,t\rangle \quad \text{Heisenberg picture}, \\ \hat{q}_S|q\rangle &= q|q\rangle \quad \text{Schrödinger picture}, \end{aligned}$$

with the following connection between these states

$$|q\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|q,t\rangle,$$

where $\hat{H}$ denotes the Hamiltonian of the system. Therefore the matrix element

$$\langle q',t'|q,t\rangle = \langle q'|\exp\left(-\frac{i}{\hbar}\hat{H}(t'-t)\right)|q\rangle \tag{B.0.1}$$

corresponds to the transition from the eigenstate $|q\rangle$ at the moment t to the eigenstate $|q'\rangle$ at the moment t', and it defines a Green function. Namely, introducing $|t\rangle$ as the solution of the Schrödinger equation $\hat{H}|t\rangle = i\hbar\partial|t\rangle/\partial t$, then

$$\langle q'|t'\rangle = \int dq\,\langle q'|\exp\left(-\frac{i}{\hbar}\hat{H}(t'-t)\right)|q\rangle\langle q|t\rangle$$

describes the time evolution of Schrödinger's wave function $\langle q|t\rangle$.

Let us first show how the matrix element (B.0.1) can be represented as a functional integral; afterwarts its relation to the Green functions (1.24) will be given. We start by representing (B.0.1) as a multiple integral from which, by some limiting procedure, the corresponding functional integral is obtained.

To begin with we divide the time interval $(t'-t)$ into $(n+1)$ equal parts with length ϵ, i.e.

$$t' = t + (n+1)\epsilon, \quad \text{and} \quad t_j = t + j\epsilon t, \quad (j = 1, ..., n).$$

Using the completeness relation at each moment t_j, $\int dq_j \, |q_j, t_j\rangle\langle q_j, t_j| = 1$, we represent the transition amplitude by

$$\langle q', t'|q, t\rangle = \int \prod_j dq_j \, \langle q', t'|q_n, t_n\rangle \cdots \langle q_{j_1}, t_{j_1}|q_{j-1}, t_{j-1}\rangle \cdots \langle q_1, t_1|q, t\rangle$$

together with

$$\langle q_j, t_j|q_{j-1}, t_{j-1}\rangle = \langle q_j|\exp\Big\{-\frac{i}{\hbar}\hat{H}\epsilon\Big\}|q_{j-1}\rangle = \langle q_j|q_{j-1}\rangle - \frac{i\epsilon}{\hbar}\langle q_j|\hat{H}|q_{j-1}\rangle + \mathcal{O}(\epsilon^2),$$

where q_0, q_{n+1}, t_0 and t_{n+1} are to be considered as q, q', t and t', respectively. Now, choosing the Hamiltonian $\hat{H} = H(\hat{p}, \hat{q})$ to be of the form $\hat{H} = T(\hat{p}) + V(\hat{q})$, we can write

$$\begin{aligned}\langle q_j|\hat{H}|q_{j-1}\rangle &= \int dp_j \, \langle q_j|p_j\rangle\langle p_j|\hat{H}|q_{j-1}\rangle \\ &= \int \frac{dp_j}{2\pi\hbar} \exp\Big\{\frac{i}{\hbar}p_j(q_j - q_{j-1})\Big\} H(p_j, q_{j-1}),\end{aligned}$$

where $H(p,q)$ is now the classical Hamiltonian. Using these equations we get

$$\begin{aligned}\langle q_j, t_j|q_{j-1}, t_{j-1}\rangle &= \int \frac{dp_j}{2\pi\hbar} \exp\Big\{\frac{i}{\hbar}p_j(q_j - q_{j-1})\Big\}\Big[1 - \frac{i}{\hbar}\epsilon H(p_j, q_{j-1})\Big] + \mathcal{O}(\epsilon^2) \\ &= \int \frac{dp_j}{2\pi\hbar} \exp\Big\{\frac{i}{\hbar}p_j(q_j - q_{j-1}) - \frac{i}{\hbar}\epsilon H(p_j, q_{j-1})\Big\} + \mathcal{O}(\epsilon^2) \qquad \text{(B.0.2)}\end{aligned}$$

and thus the following expression for the matrix element obtains (B.0.1):

$$\langle q', t'|q, t\rangle = \lim_{n\to\infty} \int \prod_{j=1}^{n} dq_j \int \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} \exp\Big\{\frac{i}{\hbar}\sum_{j=1}^{n+1}\Big[p_j(q_j - q_{j-1}) - H(p_j, q_{j-1})(t_j - t_{j-1})\Big]\Big\},$$

where the limit $n \to \infty (\epsilon \to 0)$ has been assumed, with the $\mathcal{O}(\epsilon^2)$ terms neglected.

This result will be represented in the compact form

$$\langle q', t'|q, t\rangle = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p \exp\Big\{\frac{i}{\hbar}\int_t^{t'} d\tau \, [p\dot{q} - H(p,q)]\Big\}, \qquad \text{(B.0.3)}$$

where the expression

$$\int_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p \equiv \int \prod_\tau \Big(\frac{dq(\tau)dp(\tau)}{2\pi\hbar}\Big)$$

is referred to as functional integration over the entire phase space, with the boundary conditions taken as $q(t) = q, q(t') = q'$. Let us point to the fact that the whole manifold of curves to be integrated over are given by (limits of) continuous curves $q(t)$ in configuration space and piecewise constant curves $p(t)$ in momentum space. Furthermore, we remark that this derivation has been given for a special functional form of the Hamiltonian only. The final result, however, is assumed to be true for any Hamiltonian.

If the Hamiltonian has the simple form $H = p^2/2m + V(q)$, the integration over momenta in (B.0.2) can be performed: Shifting the integration variables, $p_j \to p_j - m(\triangle q_j/\epsilon)$, we obtain by Gaussian integration

$$\int \frac{dp_j}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}(p_j \triangle q_j - \frac{p_j^2}{2m}\epsilon)\right\} = \exp\left\{\frac{i}{\hbar}\epsilon\frac{m}{2}\left(\frac{\triangle q_j}{\epsilon}\right)^2\right\}$$

where $\triangle q_j = q_j - q_{j-1}$ and

$$\frac{1}{N_j} = \int \frac{dp_j}{2\pi\hbar} \exp\left\{-\frac{i}{\hbar}\frac{p_j^2}{2m}\epsilon\right\}.$$

The final result has the form of a functional integral over the configuration space

$$\langle q', t'|q, t\rangle = \frac{1}{N} \int\limits_{q(t)=q}^{q(t')=q'} \mathcal{D}q \exp\left\{\frac{i}{\hbar}S[q]\right\}. \tag{B.0.4}$$

Here, $S[q] = \int_t^{t'} d\tau\, L(q, \dot{q})$ is the action integral over the trajectory $q(\tau)$, where $L(q, \dot{q}) = m\dot{q}^2/2 - V(q)$ is the Lagrange function, and the normalization factor N is given by

$$\frac{1}{N} = \int \mathcal{D}p \exp\left\{-\frac{i}{\hbar}\int_t^{t'} d\tau\, \frac{p^2}{2m}\right\} \quad \text{with} \quad \mathcal{D}p \equiv \prod_\tau \left(\frac{dp(\tau)}{2\pi\hbar}\right). \tag{B.0.5}$$

The matrix element $\langle q', t'|q, t\rangle$ determines all transition probabilities between quantum mechanical states. In view of further applications of the functional formalism to quantum field theories it is important also to know the path integral representation of the matrix elements of the product of position operators, corresponding to the product of field operators in quantum field theory. For the time–ordered product of n such operators the following expression holds:

$$\langle q', t'|T\Big(\hat{q}(t_1)\cdots\hat{q}(t_n)\Big)|q, t\rangle = \int\limits_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p\, q(t_1)...q(t_n) \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau\, [p\dot{q} - H(p, q)]\right\} \tag{B.0.6}$$

Let us check eq. (B.0.6) for the product of two operators: $\hat{q}(\tau_1)\hat{q}(\tau_2)$ at $\tau_1 > \tau_2$. Here again, we divide the time axis into small intervals, choosing $t_1...t_n$ in such a way that

$$\tau_1 = t_{i_1}, \quad \tau_2 = t_{i_2},$$

and then we apply the relation of completeness at each t_i. We thus have

$$\begin{aligned}
\langle q', t'|\hat{q}(\tau_1)\hat{q}(\tau_2)|q, t\rangle &= \int \prod_i dq_i \langle q', t'|q_n, t_n\rangle \cdots \langle q_{i_1}, t_{i_1}|\hat{q}(\tau_1)|q_{i_1-1}, t_{i_1-1}\rangle \cdots \\
&\qquad \cdots \langle q_{i_2}, t_{i_2}|\hat{q}(\tau_2)|q_{i_2-1}, t_{i_2-1}\rangle \cdots \langle q_1, t_1|q, t\rangle \\
&= \int \prod_i dq_i\, q_{i_1} q_{i_2} \langle q', t'|q_n, t_n\rangle \cdots \langle q_1, t_1|q, t\rangle\ .
\end{aligned}$$

Proceeding exactly as for the derivation of (B.0.3), we obtain the expression (B.0.6) for $n = 2$. Note that the last equation holds for $\tau_1 > \tau_2$. When $\tau_1 < \tau_2$, the r.h.s. of that equation corresponds to the matrix element $\langle q', t'|\hat{q}(\tau_2)\hat{q}(\tau_1)|q, t\rangle$. Therefore, the path integral, like (B.0.6), defines the matrix element of the time–ordered product of two position operators

$$\int\limits_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p\, q(t_1)q(t_2) \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau[p\dot{q} - H(p,q)]\right\} = \langle q', t'|T\Big(\hat{q}(t_1)\hat{q}(t_2)\Big)|q, t\rangle.$$

As before, it is possible to make a transition from path integrals over phase space to path integrals over configuration space.

Let us introduce also that the transition amplitude in the presence of an external source $J(\tau)$,

$$\langle q', t'|q, t\rangle^J = \int\limits_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p\ \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau[p\dot{q} - H(p,q) + J(\tau)q(\tau)]\right\}, \tag{B.0.7}$$

which corresponds to a Hamiltonian modified by a source term $H \to H - Jq$. It can be used as generating functional of the matrix elements of the position operators, which are given by the functional derivatives with respect to $J(\tau)$:

$$\langle q', t'|T\Big(\hat{q}(t_1)...\hat{q}(t_n)\Big)|q, t\rangle = \left(\frac{\hbar}{i}\right)^n \frac{\delta^n}{\delta J(t_1)...\delta J(t_n)}\langle q', t'|q, t\rangle^J|_{J=0}. \tag{B.0.8}$$

Let us now relate these matrix elements to the Green's functions, i.e. the *vacuum expection value* of the various products of position operators. Assume the Lagrangian L of the system to be (explicitly) time–independent. The energy eigenstates correspond to the wave functions $\Phi_n(q) = \langle q|n\rangle$. In particular, the ground state, or *the vacuum*, is described by the function $\Phi_0(q) = \langle q|0\rangle$. It will be convenient to use $\Phi_0(q,t)$ defined as

$$\Phi_0(q,t) = \exp\Big(-\frac{i}{\hbar}E_0 t\Big)\langle q|0\rangle = \langle q|\exp\Big(-\frac{i}{\hbar}\hat{H}t\Big)|0\rangle = \langle q, t|0\rangle.$$

We are interested in the matrix element

$$\langle 0|T\Big(\hat{q}(t_1)...\hat{q}(t_n)\Big)|0\rangle \ = \ \int dq' dq\, \Phi_0^*(q', t')\langle q', t'|T\hat{q}(t_1)...\hat{q}(t_n)|q, t\rangle\Phi_0(q,t).$$

Using for the matrix element $\langle q', t'|T\hat{q}(t_1)...\hat{q}(t_n)|q, t\rangle$ the functional form given by eq. (B.0.6) this may be written in the following way

$$\langle 0|T\Big(\hat{q}(t_1)...\hat{q}(t_n)\Big)|0\rangle = \left(\frac{\hbar}{i}\right)^n \frac{\delta^n}{\delta J(t_1)...\delta J(t_n)}Z(J)|_{J=0}, \tag{B.0.9}$$

where the generating functional $Z(J)$ is given by

$$Z(J) = \langle 0|0\rangle^J = \int dq' dq\, \Phi_0^*(q', t')\langle q', t'|q, t\rangle^J \Phi_0(q,t) \tag{B.0.10}$$

with $\langle q', t'|q, t\rangle^J$ defined by (B.0.7). However, because of the integration in eq. (B.0.10) over any value of q' and q, this is nothing else then

$$Z(J) = \int \mathcal{D}q\mathcal{D}p\ \exp\left\{\frac{i}{\hbar}\int dt\,[p\dot{q} - H(p,q) + Jq]\right\}, \tag{B.0.11}$$

where the integrations are taken over the whole space of trajectories in phase space.

The above results can be generalized to the case of more than one degree of freedom. If the number of degrees of freedom equals to N, the coordinate q should then be replaced by an N-component vector q^i. The functional integral now corresponds to the sum over all trajectories in the N-dimensional configuration space, satisfing appropriate boundary conditions.

Appendix C

Grassmann Variables, Berezinian and All That

In the Hamiltonian approach to quantum field theory fermionic fields, like Dirac fields, have be quantized by the canonical anticommutation relations. Correspondingly, in the formulation of quantum field theory by functional integrals one has to deal with (classical) anticommuting fields. These entities may be considered as fields over the Minkowski space having values in some algebra of supernumbers'. The appropriate mathematical theory is that of a Berezin algebra [47], [48] which will be introduced in the following. In addition we summarize some of their properties like differentiation, integration and change of variables, which will be relevant for a mathematical consistent formulation of quantum field theory in the functional formalism. (For a more detailed presentation see [47], [48], [49], [103].)
Let us first introduce a *Grassmann algebra* G as an associative algebra with unit over the field of complex numbers $\mathbf{C}$ which is generated by a finite (or infinite) set of anticommuting elements $\xi^\alpha, \alpha = 1, 2, \cdots, n$,

$$\xi^\alpha \xi^\beta + \xi^\beta \xi^\alpha = 0, \tag{C.0.1}$$

and being endowed with an involution. Every element of G may be written as

$$g \quad = \quad f_0 + f_\alpha \xi^\alpha + \cdots + f_{\alpha_1 \cdots \alpha_n} \xi^{\alpha_1} \cdots \xi^{\alpha_n} \quad \text{where} \quad f_0, \cdots, f_{\alpha_1 \cdots \alpha_n} \in \mathbf{C}. \tag{C.0.2}$$

The indices of the coefficients $f_0, \cdots, f_{\alpha_1 \cdots \alpha_n}$ because of (C.0.1) are assumed to be completely antisymmetric. The involution which in the operator formulation corresponds to hermition conjugation has to be required necessarily (therefore, after defining some conjugation of the generating elements, $g \in \mathsf{G}$ fulfil relations being equivalent to hermitian conjugation). The elements of that Grassmann algebra are the above mentioned supernumbers. In principle, the objects $f_0, \cdots, f_{\alpha_1 \cdots \alpha_n}$ could be elements of some function algebra, e.g. C^1 or C^∞ of continuous or infinitely differentiable functions; since we are interested in quantum field theory these functions are assumed to be defined over Minkowski space (or Euclidean space).
The *Berezin algebra* B is defined as the associative algebra with involution over the field $\mathbf{C}$ of complex numbers where the coefficients of the Grassmann variables are elements of some function algebra. Every element $\phi \in \mathsf{B}$, being a (generalized classical) field, can be represented in the form

$$\phi(x) \quad = \quad f_0(x) + f_\alpha(x) \xi^\alpha + f_{\alpha_1 \alpha_2}(x) \xi^{\alpha_1} \xi^{\alpha_2} + \cdots + f_{\alpha_1 \cdots \alpha_n}(x) \xi^{\alpha_1} \cdots \xi^{\alpha_n}, \tag{C.0.3}$$

where $\xi^\alpha,\ \alpha = 1, ..., n$ are the generating elements of a Grassmann algebra G and $f_0(x),\ f_{\alpha_1\alpha_2}(x),$ $\cdots, f_{\alpha_1\cdots\alpha_n}(x)$ are functions of the (in our case: real) variables $x^i,\ i = 1, ..., m$, belonging to some function space determined through the (physical) fields under consideration. [1]
Let us now introduce the notion of *odd* and *even* elements of the algebra B. The element $\phi_{(o)}$ whose representation (C.0.3) contains only odd powers of ξ is called d'. The element $\phi_{(e)}$ whose representation (C.0.3) involves only even powers of ξ is called even'. Note that the set of all even elements $\phi_{(e)}$ forms a subalgebra of the algebra B. Obviously, even elements commute with all elements of the algebra B, and odd elements anticommute among themselfs. For each odd $\phi_{(o)}$ (even $\phi_{(e)}$) element we introduce the quantity $\varepsilon(\phi_{(o)})$ and $\varepsilon(\phi_{(e)})$), called the *Grassmann parity*, by the rule: $\varepsilon(\phi_{(o)}) = 1$ and $\varepsilon(\phi_{(e)}) = 0$, respectively. The parity of the element $\phi_3 = \phi_1\phi_2$, when ϕ_1 and ϕ_2 have definite parities, is equal to

$$\varepsilon(\phi_3) = (\varepsilon(\phi_1) + \varepsilon(\phi_2))(\mathrm{mod}\ 2) \tag{C.0.4}$$

and the commutation relation between both elements can be presented as

$$\phi_1\phi_2 = (-1)^{\varepsilon(\phi_1)\varepsilon(\phi_2)}\phi_2\phi_1. \tag{C.0.5}$$

The set of all elements $\{\phi\}$ having definite Grassmann parity in the algebra B forms the so-called Z_2*−graded algebra*. This case is very important for purposes of quantum field theory dealing only with quantities having definite Grassmann parity. Now, and afterwards, we will assume every variable or quantity to have definite Grassmann parities. It is also convenient to introduce the Grassmann parity of indices. In what follows we denote the parity of the index A – being related to some quantity – by ε_A.
We shall now consider matrices in the algebra B which will be called *supermatrices*. The supermatrix M is characterized by its matrix elements M_{AB} which belong to B, and each of which has definite parity being characterized by the parities of their indices $(\varepsilon_A, \varepsilon_B)$. The parities of the matrix elements of the supermatrix M are assumed to obey

$$\varepsilon(M_{AB}) = \varepsilon_A + \varepsilon_B. \tag{C.0.6}$$

For supermatrices of equal size having the same order of succession of odd and even indices one can consider the operations of summation and multiplication. The results of these operations are again supermatrices. This opens the possibility to consider also regular functions $f(M)$ of a supermatrix M in an obvious way.
The *normal form* of the supermatrix M is called the supermatrix $M^{(N)}$ which is constructed from M by means of a simultaneous permutation of equally numbered rows and columns to obtain a supermatrix with a definite order of succession of indices: first come all even indices and then all odd ones. The supermatrix $M^{(N)}$ can be presented in the following block–form:

$$\left\|M^{(N)}_{AB}\right\| = \begin{pmatrix} (M_1)_{ij} & (M_2)_{i\beta} \\ (M_3)_{\alpha j} & (M_4)_{\alpha\beta} \end{pmatrix}, \tag{C.0.7}$$

where $A = (i, \alpha),\ B = (j, \beta),\ \varepsilon_i = \varepsilon_j = 0, \varepsilon_\alpha = \varepsilon_\beta = 1$, and the matrix elements of matrices $M_1,\ M_4$ are even ones whereas the matrix elements of matrices $M_2,\ M_3$ are odd ones.
The *supertrace* ($s\mathrm{Tr}M$) of a supermatrix M is defined by the rule

$$s\mathrm{Tr}M = \sum_A (-1)^{\varepsilon_A} M_{AA}. \tag{C.0.8}$$

[1] Of course, this defines only a subclass of Berezin algebras which is sufficient for our purposes.

With the help of the supertrace (C.0.8) one introduces the *superdeterminant* (sDet$M$) by

$$s\mathrm{Det}M = \exp(s\mathrm{Tr}\,\ln M). \tag{C.0.9}$$

Supertrace and superdeterminant possess many properties of trace and determinant of usual matrices. Let us now present some properties of the supertrace and the superdeterminant which are used essentially in the main text:

$$\begin{array}{rlcl}
1) & s\mathrm{Tr}(M+N) & = & s\mathrm{Tr}M + s\mathrm{Tr}N, \\
2) & s\mathrm{Tr}M & = & s\mathrm{Tr}M^{(N)} = \mathrm{Tr}M_1 - \mathrm{Tr}M_4, \\
3) & s\mathrm{Det}M & = & s\mathrm{Det}M^{(N)} = \mathrm{Det}M_1 - \mathrm{Det}^{-1}(M_4 - M_3 M_1^{-1} M_2), \\
4) & s\mathrm{Tr}MN & = & s\mathrm{Tr}NM, \\
5) & s\mathrm{Det}MN & = & s\mathrm{Det}M s\mathrm{Det}N, \\
6) & s\mathrm{Det}M^{-1} & = & s\mathrm{Det}^{-1}M.
\end{array}$$

Here we have introduced the inverse supermatrix M^{-1} of a nonsingular supermatrix M by $MM^{-1} = M^{-1}M = 1$. The conditions of nonsingularity can be expressed in the form

$$\mathrm{Det}M_1^0 \neq 0, \;\; \mathrm{Det}M_4^0 \neq 0,$$

where the matrices M_i^0 are obtained from the matrices M_i by taking the limit $\xi \to 0$. In addition the rank of a supermatrix is defined by to numbers (n_1, n_2) being given according to

$$\mathrm{rank}M = (n_1, n_2) \qquad \text{with} \qquad \mathrm{rank}M_1^0 = n_1, \;\; \mathrm{rank}M_4^0 = n_2.$$

Let us now introduce the notion of *derivation* and *integration* in the Berezin algebra B, eq. (C.0.3). Notice, first of all, that the derivation and integration with respect to the variables $\{x^i\}$ coincides with that of the ordinary derivation and integration, respectively, namely

$$\begin{aligned}
\frac{\partial \phi}{\partial x^i} &= \frac{\partial f_0(x)}{\partial x^i} + \frac{\partial f_\alpha(x)}{\partial x^i}\xi^\alpha + \frac{\partial f_{\alpha_1\alpha_2}(x)}{\partial x^i}\xi^{\alpha_1}\xi^{\alpha_2} + \cdots + \frac{\partial f_{\alpha_1\cdots\alpha_n}(x)}{\partial x^i}\xi^{\alpha_1}\cdots\xi^{\alpha_n}, \\
\int dx^i \phi &= \int dx^i f_0(x) + \Big(\int dx^i f_\alpha(x)\Big)\xi^\alpha + \Big(\int dx^i f_{\alpha_1\alpha_2}(x)\Big)\xi^{\alpha_1}\xi^{\alpha_2} + \\
&\quad + \cdots + \Big(\int dx^i f_{\alpha_1\cdots\alpha_n}(x)\Big)\xi^{\alpha_1}\cdots\xi^{\alpha_n}.
\end{aligned}$$

Derivatives with respect to the Grassmann variables ξ^α are linear operations as well, and it is sufficient to define them on products of generating elements ξ^α only. Because generating elements anticommute among themselves, there exist two types of derivatives: right and left ones. The *left derivative* is defined by the rule:

$$\frac{\partial_l}{\partial \xi^\alpha}\xi^{\alpha_1}\xi^{\alpha_2}\cdots\xi^{\alpha_k} = \sum_{i=1}^{k}(-1)^{P_i}\delta_\alpha^{\alpha_i}\xi^{\alpha_1}\cdots\xi^{\alpha_{i-1}}\xi^{\alpha_{i+1}}\cdots\xi^{\alpha_k}, \tag{C.0.10}$$

where P_i is the parity of the permutation from $(1, 2, ..., i, ..., k)$ to $(i, 1, ..., i-1, i+1, ..., k)$. The *right derivative* is defined as:

$$\frac{\partial_r}{\partial \xi^\alpha}\xi^{\alpha_1}\xi^{\alpha_2}\cdots\xi^{\alpha_k} = \sum_{i=1}^{k}(-1)^{P_{k-i+1}}\delta_\alpha^{\alpha_i}\xi^{\alpha_1}\cdots\xi^{\alpha_{i-1}}\xi^{\alpha_{i+1}}\cdots\xi^{\alpha_k}. \tag{C.0.11}$$

Let us now combine x^i and ξ^α into the common set z^A of variables: $z^A = (x^i, \xi^\alpha)$, $\varepsilon(z^A) \equiv \varepsilon_A$. Then we can present a few essential properties and relations for the derivatives with respect to variables z acting on elements of the Z_2–graded algebra:

$$
\begin{aligned}
&1) \quad \frac{\partial_l}{\partial z^A}\frac{\partial_l}{\partial z^B}\phi &=& \ (-1)^{\varepsilon_A\varepsilon_B}\frac{\partial_l}{\partial z^B}\frac{\partial_l}{\partial z^A}\phi,\\
&2) \quad \frac{\partial_r}{\partial z^A}\frac{\partial_r}{\partial z^B}\phi &=& \ (-1)^{\varepsilon_A\varepsilon_B}\frac{\partial_r}{\partial z^B}\frac{\partial_r}{\partial z^A}\phi,\\
&3) \quad \frac{\partial_l}{\partial z^A}\frac{\partial_r}{\partial z^B}\phi &=& \ \frac{\partial_r}{\partial z^B}\frac{\partial_l}{\partial z^A}\phi,\\
&4) \quad \frac{\partial_l}{\partial z^A}\phi &=& \ (-1)^{\varepsilon_A(\varepsilon(\phi)+1)}\frac{\partial_r}{\partial z^A}\phi,\\
&5) \quad \frac{\partial_l}{\partial z^A}(\phi_1\phi_2) &=& \ \frac{\partial_l\phi_1}{\partial z^A}\phi_2 + (-1)^{\varepsilon_A\varepsilon(\phi_1)}\phi_1\frac{\partial_l\phi_2}{\partial z^A},\\
&6) \quad \frac{\partial_r}{\partial z^A}(\phi_1\phi_2) &=& \ (-1)^{\varepsilon_A\varepsilon(\phi_2)}\frac{\partial_r\phi_1}{\partial z^A}\phi_2 + \phi_1\frac{\partial_r\phi_2}{\partial z^A}.
\end{aligned}
$$

Derivatives of a composite function $\Phi(z) = \phi(\varphi(z))$ of z with respect to z can be calculated as

$$\frac{\partial_l\Phi}{\partial z^A} = \frac{\partial_l\varphi^B}{\partial z^A}\frac{\partial_l\Phi}{\partial\varphi^B}, \qquad \frac{\partial_r\Phi}{\partial z^A} = \frac{\partial_r\Phi}{\partial\varphi^B}\frac{\partial_r\varphi^B}{\partial z^A}.$$

Now let us introduce the definition of the integral in the Berezin algebra B. To this end one needs, in fact, an definition of the integral over odd elements. Introducing formal symbols $d\xi^\alpha$, $\varepsilon(d\xi^\alpha) = 1$ with the following properties

$$\xi^\alpha d\xi^\beta = -d\xi^\beta\xi^\alpha, \quad d\xi^\alpha d\xi^\beta = -d\xi^\beta d\xi^\alpha,$$

the integral over odd elements is defined by the rules

$$\int d\xi^\alpha = 0, \quad \int d\xi^\alpha\xi^\alpha = 1.$$

Formally, the integral over odd elements coincides with the derivative:

$$\int d\xi^{\alpha_1}\cdots d\xi^{\alpha_k}\phi = \frac{\partial_l}{\partial\xi^{\alpha_1}}\cdots\frac{\partial_l}{\partial\xi^{\alpha_k}}\phi = \frac{\partial_r}{\partial\xi^{\alpha_k}}\cdots\frac{\partial_r}{\partial\xi^{\alpha_1}}\phi.$$

In the general case, we consider $dz^A = (dx^i, d\xi^\alpha)$, $\varepsilon(dz^A) \equiv \varepsilon_A$ with the properties

$$dz^A z^B = (-1)^{\varepsilon_A\varepsilon_B} z^B dz^A, \quad dz^A dz^B = (-1)^{\varepsilon_A\varepsilon_B} dz^B dz^A.$$

The integral over the Grassmann variables z^A possesses a number of properties of usual integrals.
(1) The integral of a total derivative is equal to zero:

$$\int dz^A\frac{\partial_r\phi}{\partial z^A} = \int dz^A\frac{\partial_l\phi}{\partial z^A} = 0, \tag{C.0.12}$$

when appropriate boundary conditions with respect to the even variables are assumed. From eq. (C.0.12) the formula of integration by parts follows:

$$\int dz^A\frac{\partial_l\phi_1}{\partial z^A}\phi_2 = -(-1)^{\varepsilon_A\varepsilon(\phi_1)}\int dz^A\phi_1\frac{\partial_l\phi_2}{\partial z^A}; \tag{C.0.13}$$

in eqs. (C.0.12), (C.0.13) no summation over repeated indices has been assumed.
(2) The integral is invariant under shifts of the integration variables:

$$\int dz\phi(z+y) = \int dz\phi(z),$$

where y^A belongs to B and does not depend on the integration variables z^A.
(3) The rules of integration formulated above allow to derive the following formula for a change of variables:

$$\int dz\ \phi(z) = \int dz\ \mathrm{Ber}\, y(z)\ \phi(y(z)),$$

where $\mathrm{Ber}\, y(z)$ is the *Berezinian* of the change of variables $y^A = y^A(z)$

$$\begin{aligned} \mathrm{Ber}\, y(z) &= s\mathrm{Det}R, \quad R^A_B = \frac{\partial_r y^A(z)}{\partial z^B} \\ &= s\mathrm{Det}L, \quad L^A_B = \frac{\partial_l y^B(z)}{\partial z^A}. \end{aligned}$$

The Berezinian can be considered as the extension of the Jacobian according to the change of variables in the case of usual integrals. The properties of the Berezinian follow from the properties of superdeterminants.
(4) Finally, we give the expression for the Gaussian integral ($\varepsilon(J_A) = \varepsilon_A$):

$$\int dz \exp\Big(-\tfrac{1}{2}z^A M_{AB} z^B + J_A z^A\Big) = (2\pi)^{l/2}(s\mathrm{Det}^{-1/2}M)\exp\Big(\tfrac{1}{2}J_A \Lambda^{AB} J_B\Big), \quad \text{(C.0.14)}$$

where the matrix M fulfills the equality $M_{AB} = (-1)^{(\varepsilon_A+\varepsilon_B+\varepsilon_A\varepsilon_B)}M_{BA}$, l is number of even components z^A, and we have used the notation

$$\Lambda^{AB} = (M^{-1})^{AB}(-1)^{\varepsilon_A}.$$

Appendix D

Functional Integrals in Perturbation Theory

Let us consider the definition of functional integrals in Quantum Field Theory sufficient to present the generating functionals of Green's functions in the framework of perturbation theory. The main object of such definition is a functional $Z(J)$ of varialbes (*sources*) $J_A,\ \varepsilon(J_A) \equiv \varepsilon_A$ given in the form of *functional integral*

$$Z(J) = \int \mathcal{D}\phi\ \exp\left\{\frac{i}{\hbar}[S(\phi) + J_A\phi^A]\right\} \equiv \int \mathcal{D}\phi\ F(\phi, J). \tag{D.0.1}$$

In Eq.(D.0.1) it is assumed that the boson fuctional S of fields $\phi^A,\ \varepsilon(\phi^A) \equiv \varepsilon_A$ can be presented in the form

$$S(\phi) = \frac{1}{2}\phi^A M_{AB}\phi^B + V(\phi) \tag{D.0.2}$$

where supermairix M with matrix elements $M_{AB},\ \varepsilon(M_{AB}) = \varepsilon_A + \varepsilon_B$ does not depend on fields ϕ^A and is not singular one. Moreover we assume the matrix M to satisfy the following properties of symmetry:

$$M_{AB} = (-1)^{\varepsilon_A+\varepsilon_B+\varepsilon_A\varepsilon_B}\ M_{BA}.$$

In Eq.(D.0.2) the functional $V(\phi)$ is considered as regular functional with respect to fields ϕ^A, i.e.

$$V(\phi) = \sum_{n>2}\frac{1}{n!}\ V_{A_1\cdots A_n}\ \phi^{A_n}\cdots\phi^{A_1}.$$

By definition the functional integral (D.0.1) within perturbation theory is presented by the following rule

$$Z(J) = \exp\left\{\frac{i}{\hbar}V\left(\frac{\hbar}{i}\frac{\delta}{\delta J}\right)\right\}\ Z_0(J), \tag{D.0.3}$$

where the functional $Z_0(J)$ has the Gaussian form

$$Z_0(J) = \int \mathcal{D}\phi\ \exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}\phi^A M_{AB}\phi^B + J_A\phi^A\right]\right\}$$

and is defined as

$$Z_0(J) = (s\mathrm{Det}M)^{-1/2} \exp\Big\{ -\frac{i}{2\hbar} J_A \Lambda^{AB} J_B \Big\}. \tag{D.0.4}$$

In Eq.(D.0.4) we have used the following notation

$$\Lambda^{AB} = (M^{-1})^{AB}(-1)^{\varepsilon_B}$$

where supermatrix M^{-1} is inverse to M. Notice, the superdeterminat $sDetM$ in Eq.(D.0.4) is some numerical factor which does not depend on variables. We can omit the numerical factors which appear as a result of integration by the definitions (D.0.3), (D.0.4) and do not contain parameters essential for the theory. The reason is the fact that only relative (normalized) quantities in which these factors vanish are of actual interest for Quantum Field Theory.

From definitions (D.0.3), (D.0.4) one can derive the basic properties of the functional integrals. Here we restrict ourself only by enumerations of them omitting all proofs (for details of proofs see, for example, [80], [103]).

The integral (D.0.1) is invariant under the shifts of integration variables

$$\int \mathcal{D}\phi\, F(\phi, J) = \int \mathcal{D}\phi\, F(\phi + \varphi, J). \tag{D.0.5}$$

The integral of the total derivative over any of the integration field ϕ^A is equal to zero

$$\int \mathcal{D}\phi\, \frac{\delta}{\delta\phi^A}\, F(\phi, J) = 0. \tag{D.0.6}$$

From this property formulas of integration by parts follow

$$\int \mathcal{D}\phi\, F(\phi, J)\, \frac{\delta G(\phi, J)}{\delta\phi^A} = -\int \mathcal{D}\phi\, (-1)^{\varepsilon_A \varepsilon(G)} \frac{\delta F(\phi, J)}{\delta\phi^A}\, G(\phi, J) \tag{D.0.7}$$

where derivatives with respect to ϕ^A are considered as right ones.

The formula for the change of variables holds:

$$\int \mathcal{D}\phi\, F(\phi, J) = \int \mathcal{D}\phi\, F(\varphi(\phi), J)\, \mathsf{Ber}[\varphi(\phi)], \tag{D.0.8}$$

where $\mathsf{Ber}[\varphi(\phi)]$ is the Berezinian of the change of variables

$$\mathsf{Ber}[\varphi(\phi)] = s\mathrm{Det}R,\ R^A_B = \frac{\delta\varphi^A(\phi)}{\delta\phi^B}. \tag{D.0.9}$$

In Eqs.(D.0.8), (D.0.9) $\varepsilon(\varphi^A) = \varepsilon(\phi^A)$ and supermatrix R is nonsingular one.

Finally, formula for functional δ-fuction, $\delta(J)$, holds

$$\delta(J) = \int \mathcal{D}\phi\, \exp\Big\{\frac{i}{\hbar}[J_A\phi^A]\Big\}. \tag{D.0.10}$$

The δ-function (D.0.10) obeys the usual property for δ-functions

$$\int \mathcal{D}J\, F(\phi, J)\delta(J) = F(\phi, 0).$$

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